CONFIDENCE BOUNDS FOR THE EXPONENTIAL MEAN IN TIME-TRUNCATED LIFE TESTS

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1. Introduction

A commonly occurring life-test situation is: a time T is specified, n units are put on test without replacement and the successive ordered times-to-failure $X_1 \leq \cdots \leq X_r < T$, $r \leq n$, are observed. This life testing procedure is commonly referred to as Type 1 censoring, which will be assumed throughout this paper.

Here we suppose that each of the n units tested has the same one-parameter exponential life-time distribution of which the mean is θ. Computing methods will be developed for the lower confidence bound on θ based on the maximum-likelihood estimate (MLE).

The MLE, say $\hat{\theta}$, has been given by Halperin (1950), Bartlett (1953a,b), Deemer and Votaw (1955) and Bartholomew (1957):

(1) $$\hat{\theta} = [\sum_{i=1}^{r} X_i + (n-r)T]/r, \quad r \geq 1 \quad .$$

When $r=0$, $\hat{\theta}$ is undefined.

Approximate confidence intervals have been studied by Bartlett (1953a,b) and Bartholomew (1963), and the asymptotic properties of $\hat{\theta}$ have been investigated by Deemer and Votaw (1955) and Yang and Sirvanci (1977). In particular, $\hat{\theta}$ is consistent and asymptotically unbiased, i.e., $\lim_{n\to\infty} E_c(\hat{\theta}) = \theta$, where E_c denotes the conditional expectation on $r>0$.

The exact distribution of $\hat{\theta}$ has had an interesting history: Halperin (1950) gave the distribution of $\hat{\theta}$, conditional on r, and described very briefly how the unconditional distribution could be obtained; Halperin (1960) gave the distribution of $r\hat{\theta}$; Bartholomew (1963) was the first to give the distribution of $\hat{\theta}(r\geq 1)$; Hoem (1969) essentially presented the distribution of $\hat{\theta}$ again, along with other results.

Barlow, et al. (1968) developed a computer program for obtaining interval estimates of θ, and Spurrier and Wei (1980) presented a hypothesis test procedure based on $\hat{\theta}$, in which $r=0$ is not conditioned out. In each of these two papers the exact distribution of $\hat{\theta}$ was used. Virtually all authors have commented on the computational complexity of the exact distribution. For example, it was seventeen years (1963 to 1980) from the availability of the exact distribution of $\hat{\theta}$ to the development of hypothesis tests for this most important life-testing situation.

The inclusion of $r=0$ in the hypothesis test procedure of Spurrier and Wei (1980) is tantamount to taking $\hat{\theta}=\infty$ (i.e., always accept $H_o: \theta \geq \theta_o$) when $r=0$. In that sense it should be noted that the two distribution functions are simply related:

$$(2) \qquad P(\hat{\theta}\geq t \mid r\geq 1) = \frac{P(\hat{\theta}\geq t \mid r\geq 0) - e^{-(n\theta^{-1}T)}}{1 - e^{-(n\theta^{-1}T)}} .$$

As indicated by the numerator in (2) the inclusion of $r=0$ in testing $H_o: \theta \leq \xi$ ($\theta \geq \xi$) precludes test sizes $\alpha \leq \exp(-n\xi^{-1}T)$ ($(1-\alpha) < \exp(-n\xi^{-1}T)$). On the other hand, inclusion of $r = 0$ means that the test is always applicable.

Since $\hat{\theta}$ is undefined when $r = 0$, in this case one can use the fact that the number of failures r is a binomial random variable with parameters n and $p = 1 - \exp(-T/\theta)$ to obtain a confidence bound. Hence, for confidence level $1 - \alpha$,

$$\theta_*/T = n(-\ell n\alpha)^{-1} \quad ,$$

is a $(1 - \alpha)$-level lower confidence bound for θ/T.

The case of a random sample of size n from an exponential distribution (right) truncated at T is different from that considered here. The former case has been considered in some detail by, among others, Bain, et al. (1977) and Deemer and Votaw (1955).

2. Exact Confidence Bounds for θ

Hypothesis tests involve a critical region. Hence the results of Spurrier and Wei (1980) cannot be used for obtaining interval estimates which require direct use of the distribution function as in the computer program of Barlow, et al. (1968).

