CHAPTER 2

Preliminaries on partitions and homogeneous symmetric polynomials

In this chapter we establish appropriate notations for partitions and homogeneous symmetric polynomials and summarize basic facts about them. They are needed for derivation of zonal polynomials in Chapter 3. It is important to check the definitions and notational conventions given in this chapter since various notational conventions on partitions and homogeneous symmetric polynomials are found in the literature. A large part of the material in this chapter is found in Macdonald (1979), Chapter 1.

§ 2.1 PARTITIONS

A set of positive integers $p = (p_1, \ldots, p_\ell)$ is called a *partition* of n if $n{=}p_1 + \cdots + p_\ell$. To denote p uniquely we order the elements as $p_1 \geq p_2 \geq \cdots \geq p_\ell$. $p_1, \ldots, p_\ell$ are called *parts* of p; ℓ, p_1, n are

$$\begin{aligned} \ell &= \ell(p) = \textit{length of } p = \textit{number of parts}, \\ p_1 &= h(p) = \textit{height of } p, \\ n &= |p| = \textit{weight of } p. \end{aligned} \tag{1}$$

respectively. The multiplicity m_i of i, $(i = 1, 2, \ldots)$ in p is defined as

$$m_i = \textit{number of } j \textit{ such that } p_j = i. \tag{2}$$

Using the m_i's p is often denoted as $p{=}(1^{m_1}2^{m_2}\ldots)$. The set of all partitions of n is denoted by P_n $(=\{p\ :\ |p|{=}n\})$.

It is often convenient to look at p as having any number of additional zeros $p=(p_1,\ldots,p_\ell,0,\ldots,0)$. In this case it is understood that $p_k=0$ for $k>\ell(p)$. With this convention addition of two partitions is defined by $(p+q)_i=p_i+q_i$, $i{=}1,2,\ldots$.

A nice way of visualizing partitions is to associate the following diagrams to them. For $p=(p_1,\ldots,p_\ell)$ we associate a diagram which has p_i dots (or squares) in *i-th* row. For example the diagram of (4,2,2,1) is given by

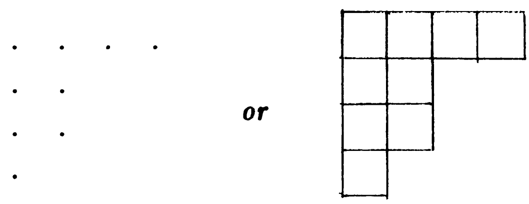

Figure 2.1.

We define the ***conjugate partition*** p' of p by means of this diagram, namely p' is a partition whose diagram is the transpose of the diagram of p. From Figure 2.1 we see $(4,2,2,1)' = (4,3,1,1)$. Clearly $p''=(p')'=p$. Furthermore $|p|=|p'|$, $\ell(p)=h(p')$, $h(p)=\ell(p')$. More explicitly p' is determined by

$$m_i(p')=p_i-p_{i+1},\qquad i=1,\ldots,\ell. \tag{3}$$

Therefore for example

$$\begin{aligned}\ell(p')&=m_1(p')+m_2(p')+\cdots\\&=(p_1-p_2)+(p_2-p_3)+\cdots\\&=p_1=h(p).\end{aligned} \tag{4}$$

Let $s\geq h(p)$, $t\geq \ell(p)$. We define

$$p^*_{s,t}=(s-p_t,s-p_{t-1},\cdots,s-p_1). \tag{5}$$

From Figure 2.2 we have $(4,2,2,1)^*_{4,5}=(4,3,2,2,0)$. Note that

$$|p^*_{s,t}|=st-|p|. \tag{6}$$

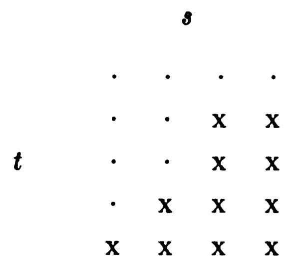

Figure 2.2.

