SCREEN TESTING AND CONDITIONAL PROBABILITY OF SURVIVAL

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1. Introduction

There have been only a few parametric models extensively examined for application to reliability; these include the exponential distribution of Epstein-Sobel (1953) and the Weibull distribution (1961). The one most widely utilized for electronic components has been the exponential model, not only because of its simple and intuitive properties but also because of the extent of the estimation and sampling procedures which have been developed from the theory. However, neither of these models is applicable to the study of screen testing.

One of the early discoveries was that mixtures of exponentially distributed random variables have a decreasing failure rate (Proschan, 1963). Thus any two groups of components with constant, but different, failure rates would, if mixed and sampled at random, exhibit a decreasing failure rate. As a consequence, the family of life lengths with decreasing failure rate certainly arises in practice and particular subsets of this family could be of great utility for specific applications, see, e.g., Cozzolino (1968). We examine one such model with shape and scale parameters α and β, respectively, which is based upon a gamma mixture of exponential distributions. This family was introduced by Afanas'ev (1940) and later by Lomax (1954) as a generalization of a Pareto distribution.

A very important property of this gamma-mixed exponential distribution is that the conditional life remaining after a time τ is again distributed as a gamma-mixed exponential. This is shown in Section 3 and its application to screen testing is discussed.

Kulldorff and Vännman (1973) and Vännman (1976) have studied a variant of the gamma-mixed exponential model containing a location parameter. They obtained a best linear unbiased estimate of the scale parameter assuming that the shape parameter α was known and in a region restricted so that both the mean and the variance exist, namely $\alpha > 2$. When this restriction of $\alpha > 2$ cannot be met, an estimate based on a few order statistics, which are optimally spaced, is given, and tables of the weights as functions of the number of spacings are provided. The estimate based on these order statistics for the $\alpha > 2$ case is claimed to be an asymptotically best linear unbiased estimate. In all cases, the shape parameter was assumed known and the sample was either complete or Type II censored. It is contended that BLUE estimates of the shape parameter are not attainable.

Harris and Singpurwalla (1968) examined the method of moments as an estimation procedure for this same model but again with the shape parameter restricted to $\alpha > 2$ and with a complete sample.

Harris and Singpurwalla (1969) also exhibited the maximum likelihood equations but only for complete samples and without resolving the question of existence of solutions. In Kulldorff and Vännman (1973) there is a brief bibliography of results on parametric estimation for this distribution under various assumptions.

As a consequence of the widespread adoption of integrated circuitry, life testing in electronic manufacturing virtually always provides incomplete samples. This is because of censored tests, the expense and the paucity of failures owing to the high reliability of integrated circuit devices. Such service life data cannot be adequately treated by any of the presently known statistical techniques without employing Bayesian arguments, with their utilization of subjective information. This would indicate the need for an objective estimation procedure making use of the only type of data available and without the potential for bias inherent in Bayesian priors.

In this paper the maximum likelihood estimates are obtained for both the shape and scale parameters of the gamma-mixed exponential (Lomax distribution), and sufficient conditions are given for their existence. These estimates are derived for censored data and *a fortiori* for complete samples, even with a paucity of failure observations.

The existence conditions obtained here for the maximum likelihood estimates apply even to the case where the variance and possibly the mean do not exist: $0 < \alpha \leq 2$. Moreover, the estimates of the shape parameter α which have been obtained from actual data indicate that this region $0 < \alpha \leq 2$ is important because all the estimates obtained of α have been less than unity.

2. The Model

We postulate that the underlying process which determines the length of life of a component under consideration is the following: The quality of construction determines a level of resistance to stress which the component can tolerate. The service environment provides shocks of varying magnitude to the component, and failure takes place when, for the first time, the stress from an environmentally induced shock exceeds the strength of the component.

If the time between shocks of any magnitude is exponentially distributed with a mean depending upon that magnitude, then the life length of each component will be exponentially distributed with a failure rate which is determined by the quality of assembly. It follows that each component has a constant failure rate but that the variability in manufacture and inspection techniques forces some components to be extremely good while a few others are bad and most are in-between.

Let X_λ be the life length of a component in such a service environment, with a constant failure rate λ which is unknown. The variability of manufacture determines various percentages of the λ-values and this variability can be described by some distribution, say G.