Following Barlow, et al., if the confidence coefficient is $1 - \alpha$, then based on Bartholomew's (1963) result θ_* must satisfy

$$1 - \alpha = (1 - e^{-nT/\theta_*})^{-1} \sum_{k=1}^{n} \binom{n}{k} e^{-(n-k)T/\theta_*} \sum_{i=0}^{k} \binom{k}{i} (-1)^i e^{-iT/\theta_*}$$

$$\times\ \phi(2[k\hat{\theta} - (n - k + i)T]/\theta_*,\ 2k) \quad , \tag{3}$$

where

$$\phi(u,\nu) = \begin{cases} [2^{\nu/2}\,\Gamma(\nu/2)]^{-1} \int_0^u e^{-t/2}\, t^{(\nu/2)-1}\, dt & u \geq 0 \\ 0, & u < 0 \end{cases} ,$$

and $r \geq 1$.

Inspecting (3), one notes that five quantities must be specified to obtain θ_*: $n, T, r, \hat{\theta}$ and $1-\alpha$ and thus it appears that a computer program must be used for each different estimating situation. However, the following points are noted:

i) the random variable r is needed solely to compute $\hat{\theta}$.

ii) the variables T and θ_* appear always as the ratio $(\theta_*/T)^{-1}$.

iii) $[k\hat{\theta} - (n-k+i)T]/\theta_* = [\frac{k\hat{\theta}}{T} - (n-k+i)](T/\theta_*)$

and hence $\hat{\theta}$ appears only with T as $\hat{\theta}/T$ and, again,

θ_* and T appear only as the ratio $(\theta_*/T)^{-1}$.

iv) only a few confidence coefficients (i.e., 0.90, 0.95) are commonly used.

v) n, $\hat{\theta}$ and T are known.

Thus, $\hat{\theta}/T$ can be an entry variable (with n and $1-\alpha$) to obtain θ_*/T. Multiplication by T then yields θ_*.

3. Computational Aspects and an Approximation

A computer program along the lines given by Barlow, et al. (1968) was written for the purpose of tabulating values of θ_*/T. It produced results agreeing exactly with their results (to four significant digits) for the two examples given by those authors, namely $\theta_* = 28.49$ and $\theta_* = 32.09$, for $\hat{\theta} = 51.166$, $T = 50$, $n = 10$ and $\alpha = 0.10$ and 0.05, respectively. Results of the computer program also agreed rather well with the asymptotic normal and chi-square approximations in an example given by Bartholomew (1963) for $\hat{\theta}/T = 0.705$, $n = 20$, $\alpha = 0.025$. Bartholomew's asymptotic lower confidence bounds on θ/T are 0.45 and 0.46. The computer program yields 0.39.

It was found, however, that in another example given by Bartholomew (1963) for $\hat{\theta}/T = 3.35$, $n = 40$, $\alpha = 0.025$, the asymptotic lower bound is 1.9, while the computer program yields 0.63 for θ_*/T, the "exact" lower confidence bound.

The vast discrepancy in these latter results appears to be due to a combination of factors. First the chi-square subroutine used in the computer program produces results that are accurate to about 10^{-3} for $\hat{\theta}/T$ less than 1. As $\hat{\theta}/T$ approaches n (where $\hat{\theta}/T < n$), the accuracy in the output of the subroutine declines. Second, $(n+1)(n+2)/2$ terms involving these chi-square evaluations are summed. Thus, when $n = 40$, there are 861 such terms to be summed. Another possible explanation, of course, is that a sample size of 40 is too small for asymptotic results to apply.

To attempt to ascertain the level of accuracy of the estimate provided by the computer program, three avenues are explored. A more presice chi-square evaluation, from IBM, with 10^{-9} accuracy was incorporated in the program. In addition, a method was found for calculating approximate confidence bounds which do not depend on asymptotic values. This allowed for comparisons to be made for small values of n. Finally a simulation study was performed.

We noted above the asymptotic chi-square approximation used by Bartholomew. This is simply a two-moment fit that uses the conditional (on $r > 0$) mean m and variance v of $\hat{\theta}$ (see Patnaik (1949)): $2m\hat{\theta}/v$ is approximately a chi-square variate with $\nu = 2m^2/v$ degrees of freedom. Bartholomew used the asymptotic conditional mean and variance, respectively, of $\hat{\theta}$ for m and v and noted that for T infinite, $2m\hat{\theta}/v$ is an exact chi-square variate with 2n degrees of freedom since, in this case, the expectation and variance of $\hat{\theta}$ are θ and θ^2/n, respectively.