Now we introduce two orderings in $\mathcal{P}_n$. The first one is called the *lexicographic ordering* ($>$). In this ordering p is said to be *higher than* q ($p > q$) if

$$(7) \qquad p_1 = q_1, \ldots, p_{k-1} = q_{k-1}, p_k > q_k \quad \textit{for some } k.$$

This is a total ordering. For example $\mathcal{P}_4$ is ordered as (4)>(3,1)>(2,2)>(2,1,1)>(1,1,1,1).

This ordering is preserved by addition.

Lemma 1. *If* $p^1 \geq q^1$, $p^2 \geq q^2$ *then* $p^1 + p^2 \geq q^1 + q^2$ *with equality iff* $p^1 = q^1, p^2 = q^2$.

Proof is easy and omitted.

Another ordering is the *majorization* ordering. p *majorizes* q ($p \succ q$) if and only if

$$(8) \qquad p_1 \geq q_1,\ p_1 + p_2 \geq q_1 + q_2,\ \ldots\ , p_1 + \cdots + p_k \geq q_1 + \cdots + q_k, \ldots$$

Note that for $k \geq max(\ell(p), \ell(q))$ the equality holds because both sides are equal to the weight n. Majorization is a partial ordering and it is stronger than the lexicographic ordering:

Lemma 2. *If* $p \succ q$ *then* $p \geq q$.

Proof. Suppose $p_1 = q_1, \cdots, p_{k-1} = q_{k-1}, p_k \neq q_k$. Then $p_1 + \cdots + p_k \geq q_1 + \cdots + q_k$ implies $p_k > q_k$. Hence $p > q$. ∎

Remark 1. The converse of Lemma 2 is false. For example (3,1,1,1)> (2,2,2) but there is no majorization between these two.

Analogous to Lemma 1 we have

Lemma 3. *If $p^1 \succ q^1$, $p^2 \succ q^2$, then $p^1 + p^2 \succ q^1 + q^2$ with equality iff $p^1 = q^1$, $p^2 = q^2$.*

Proof. For any k

$$(p_1^1 + p_1^2) + \cdots + (p_k^1 + p_k^2) \geq (q_1^1 + q_1^2) + \cdots + (q_k^1 + q_k^2)$$

with equality iff $p_1^i + \cdots + p_k^i = q_1^i + \cdots + q_k^i$, $i = 1, 2$. ∎

The last lemma in this section is the following:

Lemma 4. *Let $p, q \in P_n$ and let s, t be such that $s \geq h(p), s \geq h(q), t \geq \ell(p), t \geq \ell(q)$. Then $p \succ q$ if and only if $p^*_{s,t} \succ q^*_{s,t}$.*

Proof. (8) holds if and only if $p_1 - n \geq q_1 - n, p_1 + p_2 - n \geq q_1 + q_2 - n, \ldots$. Noting that $n = p_1 + \cdots + p_t = q_1 + \cdots + q_t$, these inequalities in the reversed order imply $p^*_{s,t} \succ q^*_{s,t}$.

§ 2.2 HOMOGENEOUS SYMMETRIC POLYNOMIALS

Let $f(x_1, \ldots, x_k)$ be a polynomial in $x_1, \ldots, x_k$. f is ***homogeneous (of degree n)*** if f has only *n-th* degree terms. f is *symmetric* if

$$f(x_1, \ldots, x_k) = f(x_{i_1}, \ldots, x_{i_k}), \tag{1}$$

where $(i_1, \ldots, i_k)$ is any permutation of $(1, \ldots, k)$. Let V_n denote the set of all *n-th* degree homogeneous symmetric polynomials including the constant $f \equiv 0$. We look at V_n as a vector space where addition is the usual addition of polynomials. Let $f \in V_n$ and suppose that f has a term $a x_1^{p_1} \cdots x_\ell^{p_\ell}$ $((p_1, \ldots, p_\ell) \in P_n)$, then by symmetry it also has a term $a x_{i_1}^{p_1} \cdots x_{i_\ell}^{p_\ell}$ where $i_1, \ldots, i_\ell$ are distinct integers taken from (1,...,k). Counting all different terms we see that

f can be written as a linear combination of *monomial symmetric functions* $\mathcal{M}_p,\ p \in \mathcal{P}_n$,

$$f = \sum_{p \in \mathcal{P}_n} a_p \mathcal{M}_p, \tag{2}$$

where

$$\mathcal{M}_p(x_1, \ldots, x_k) = \sum_{(i_1,\ldots,i_\ell) \subset (1,\ldots,k)} x_{i_1}^{p_1} \cdots x_{i_\ell}^{p_\ell}. \tag{3}$$