Let T be the life length of one of the components which is selected at random from the population of manufactured components. We denote the reliability of this component by R and we have

$$R(t) = P[T > t] \quad \text{for } t > 0 \quad .$$

Let Λ be the random variable which has distribution G. We can write

$$\text{(1)} \qquad R(t) = E\, P[X_\lambda > t \mid \Lambda = \lambda] = \int_0^\infty e^{-\lambda t}\, dG(\lambda) \quad .$$

Because of having a form which can fit a wide variety of practical situations when both scale and shape parameters are disposable, it is assumed that G is a gamma distribution, i.e., for some $\alpha > 0$, $\beta > 0$,

$$g(\lambda) = \frac{\lambda^{\alpha-1} e^{-\lambda/\beta}}{\Gamma(\alpha)\beta^\alpha} \quad \text{for } \lambda > 0 \quad .$$

That this assumption is robust, even when mixing as few as five equally weighted λ's, has been shown by Sunjata (1974) in an unpublished thesis. It follows from equation (1) that the reliability function is

$$\text{(2)} \qquad R(t) = \frac{1}{(1 + t\beta)^\alpha} = e^{-\alpha\, \ell n(1 + t\beta)} \quad .$$

The failure rate, hazard rate, can be shown to be

$$\text{(3)} \qquad q(t) = \frac{\alpha\beta}{1 + t\beta} \quad ,$$

which is a decreasing function of $t > 0$.

Maximum likelihood estimates for α, β and hence R(t) and q(t) are given in Section 4.

3. Residual Life Property of the Model

An important property of this model is that residual life on a component is distributed as a gamma-mixed exponential. Thus a "burn-in" test of a component will yield a residual life which is also in the same family. This property also holds for mixed exponential distributions where the mixing distribution is other than gamma.

The residual life T_h of a component is defined to be the life remaining after time h, given that the component is alive at time h. It can be shown that: A burn-in for h units of time on a component with initial life determined by a gamma-mixed exponential distribution with parameters α and β will yield a residual life T_h and will be distributed as a gamma-mixed exponential with parameters α and $\beta_h = \frac{\beta}{1+\beta h}$.

It follows that this life length model is "used better than new" or "new worse than used" in the sense that T_h is stochastically larger than T for all $h > 0$.

An important consequence of this property is that one can calculate the value of the increased reliability attained by burn-in procedures as compared with the cost of conducting them. It has long been the practice to burn-in electronic components based on intuitive ideas of "infant mortality" in order to provide reasonable assurance of having detected all defectively assembled units. This model, whenever it is applicable, makes possible an economic analysis. A variation of this result has been discussed by Bhaltacharya (1963).

Consider the following data from a screen test of flight control electronic packages:

Failure times: 1, 8, 10

Alive times: 59, 72, 76, 113, 117, 124, 145, 149, 153, 182, 320 .

Each package has recorded, in minutes, either a failure time or an alive time. An alive time is the time the test was terminated with the package still functioning. In this example the reason for the censoring was that the test equipment was needed elsewhere, that funds for testing were depleted, etc. We define this censoring to be *random*, that is, when a package is operating the time of censoring depends on some censoring distribution which is independent of the failure distribution. Type I censoring at a preassigned time is a special case of random censoring. Random censoring differs, however, from Type II censoring, where censor occurs at some preassigned number of failures. For this example the censoring is random in that time on test equals minimum {failure time, random censor time}.

The example data do not exhibit a constant failure rate. Even if we assume a fourth failure rate at 59 minutes and Type II censoring at the fourth failure, we reject the hypothesis of a constant or increasing failure rate in favor of a decreasing failure rate (using F-criteria suggested by Gnedenko, et al. (1969)). If we assume that the data are from a gamma-mixed exponential, we find (using equations in Section 4) that $\hat{\alpha} = 0.0453$ and $\hat{\beta} = 1.03$.