Yang and Sirvanci (1977) have provided expressions for the exact conditional mean and variance of $\hat{\theta}$. These can readily be converted to exact conditional moments of $\hat{\theta}/T$, involving only the parameter θ/T. Dividing the expressions for the conditional mean and variance of $\hat{\theta}$ given by those authors by T and T^2, respectively, we obtain

$$E_c(\hat{\theta}/T) = \theta/T - 1/p + nE_c(1/r)$$

and

$$\mathrm{Var}_c(\hat{\theta}/T) = E_c(1/r)(\theta^2/T^2 - q/p^2) + n^2 \mathrm{Var}_c(1/r) ,$$

where

$$q = e^{-T/\theta}, \quad p = 1 - q ,$$

$$E_c(1/r) = \sum_{k=1}^{n} \binom{n}{k} p^k q^{n-k}/[k(1-q^n)] ,$$

and

$$E_c(1/r^2) = \sum_{k=1}^{n} \binom{n}{k} p^k q^{n-k}/[k^2(1-q^n)] .$$

In these expressions $E_c(\cdot)$ and $\mathrm{Var}_c(\cdot)$ indicate conditioning on $r > 0$. In the sequel we let $m = E_c(\hat{\theta}/T)$ and $v = \mathrm{Var}_c(\hat{\theta}/T)$.

Use of the chi-square approximation results generally in noninteger degrees of freedom. Thus, to obtain iteratively the appoximate lower confidence bound for θ/T we use the Wilson-Hilferty (1931) transformation of chi-square to normality, namely, for $\nu = 2m^2/v$,

$$3\sqrt{\nu/2}\,\left[\left(\frac{\hat{\theta}/T}{m}\right)^{1/3} + 2/(9\nu) - 1\right]$$

is approximately $N(0,1)$. This transformation has been used in several instances by Mann, Schafer and Singpurwalla (1974), particularly in obtaining approximate lower confidence bounds for scale parameters. McGinnis and Sammons (1970), in an investigation of gamma approximations, showed that the Wilson-Hilferty and Severo and Zelen (1960) equations are most effective for our purposes. Their results and results of Mann, Schafer and Singpurwalla indicate that the Wilson-Hilferty approximation yields acceptable results (at least two good significant figures) as long as the number of degrees of freedom $\nu = 2m^2/v$ is 3.5 or greater.

A comparison was made of the exact lower confidence bounds on θ/T (with confidence level $1 - \alpha$) obtained from the computer program with the IBM chi-square subroutine and the bounds based on the chi-square approximation, for

$n = 5(5)20$, $\hat{\theta}/T = n/2$, $n/4$, 1, $1/n$, $2/n$, $4/n$ and $\alpha = 0.10$, 0.05 and 0.01. This resulted in discrepancies of 2 or less in the second to fourth significant figures in the corresponding values of θ_*/T for $n = 10$, 15, 20 and $\hat{\theta}/T$ less than or equal to 1, and very large discrepancies for large values of $\hat{\theta}/T$. In all of the cases evaluated, $\nu = 2m^2/v > 3.5$.

A simulation study was then undertaken to determine the accuracy of the values of θ_*/T computed by the two methods. In the study we evaluated the probability p associated with obtaining values less than $\hat{\theta}/T$ when θ_*/T is the true value of θ/T. When θ_*/T (associated with confidence level $1 - \alpha$) is the correct value, p is equal to α.

A Monte Carlo sample size of 5000 was used with various combinations of specified values of n, $\hat{\theta}/T$ and α, including those chosen for the earlier comparison of the two methods. Particular attention was paid to very large and very small values of $\hat{\theta}/T$.

The specified values for the confidence levels are essentially correct (i.e., deviations in $\hat{p}$, the calculated p, from the specified value of α are within expected bounds) for the "exact" values of θ_*/T calculated by means of the computer program for $\alpha = 0.1$, 0.05 and 0.01 when $n = 2(1)10$ and $\hat{\theta}/T < n - 0.2$ and when $n = 15$, 20 and $\hat{\theta}/T \leq 1$. Thus, it appears that an increase in sample size beyond 10 causes problems for the computer-program estimator of θ_*/T unless $\hat{\theta}/T$ is quite small. Tabulations for $n = 2(1)10$ appear in Table 1.