In (3) we count only distinguishable terms. For example

$$\mathcal{M}_{(1,1)} = \sum_{i<j} x_i x_j. \tag{4}$$

Sometimes it is more convenient to use *augmented monomial symmetric function* $\mathcal{A}\mathcal{M}_p$ for which the summation in (3) is over all permutations of ℓ different integers from (1,...,k). Therefore for example

$$\mathcal{A}\mathcal{M}_{(1,1)} = \sum_{i \neq j} x_i x_j = 2\mathcal{M}_{(1,1)}. \tag{5}$$

In general

$$\mathcal{A}\mathcal{M}_p = (\prod_{i=1}^{h(p)} m_i!)\mathcal{M}_p. \tag{6}$$

where $(p_1, \ldots, p_\ell) = (1^{m_1} 2^{m_2} \ldots)$.

We note that in (2) the number of variables k does not play an explicit role. Actually $\mathcal{M}_p$ can be defined for any number of variables by (3) and

$$\mathcal{M}_p(x_1, \ldots, x_k, 0, \ldots, 0) = \mathcal{M}_p(x_1, \ldots, x_k). \tag{7}$$

Hence it suffices to consider $\mathcal{M}_p$ which is defined for sufficiently large number of variables. Now suppose

$$\sum_{p \in \mathcal{P}_n} a_p \mathcal{M}_p = 0. \tag{8}$$

We look at terms of the form $x_1^{q_1} \cdots x_\ell^{q_\ell}$. Differentiating (8) p_i times with respect to x_i, $i=1,\ldots,\ell$ we have $(\prod p_i!)a_p = 0$. Hence $a_p = 0$ for all $p \in \mathcal{P}_n$ and $\mathcal{M}_p, p \in \mathcal{P}_n$ are linearly independent in V_n. (Of course if $k < \ell(p)$ then $\mathcal{M}_p(x_1,\ldots,x_k) = 0$ which is linearly dependent in a trivial sense. But as above we consider k to be sufficiently large. For more detail see Section 4.1.) From (2) and (8) it follows that $\{ \mathcal{M}_p,\ p \in \mathcal{P}_n \}$ forms a basis of V_n. This is a rather obvious basis. We want to consider other bases. The following lemma is useful for this purpose.

Lemma 1. *If A is an upper triangular matrix with nonzero diagonal elements, then A^{-1} has the same property. Furthermore if A has diagonal elements* 1 *and integral offdiagonal elements, then A^{-1} has the same property.*

Proof. The first statement is obvious. For the second statement note $|A|=1$. Hence $A^{-1} = (a^{ij}) = (\Delta_{ji})$, where Δ_{ij} is a cofactor of A. But Δ_{ij}'s are integers. ∎

Now we consider products of elementary symmetric functions. Let

$$u_r = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r} \tag{9}$$

be the *r-th* elementary symmetric function. For $p \in \mathcal{P}_n$ we define

$$\mathcal{U}_p = u_1^{p_1 - p_2} u_2^{p_2 - p_3} \cdots u_\ell^{p_\ell}. \tag{10}$$

The degree of $\mathcal{U}_p$ is

$$(p_1 - p_2) + 2(p_2 - p_3) + \cdots + \ell p_\ell = p_1 + \cdots + p_\ell = n. \tag{11}$$

Hence $\mathcal{U}_p \in V_n$. $\mathcal{U}_p$ defined by (10) corresponds to $\mathcal{U}_{p'}$ in Macdonald's notation (1979).