The question is: how long should packages of this type be subjected to a screen test? Let $\hat{\beta}_h$ denote the estimated scale parameter after a burn-in for h units of time. For this package 96 hours of burn-in time has been used. Figure 1 shows $\hat{\beta}_h$ as a function of h, with $\hat{\beta}$ as above. If it could be assumed that burn-in tests were equivalent to actual use tests, then one could estimate reliability at time t as a function of burn-in time h. For example

$$\hat{R}(t) = [1 + t\hat{\beta}_h]^{-\hat{\alpha}} \quad .$$

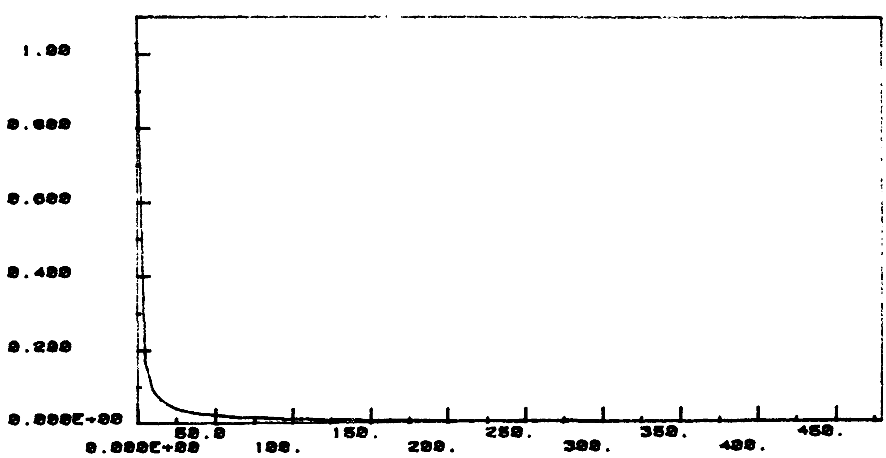

FIGURE 1. Graph of $\hat{\beta}_h$ as a function of burn-in time h in minutes.

Figure 2 shows the change in estimated reliability at 20 minutes as a function of burn-in time h, in minutes, for $\hat{\alpha}$ and $\hat{\beta}$ as above.

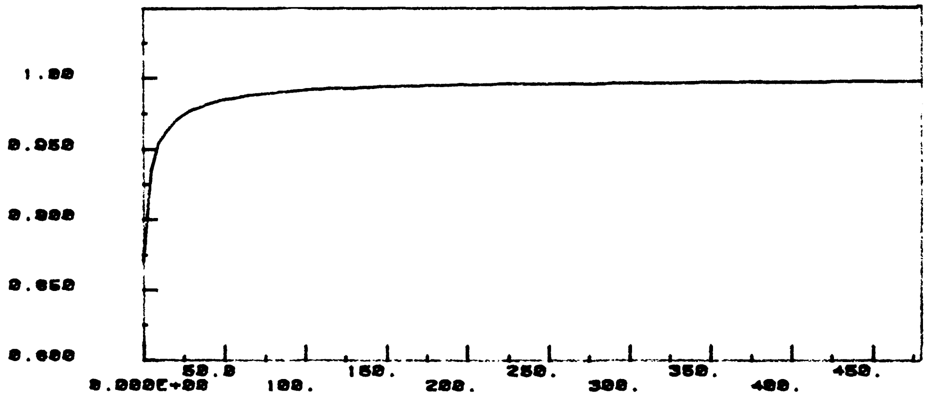

FIGURE 2. Graph of estimated burn-in reliability at 20 minutes as a function of previous burn-in time h in minutes.

Since burn-in reliability is usually proportional to in-use reliability, one must question accepted burn-in times of 96 hours for equipment which exhibit burn-in data of the type given in the example.

4. Estimation of Parameters Using Incomplete Samples

When components having a decreasing failure rate are tested, the samples are virtually always incomplete in the sense that testing is stopped before all components have failed. A datum on a component that "failure has not yet occurred after a specified life" is called an alive time. Samples containing such observations are censored. In our experience with electronic components, Type I and Type II censoring occur less frequently than random censoring. However, until recently Type I and Type II censoring were the cases addressed in the literature. The results which follow apply to Type I, Type II, or random censoring, as defined above. Complete samples are a special case.

It is assumed in this section that we are given a sample $\underline{t} = (t_1, \ldots, t_k, \ldots, t_n)$, where $t_1, \ldots, t_k$ are ordered observations of times of failures while $t_{k+1}, \ldots, t_n$ are ordered observed alive times where censoring is one of the types described above. It is assumed that $k \geq 1$.

Some results will now be given on maximum likelihood estimation of the unknown shape and scale parameters for the gamma-mixed exponential in the case of censored samples.

Remark 1: When the scale parameter β is known, there exists a m.l.e. of α given by

$$\hat{\alpha} = \frac{k}{\sum_{1}^{n} \ln(1+\beta t_i)} .$$

This result is not new. If β is known then the values $y_i = \ln(1+\beta t_i)$ are observations from an exponential distribution with unknown failure rate α.