The asymptotic estimator of θ_*/T also works best for small values of $\hat{\theta}/T$, as indicated earlier by the comparisons made with the computer-program estimates. A simulation made to examine the asymptotic bounds for $n = 40$ gave results that confirmed this conclusion. For $\hat{\theta}/T = 0.1$, 0.2, 0.5, 1(1)5, $\hat{p}$ is within expected bounds for $\alpha = 0.1$ and 0.05. For $\alpha = 0.01$ and for values of $\hat{\theta}/T$ 10 or larger the values obtained for θ_*/T are not accurate.

TABLE 1. Values of θ_*/T, lower confidence bound at level $1-\alpha$, corresponding to $\hat{\theta}/T$, for $n=2(1)10$, $\alpha=0.1$, 0.05, 0.01.

n	$\hat{\theta}/T$	α=0.1	α=0.05	α=0.01
2	1.8	1.4575	0.7913	0.4127
2	1.7	0.9256	0.6120	0.3580
2	1.6	0.7259	0.5180	0.3217
2	1.5	0.6118	0.4552	0.2935
2	1.4	0.5330	0.4075	0.2694
2	1.3	0.4724	0.3681	0.2476
2	1.2	0.4224	0.3336	0.2271
2	1.1	0.3790	0.3023	0.2076
2	1.0	0.3397	0.2729	0.1885
2	0.9	0.3388	0.2725	0.1883
2	0.8	0.3345	0.2701	0.1876
2	0.7	0.3230	0.2625	0.1842
2	0.6	0.2991	0.2445	0.1740
2	0.5	0.2572	0.2108	0.1507
2	0.4	0.2056	0.1686	0.1206
2	0.3	0.1542	0.1264	0.0904
2	0.2	0.1028	0.0843	0.0603
2	0.1	0.0514	0.0421	0.0301
3	2.8	2.2646	1.2154	0.6347
3	2.7	1.4559	0.9543	0.5579
3	2.6	1.1540	0.8209	0.5092
3	2.5	0.9885	0.7341	0.4728
3	2.4	0.8780	0.6703	0.4436
3	2.3	0.7959	0.6196	0.4181
3	2.2	0.7304	0.5770	0.3952
3	2.1	0.6757	0.5399	0.3741
3	2.0	0.6282	0.5066	0.3542
3	1.5	0.6289	0.5062	0.3540
3	1.4	0.6254	0.5043	0.3526
3	1.3	0.6118	0.4955	0.3478
3	1.2	0.5846	0.4761	0.3367
3	1.1	0.5418	0.4439	0.3167
3	1.0	0.4844	0.3991	0.2871
3	0.9	0.4273	0.3541	0.2566
3	0.8	0.3798	0.3161	0.2301
3	0.7	0.3383	0.2823	0.2059
3	0.6	0.3001	0.2512	0.1836
3	0.5	0.2655	0.2235	0.1651
3	0.4	0.2240	0.1894	0.1418
3	0.3	0.1690	0.1429	0.1071
3	0.2	0.1127	0.0953	0.0714