Lemma 2.

$$\mathcal{U}_p = \mathcal{M}_p + \sum_{q<p} a_{pq} \mathcal{M}_q, \tag{12}$$

where a_{pq} are integers.

Proof. Consider monomial terms of the form $x_1^{q_1}x_2^{q_2}\cdots x_k^{q_k}$, $q=(q_1,\ldots,q_k)\in\mathcal{P}_n$. Now

$$\mathcal{U}_p=(x_1+\cdots)^{p_1-p_2}(x_1x_2+\cdots)^{p_2-p_3}\cdots(x_1\cdots x_\ell+\cdots)^{p_\ell}.$$

Hence the highest order term obtained by expanding $\mathcal{U}_p$ is

$$x_1^{p_1-p_2}(x_1x_2)^{p_2-p_3}\cdots(x_1\cdots x_\ell)^{p_\ell}=x_1^{p_1}x_2^{p_2}\cdots x_\ell^{p_\ell},$$

which has coefficient 1. It is clear that other terms are lower in the lexicographic ordering and have integral coefficients. ∎

Remark 1. For a stronger result see Lemma 4.1.1.

We order $\mathcal{M}_p,\mathcal{U}_p$, $p\in\mathcal{P}_n$ according to the lexicographic ordering and form two vectors:

$$\mathcal{M}=\begin{pmatrix}\mathcal{M}_{(n)}\\ \mathcal{M}_{(n-1,1)}\\ \cdot\\ \cdot\\ \cdot\\ \mathcal{M}_{(1^n)}\end{pmatrix},\qquad \mathcal{U}=\begin{pmatrix}\mathcal{U}_{(n)}\\ \mathcal{U}_{(n-1,1)}\\ \cdot\\ \cdot\\ \cdot\\ \mathcal{U}_{(1^n)}\end{pmatrix}. \tag{13}$$

Then Lemma 2 implies that

$$\mathcal{U}=\boldsymbol{A}\mathcal{M},\qquad \boldsymbol{A}=(a_{pq}), \tag{14}$$

where $\boldsymbol{A}$ is a matrix satisfying the condition of Lemma 1. Therefore considering $\boldsymbol{A}^{-1}=(a^{pq})$ we obtain

$$\mathcal{M}_p=\mathcal{U}_p+\sum_{q<p}a^{pq}\mathcal{U}_q, \tag{15}$$

where a^{pq} are integers. We see that $\{\mathcal{U}_p, p\in\mathcal{P}_n\}$ forms another basis of V_n.

Product of $\mathcal{U}$ functions corresponds to the addition of partitions.

Lemma 3.

$$\mathcal{U}_p\mathcal{U}_q = \mathcal{U}_{p+q}. \tag{16}$$

Proof is easy and omitted.

The third basis of V_n is given by product of power sums. Let

$$t_r = \sum x_i^r. \tag{17}$$

For $p \in \mathcal{P}_n$ we define

$$\mathcal{T}_p = t_1^{p_1-p_2} t_2^{p_2-p_3} \cdots t_\ell^{p_\ell}. \tag{18}$$

$\mathcal{T}_p$ defined by (18) corresponds to $\mathcal{T}_{p'}$ in Macdonald (1979) and in Saw (1977). Here we prefer the above definition because of the simpler relation between $\mathcal{U}_p$ and $\mathcal{T}_p$.

Let

$$U(s) = \prod(1 + sx_i) = 1 + u_1 s + u_2 s^2 + \ldots \tag{19}$$

be a generating function of u's. Then

$$\begin{aligned} \log U(s) &= \sum \log(1 + sx_i) \\ &= st_1 - \frac{s^2}{2}t_2 + \cdots + (-1)^{r-1}\frac{s^r}{r}t_r + \cdots. \end{aligned} \tag{20}$$

On the other hand

$$\log U(s) = (u_1 s + u_2 s^2 + \cdots) - \frac{1}{2}(u_1 s + u_2 s^2 + \cdots)^2 + \cdots. \tag{21}$$