THEOREM 2: For given $\alpha, \underline{t}$ there exist a unique m.l.e. of β, denoted by $\hat{\beta}$. It is given implicitly as the positive root of the equation

$$(4) \qquad 1 - \frac{1}{k} \sum_{1}^{k} \frac{\beta t_j}{1+\beta t_j} - \frac{\alpha}{k} \sum_{1}^{n} \frac{\beta t_i}{1+\beta t_i} = 0 .$$

PROOF: We have the vector $\underline{t} = (t_1, \ldots, t_k, \ldots, t_n)$ corresponding to the observed events $[T_i = t_i]$ for $i = 1, \ldots, k$ and $[T_i > t_i]$ for $i = k+1, \ldots, n$, where $k \geq 1$. By definition the log-likelihood is given by

$$e^L = \prod_{j=1}^{k} q(t_j) \prod_{i=1}^{n} R(t_i) .$$

Substituting, taking logarithms, and simplifying we have

$$L(\beta|\alpha,\underline{t}) = k\ln(\alpha\beta) - \sum_{j=1}^{k} \ln(1+\beta t_j) - \alpha \sum_{i=1}^{n} \ln(1+\beta t_i) .$$

Dividing by k we write

$$L(\beta|\alpha,\underline{t}) = \ln\alpha + \ln\beta - \frac{1}{k} \sum_{j=1}^{k} \ln(1+\beta t_j) - \frac{\alpha}{k} \sum_{i=1}^{n} \ln(1+\beta t_i) .$$

$$L'(\beta|\alpha,\underline{t}) = \frac{1}{\beta} \left[1 - \frac{1}{k} \sum_{j=1}^{k} \frac{\beta t_j}{1+\beta t_j} - \frac{\alpha}{k} \sum_{i=1}^{n} \frac{\beta t_i}{1+\beta t_i} \right] .$$

For $k \geq 1$, the m.l.e. for β is given as the positive root of

$$(6) \qquad A(\beta) = 1 - \frac{1}{k}\sum_{j=1}^{k}\frac{\beta t_j}{1+\beta t_j} - \frac{\alpha}{k}\sum_{i=1}^{n}\frac{\beta t_i}{1+\beta t_i} = 0 \quad .$$

The root of equation (6) exists and is unique since $A(0) = 1$, $A(\infty) < 0$ and A is strictly decreasing over $[0,\infty]$.

When α and β are both unknown there are sets of n positive numbers (say $\underline{t} = (t_1,\ldots,t_k,\ldots,t_n)$ with $k \leq n$ designated as alive times) that cannot be used to estimate both unknown parameters. It is shown that both m.l.e.'s exist whenever the sample satisfies the condition

$$(7) \qquad 2\left(\sum_{j=1}^{k}\frac{t_j}{k}\right)\left(\sum_{i=1}^{n}\frac{t_i}{n}\right) < \sum_{i=1}^{n}\frac{t_i^2}{n} \quad .$$

If the sample fails to satisfy this condition then the model may not be appropriate and a constant failure rate model or a convex failure rate model may be indicated rather than a decreasing failure rate model.

One can check that a complete sample of failure times, i.e., with $k = n$ will satisfy (7) if the sample standard deviation exceeds the mean. For decreasing failure rate distributions the standard deviation does exceed the mean, when they both exist.

THEOREM 3: For a given sample $\underline{t}$, with $k \geq 1$, both α,β unknown, the m.l.e. of $\beta,\hat{\beta}$, exists as the smallest positive root of

$$(8) \qquad 1 - \frac{1}{k}\sum_{j=1}^{k}\frac{\beta t_j}{1+\beta t_j} - \frac{\sum_{i=1}^{n}\frac{\beta t_i}{1+\beta t_i}}{\sum_{i=1}^{n}\ell n(1+\beta t_i)} = 0 \ ,$$

if the sample satisfies

$$2\left(\sum_{i=1}^{n} \frac{t_i}{n}\right)\left(\sum_{j=1}^{k} \frac{t_j}{k}\right) < \sum_{i=1}^{n} \frac{t_i^2}{n} .$$