3	0.1	0.0563	0.0476	0.0357
4	3.8	3.0718	1.6410	0.8567
4	3.7	1.9890	1.2958	0.7574
4	3.6	1.5826	1.1227	0.6959
4	3.5	1.3641	1.0117	0.6513
4	3.4	1.2213	0.9312	0.6156
4	3.3	1.1166	0.8684	0.5855
4	3.2	1.0346	0.8166	0.5593
4	3.1	0.9672	0.7724	0.5355
4	3.0	0.9096	0.7333	0.5136
4	2.0	0.9091	0.7338	0.5124
4	1.9	0.9026	0.7289	0.5103
4	1.8	0.8800	0.7119	0.5017
4	1.7	0.8385	0.6811	0.4842
4	1.6	0.7789	0.6365	0.4563
4	1.5	0.7060	0.5812	0.4207
4	1.4	0.6395	0.5303	0.3874
4	1.3	0.5895	0.4912	0.3609
4	1.2	0.5510	0.4606	0.3391
4	1.1	0.5219	0.4372	0.3218
4	1.0	0.5011	0.4212	0.3107
4	0.9	0.4749	0.4015	0.2989
4	0.8	0.4318	0.3667	0.2753
4	0.7	0.3792	0.3228	0.2437
4	0.6	0.3282	0.2803	0.2123
4	0.5	0.2809	0.2406	0.1830
4	0.4	0.2329	0.2002	0.1534
4	0.3	0.1794	0.1546	0.1193
4	0.2	0.1197	0.1032	0.0797
4	0.1	0.0599	0.0516	0.0398
5	4.8	3.7948	2.0679	1.0784
5	4.7	2.5209	1.6372	0.9567
5	4.6	2.0113	1.4243	0.8823
5	4.5	1.7396	1.2888	0.8290
5	4.4	1.5641	1.1915	0.7869
5	4.3	1.4366	1.1161	0.7521
5	4.2	1.3376	1.0547	0.7220
5	4.1	1.2570	1.0029	0.6952
5	4.0	1.1889	0.9577	0.6708
5	2.5	1.1889	0.9567	0.6701
5	2.4	1.1807	0.9489	0.6666
5	2.3	1.1489	0.9250	0.6532
5	2.2	1.0925	0.8839	0.6288
5	2.1	1.0158	0.8275	0.5938
5	2.0	0.9269	0.7618	0.5524
5	1.9	0.8493	0.7041	0.5155
5	1.8	0.7933	0.6619	0.4876
5	1.7	0.7529	0.6309	0.4661
5	1.6	0.7262	0.6098	0.4507
5	1.5	0.7138	0.6003	0.4437
5	1.4	0.7056	0.5942	0.4408
5	1.3	0.6826	0.5759	0.4303
5	1.2	0.6389	0.5410	0.4071
5	1.1	0.5832	0.4958	0.3750
5	1.0	0.5284	0.4510	0.3423
5	0.9	0.4841	0.4144	0.3152
5	0.8	0.4446	0.3820	0.2917
5	0.7	0.4017	0.3468	0.2668
5	0.6	0.3508	0.3037	0.2354
5	0.5	0.2964	0.2573	0.2004
5	0.4	0.2429	0.2115	0.1656
5	0.3	0.1865	0.1629	0.1283
5	0.2	0.1251	0.1093	0.0862
5	0.1	0.0625	0.0546	0.0431
6	5.8	4.4991	2.4950	1.2999
6	5.7	3.0492	1.9783	1.1558
6	5.6	2.4395	1.7256	1.0685