Comparing coefficients of s^r in (20) and (21) we see

$$\begin{aligned} t_r &= (-1)^{r-1} r \ \{u_r + \sum_{q>(1^r), q\in\mathcal{P}_r} a_{rq}\mathcal{U}_q\} \\ &= (-1)^{r-1} r \ \{\mathcal{U}_{(1^r)} + \sum_{q>(1^r), q\in\mathcal{P}_r} a_{rq}\mathcal{U}_q\}. \end{aligned} \tag{22}$$

Actually

$$
\begin{aligned}
a_{rq} &= \frac{(-1)^{q_1-1}}{q_1}\binom{q_1}{q_1-q_2, q_2-q_3, \ldots, q_{\ell(q)}} \\
&= \frac{(-1)^{q_1-1}(q_1-1)!}{(q_1-q_2)!\cdots q_{\ell(q)}!}.
\end{aligned} \tag{23}
$$

This follows from the fact that $\mathcal{U}_q$ being a product of q_1 elementary symmetric functions comes only from the q_1-*th* power term in the expansion of log in (21).

Now

$$
\begin{aligned}
\mathcal{T}_p &= \prod_{r=1}^{\ell(p)} t_r^{p_r-p_{r+1}} \\
&= \prod_{r=1}^{\ell(p)} \left[(-1)^{r-1} r \ \{\mathcal{U}_{(1^r)} + \sum_{q>(1^r), q\in\mathcal{P}_r} a_{rq}\mathcal{U}_q\}\right]^{p_r-p_{r+1}}
\end{aligned} \tag{24}
$$

By Lemma 2.1.1 and Lemma 3 the lowest order term in (24) is given by

$$
\begin{aligned}
&\prod_{r=1}^{\ell} [(-1)^{r-1} r\,\mathcal{U}_{(1^r)}]^{p_r-p_{r+1}} \\
&\quad= \prod_{r=1}^{\ell} [(-1)^{r-1} r]^{p_r-p_{r+1}} \, u_1^{p_1-p_2} u_2^{p_2-p_3} \cdots u_\ell^{p_\ell} \\
&\quad= (-1)^{|p|-p_1} (\prod_{r=1}^{\ell} r^{p_r-p_{r+1}}) \mathcal{U}_p.
\end{aligned} \tag{25}
$$

Hence

Lemma 4.

$$
\mathcal{T}_p = \sum_{q\geq p} a_{pq}\mathcal{U}_q, \tag{26}
$$

where

$$
a_{pp} = (-1)^{|p|-p_1} \prod_{r=1}^{\ell(p)} r^{p_r-p_{r+1}} \neq 0. \tag{27}
$$

Let

$$\mathcal{T} = \begin{pmatrix} \mathcal{T}_{(n)} \\ \mathcal{T}_{(n-1,1)} \\ \cdot \\ \cdot \\ \cdot \\ \mathcal{T}_{(1^n)} \end{pmatrix}.$$

Then Lemma 4 shows that

$$\mathcal{T} = \boldsymbol{F}\mathcal{U}, \tag{28}$$

where $\boldsymbol{F}$ is lower triangular with nonzero diagonal elements. Hence $\{ \mathcal{T}_p, p \in \mathcal{P}_n \}$ forms a basis of V_n.

Remark 2. To show that $\{ \mathcal{T}_p, p \in \mathcal{P}_n \}$ is a basis it is much easier to note

$$\mathcal{T}_{p'} = \mathcal{A}\mathcal{M}_p + \sum_{q>p} a_{pq}\mathcal{A}\mathcal{M}_q,$$

where a_{pq} are integers. But we will use Lemma 4 in Section 4.6.

We study symmetric functions further in Section 4.1 and Section 5.3. However the material covered so far suffices to derive zonal polynomials which form another basis of V_n.

Remark 3. For the coefficients of basis functions we generally use $a_p, b_p, \cdots, a_{pq}, b_{pq}$, etc. Since there are many instances of this, it is impossible to use different symbols for each case. For example a_{pq} in Lemma 2 and in Lemma 4 are different.