Given $\hat{\beta}$, the m.l.e. of α is given by

$$\hat{\alpha} = \frac{k}{\sum_{i=1}^{n} \ell n(1+\hat{\beta}t_i)} .$$

PROOF: Consider the log-likelihood function defined in (5). Since α is unknown, we write (5) as $L(\alpha,\beta|\underline{t})$. All stationary points, which are determined by $\underline{t}$, can be found by the simultaneous solution of $\partial L(\alpha,\beta|\underline{t})/\partial \alpha = 0$ and $\partial L(\alpha,\beta|\underline{t})/\partial \beta = 0$. This yields two equations in α and β:

(9) $$\frac{1}{\alpha} = \frac{1}{k} \sum_{i=1}^{n} \ell n(1+\beta t_i); \quad 1 - \frac{1}{k} \sum_{j=1}^{k} \frac{\beta t_j}{1+\beta t_j} = \frac{\alpha}{k} \sum_{i=1}^{n} \frac{\beta t_i}{1+\beta t_i} .$$

Combining these into a single equation in β, we seek $\hat{\beta}$ as the root of equation (8), i.e., the root of

$$B(\beta) = 1 - \frac{1}{k} \sum_{j=1}^{k} \frac{\beta t_j}{1+\beta t_j} - \frac{\sum_{i=1}^{n} \frac{\beta t_i}{1+\beta t_i}}{\sum_{i=1}^{n} \ell n(1+\beta t_i)} = 0 .$$

Note that $\lim_{\beta\to 0} B(\beta) = 0$. We want to find a sufficient condition for the equation $B(\beta) = 0$ to have a positive root. It is clear that $B(\beta) \underset{\beta\to\infty}{\to} 0$ as a negative quantity. We will show that if the sample satisfies condition (7) then $B(\beta)$ is positive in a neighborhood of zero. Under these conditions there exists a β such that $B(\beta) = 0$.

To show that if condition (7) holds then $B(\beta)$ is positive in a neighborhood of zero consider the function

$$\frac{1}{\beta} B(\beta) = - \left[\frac{\sum_{i=1}^{n} \frac{t_i}{1+\beta t_i} - \frac{1}{k} \sum_{j=1}^{k} \frac{1}{1+\beta t_j} \sum_{i=1}^{n} \frac{\ln(1+\beta t_i)}{\beta}}{\sum_{i=1}^{n} \ln(1+\beta t_i)} \right] .$$

Repeated application of l'Hôpital's rule shows that

$$\lim_{\beta \to 0} \frac{1}{\beta} (B(\beta)) = \frac{\sum_{i=1}^{n} t_i^2/2 - \frac{1}{k} \sum_{j=1}^{k} t_j \sum_{i=1}^{n} t_i}{\sum_{i=1}^{n} t_i} ,$$

which is positive if $\sum_{i=1}^{n} \frac{t_i^2}{n} > 2 \sum_{j=1}^{k} \frac{t_j}{k} \sum_{i=1}^{n} \frac{t_i}{n}$.

The solution for $\hat{\alpha}$ follows immediately from the first equation in (9).

Computationally equations (4) and (8) are not difficult to solve. In fact, their solutions are obtainable using a simple programmable calculator.

5. Conclusion

If screen tests are effective we should be observing a decreasing failure rate as a function of time on test. In practice it is often assumed that as the result of screen tests the surviving components are exponentially lived. Of course, this is not always the case. The important question is how long should a component be burned-in in order to make its residual life distribution acceptable? The decreasing failure rate gamma-mixed exponential model, when applicable, allows estimation of the improvement in reliability as a function of screen test time.

This study suggests that if a component has a life distribution with decreasing failure rate it is the alive times within the data which contribute principally to the estimation of the parameters (and thereby to the determination of reliability) since only one failure observation is required even to estimate two parameters, presuming the data of alive times are ample.

The usual justification for using maximum likelihood estimates is owing to their asymptotically optimal properties and to their asymptotic normality. The problem of obtaining exact sampling distributions of the maximum likelihood estimators of the parameters for the model studied seems to be difficult because the estimates are only implicitly defined. Myhre and Saunders (1981) have shown that when they exist, the m.l.e.'s for α and β based on Type I or on random sampling are consistent and are asymptotically normally distributed. In addition, Lucke and Myhre (1980) have shown that the distribution function estimated using the joint m.l.e.'s of the parameters is closer to the true distribution function for regions of interest in reliability theory than is the estimated distribution function using a known shape parameter and the BLUE estimate of Vännman (1976) for the scale parameter.

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