TABLE 1 (continued)

n	$\hat{\theta}/T$	α=0.1	α=0.05	α=0.01	n	$\hat{\theta}/T$	α=0.1	α=0.05	α=0.01
6	5.5	2.1148	1.5656	1.0064	7	2.3	1.1101	0.9301	0.6935
6	5.4	1.9065	1.4513	0.9578	7	2.2	1.1105	0.9285	0.6932
6	5.3	1.7562	1.3634	0.9181	7	2.1	1.0944	0.9230	0.6873
6	5.2	1.6401	1.2924	0.8840	7	2.0	1.0589	0.9005	0.6710
6	5.1	1.5461	1.2327	0.8539	7	1.9	1.0011	0.8541	0.6397
6	5.0	1.4671	1.1810	0.8268	7	1.8	0.9310	0.7957	0.6008
6	3.0	1.4730	1.1776	0.8266	7	1.7	0.8660	0.7416	0.5641
6	2.9	1.4610	1.1676	0.8213	7	1.6	0.8205	0.7026	0.5372
6	2.8	1.4168	1.1380	0.8032	7	1.5	0.7925	0.6780	0.5198
6	2.7	1.3442	1.0867	0.7723	7	1.4	0.7690	0.6607	0.5071
6	2.6	1.2508	1.0187	0.7303	7	1.3	0.7338	0.6342	0.4881
6	2.5	1.1465	0.9421	0.6822	7	1.2	0.6813	0.5905	0.4572
6	2.4	1.0575	0.8770	0.6416	7	1.1	0.6259	0.5432	0.4231
6	2.3	0.9949	0.8308	0.6118	7	1.0	0.5783	0.5031	0.3937
6	2.2	0.9510	0.7980	0.5905	7	0.9	0.5347	0.4662	0.3668
6	2.1	0.9228	0.7767	0.5754	7	0.8	0.4842	0.4243	0.3355
6	2.0	0.9116	0.7682	0.5694	7	0.7	0.4295	0.3774	0.2997
6	1.9	0.9121	0.7671	0.5696	7	0.6	0.3751	0.3308	0.2641
6	1.8	0.9088	0.7625	0.5682	7	0.5	0.3188	0.2821	0.2265
6	1.7	0.8913	0.7501	0.5601	7	0.4	0.2592	0.2299	0.1856
6	1.6	0.8495	0.7190	0.5389	7	0.3	0.1976	0.1757	0.1425
6	1.5	0.7897	0.6716	0.5061	7	0.2	0.1329	0.1182	0.0961
6	1.4	0.7266	0.6205	0.4702	7	0.1	0.0665	0.0591	0.0480
6	1.3	0.6744	0.5776	0.4396	8	7.8	6.9127	3.3347	1.7392
6	1.2	0.6378	0.5471	0.4176	8	7.7	4.0841	2.6551	1.5504
6	1.1	0.6068	0.5213	0.3998	8	7.6	3.2888	2.3237	1.4382
6	1.0	0.5686	0.4901	0.3782	8	7.5	2.8615	2.1167	1.3594
6	0.9	0.5169	0.4477	0.3472	8	7.4	2.5879	1.9689	1.2982
6	0.8	0.4637	0.4027	0.3134	8	7.3	2.3930	1.8555	1.2488
6	0.7	0.4148	0.3614	0.2828	8	7.2	2.2427	1.7653	1.2064
6	0.6	0.3644	0.3188	0.2511	8	7.1	2.1222	1.6902	1.1696
6	0.5	0.3093	0.2713	0.2151	8	7.0	2.0211	1.6254	1.1368
6	0.4	0.2517	0.2215	0.1763	8	4.0	2.0256	1.6210	1.1375
6	0.3	0.1925	0.1698	0.1358	8	3.9	1.9982	1.6135	1.1283
6	0.2	0.1293	0.1142	0.0915	8	3.8	1.9327	1.5703	1.1002
6	0.1	0.0647	0.0571	0.0458	8	3.7	1.8346	1.4963	1.0558
7	6.8	5.7191	2.9202	1.5206	8	3.6	1.7136	1.4030	1.0007
7	6.7	3.5715	2.3176	1.3542	8	3.5	1.5811	1.3034	0.9420
7	6.6	2.8672	2.0259	1.2535	8	3.4	1.4704	1.2218	0.8935
7	6.5	2.4891	1.8419	1.1835	8	3.3	1.3946	1.1664	0.8593
7	6.4	2.2480	1.7106	1.1285	8	3.2	1.3431	1.1285	0.8349
7	6.3	2.0752	1.6102	1.0837	8	3.1	1.3108	1.1047	0.8191
7	6.2	1.9420	1.5294	1.0456	8	3.0	1.2971	1.0963	0.8145
7	6.1	1.8345	1.4618	1.0121	8	2.6	1.3091	1.0924	0.8161
7	6.0	1.7446	1.4036	0.9822	8	2.5	1.2898	1.0941	0.8121
7	3.5	1.7565	1.3979	0.9823	8	2.4	1.2602	1.0817	0.8009
7	3.4	1.7326	1.3885	0.9754	8	2.3	1.2138	1.0407	0.7751
7	3.3	1.6764	1.3532	0.9523	8	2.2	1.1440	0.9785	0.7357
7	3.2	1.5905	1.2911	0.9140	8	2.1	1.0686	0.9137	0.6934
7	3.1	1.4828	1.2104	0.8653	8	2.0	1.0087	0.8606	0.6580
7	3.0	1.3642	1.1227	0.8123	8	1.9	0.9703	0.8262	0.6350
7	2.9	1.2644	1.0495	0.7678	8	1.8	0.9466	0.8100	0.6217
7	2.8	1.1953	0.9988	0.7358	8	1.7	0.9247	0.8012	0.6128
7	2.7	1.1476	0.9637	0.7127	8	1.6	0.8986	0.7776	0.5978
7	2.6	1.1172	0.9413	0.6984	8	1.5	0.8511	0.7330	0.5683
7	2.5	1.1050	0.9331	0.6926	8	1.4	0.7896	0.6819	0.5320

TABLE 1 (continued)

n	$\hat{\theta}/T$	α=0.1	α=0.05	α=0.01	n	$\hat{\theta}/T$	α=0.1	α=0.05	α=0.01
8	1.3	0.7368	0.6395	0.5002	9	0.7	0.4525	0.4027	0.3270
8	1.2	0.6960	0.6071	0.4758	9	0.6	0.3941	0.3519	0.2870
8	1.1	0.6539	0.5725	0.4505	9	0.5	0.3336	0.2988	0.2450
8	1.0	0.6022	0.5276	0.4183	9	0.4	0.2711	0.2434	0.2004
8	0.9	0.5479	0.4817	0.3835	9	0.3	0.2060	0.1853	0.1532
8	0.8	0.4970	0.4379	0.3503	9	0.2	0.1384	0.1247	0.1034
8	0.7	0.4424	0.3917	0.3146	9	0.1	0.0693	0.0624	0.0517
8	0.6	0.3852	0.3420	0.2761	10	9.8	7.2291	4.0400	2.1375
8	0.5	0.3267	0.2910	0.2363	10	9.7	5.0197	3.2750	1.9199
8	0.4	0.2656	0.2372	0.1936	10	9.6	4.0725	2.8907	1.7921
8	0.3	0.2021	0.1808	0.1482	10	9.5	3.5632	2.6405	1.6932
8	0.2	0.1358	0.1217	0.1000	10	9.4	3.2425	2.4694	1.6269
8	0.1	0.0680	0.0609	0.0500	10	9.3	3.0072	2.3331	1.5681
9	8.8	8.0543	3.7199	1.9450	10	9.2	2.8279	2.2286	1.5205
9	8.7	4.5771	2.9826	1.7424	10	9.1	2.6885	2.1405	1.4782
9	8.6	3.7006	2.6184	1.6211	10	9.0	2.5645	2.0611	1.4396
9	8.5	3.2250	2.3868	1.5324	10	5.0	2.5532	2.0766	1.4407
9	8.4	2.9212	2.2236	1.4653	10	4.9	2.5196	2.0631	1.4275
9	8.3	2.7061	2.0996	1.4111	10	4.8	2.4393	1.9999	1.3911
9	8.2	2.5400	1.9987	1.3653	10	4.7	2.3179	1.9025	1.3342
9	8.1	2.4062	1.9172	1.3259	10	4.6	2.1703	1.7854	1.2685
9	8.0	2.2953	1.8452	1.2901	10	4.5	2.0103	1.6629	1.1987
9	4.5	2.2916	1.8498	1.2906	10	4.4	1.8783	1.5643	1.1426
9	4.4	2.2605	1.8398	1.2795	10	4.3	1.7891	1.4991	1.1031
9	4.3	2.1868	1.7877	1.2470	10	4.2	1.7306	1.4553	1.0761
9	4.2	2.0775	1.7014	1.1963	10	4.1	1.6958	1.4282	1.0602
9	4.1	1.9429	1.5951	1.1352	10	4.0	1.6819	1.4184	1.0537
9	4.0	1.7967	1.4838	1.0710	10	3.3	1.6663	1.4298	1.0557
9	3.9	1.6754	1.3937	1.0186	10	3.2	1.6331	1.4304	1.0499
9	3.8	1.5929	1.3335	0.9821	10	3.1	1.6103	1.4053	1.0384
9	3.7	1.5377	1.2926	0.9565	10	3.0	1.7191	1.3563	1.0142
9	3.6	1.5038	1.2671	0.9401	10	2.9	1.6053	1.2841	0.9699
9	3.5	1.4892	1.2581	0.9356	10	2.8	1.4678	1.2062	0.9194
9	3.0	1.5013	1.2555	0.9376	10	2.7	1.3716	1.1404	0.8769
9	2.9	1.4795	1.2632	0.9354	10	2.6	1.3166	1.0963	0.8477
9	2.8	1.4509	1.2590	0.9273	10	2.5	1.2748	1.0757	0.8319
9	2.7	1.4201	1.2256	0.9090	10	2.4	1.2332	1.0826	0.8237
9	2.6	1.3705	1.1671	0.8746	10	2.3	1.2089	1.0798	0.8188
9	2.5	1.2934	1.0952	0.8287	10	2.2	1.2861	1.0482	0.8126
9	2.4	1.2140	1.0282	0.7850	10	2.1	1.2173	1.0105	0.7948
9	2.3	1.1582	0.9783	0.7530	10	2.0	1.1315	0.9640	0.7584
9	2.2	1.1237	0.9502	0.7330	10	1.9	1.0471	0.9209	0.7147
9	2.1	1.0937	0.9442	0.7230	10	1.8	0.9836	0.8784	0.6803
9	2.0	1.0722	0.9418	0.7167	10	1.7	0.9647	0.8431	0.6585
9	1.9	1.0770	0.9157	0.7061	10	1.6	0.9532	0.8128	0.6457
9	1.8	1.0350	0.8743	0.6805	10	1.5	0.8932	0.7832	0.6202
9	1.7	0.9609	0.8219	0.6434	10	1.4	0.8273	0.7351	0.5809
9	1.6	0.8927	0.7758	0.6063	10	1.3	0.7779	0.6873	0.5465
9	1.5	0.8448	0.7422	0.5789	10	1.2	0.7396	0.6478	0.5194
9	1.4	0.8164	0.7127	0.5589	10	1.1	0.6862	0.6066	0.4879
9	1.3	0.7774	0.6758	0.5350	10	1.0	0.6284	0.5584	0.4505
9	1.2	0.7190	0.6299	0.5006	10	0.9	0.5758	0.5122	0.4150
9	1.1	0.6646	0.5852	0.4659	10	0.8	0.5198	0.4635	0.3779
9	1.0	0.6165	0.5445	0.4351	10	0.7	0.4616	0.4127	0.3379
9	0.9	0.5645	0.4990	0.4015	10	0.6	0.4017	0.3606	0.2966
9	0.8	0.5084	0.4510	0.3644	10	0.5	0.3398	0.3058	0.2526

Table 1 (continued)

n	$\hat{\theta}/T$	α=0.1	α=0.05	α=0.01
10	0.4	0.2758	0.2488	0.2067
10	0.3	0.2094	0.1893	0.1579
10	0.2	0.1407	0.1273	0.1064
10	0.1	0.0704	0.0637	0.0532

Table 2 gives values of θ_*/T calculated iteratively, by means of the chi-square approximation in conjunction with the Wilson-Hilferty approximation, for selected values of $\hat{\theta}/T \leq 5$, $n = 40$, $\alpha = 0.1$ and 0.05. Here, as well as for the values of $\hat{\theta}/T$ and θ_*/T covered in Table 1 for a fixed combination of α and n, one can interpolate by converting to $\ell n(\hat{\theta}/T)$ and using linear interpolation to determine a corresponding $\ell n(\theta_*/T)$. An example is given in Section 5.

TABLE 2. Approximate chi-square values of θ_*/T. Lower confidence bound at level $1 - \alpha$, corresponding to $\hat{\theta}/T$, for $n = 40$, $\alpha = 0.1$, 0.05.

$\hat{\theta}/T$	θ_*/T	
	$\alpha = 0.1$	$\alpha = 0.5$
5.0	3.065	2.824
4.0	2.591	2.396
3.0	2.057	1.912
2.0	1.460	1.366
1.0	0.783	0.739
0.5	0.407	0.385
0.2	0.166	0.157
0.1	0.083	0.078

We are at present investigating methods combining simulation techniques and smoothing procedures, so that tabulations can be made over the range $1 < \hat{\theta}/T < n$ for $n > 10$. Results of these investigations will appear in a later paper.

4. Tabulation of the Confidence Bounds

Values of θ_*/T calculated by means of the computer program appear in Table 1 for $n = 2(1)10$, $\alpha = 0.1$, 0.05 and 0.01. These have been checked for accuracy, as described in Section 3, and should be correct to within a unit in the second significant figure. Some values have four significant figures of accuracy.

The range of values of $\hat{\theta}/T$ exhibited in Table 1 reflects the range of values that this estimator is able to take on. For example, if $n = 9$ and the number of failures r is equal to 1, then, from (1), one sees that $\hat{\theta}$ is equal to 8T, plus an increment that ranges from zero to slightly less than T. If $r = 2$, then $\hat{\theta}$ must be less than 4.5T, but no less than 3.5T. If $r \geq 3$, $\hat{\theta}$ may range from zero to 3T. In fact, for $n = 2$, $0.0 < \hat{\theta}/T < 2.0$; for $n = 3(1)6$, $0 < \hat{\theta}/T < n/2$ or $n - 1 < \hat{\theta}/T < n$; and for $n = 7(1)10$, $0 < \hat{\theta}/T < n/3$ or $n/2 - 1 < \hat{\theta}/T < n/2$ or $n - 1 < \hat{\theta}/T < n$.

5. An Example

Consider exponential failure times generated by a sample of size 5 of electronic parts that have been "burnt in". Here, T is equal to 106 days and there are four observed failure times, namely, 1.2, 19.6, 45.1 and 91.3 days, so that $\hat{\theta} = [1.2 + 19.6 + 45.1 + 91.3 + 106]/4 = 65.8$ days and $\hat{\theta}/T = 0.621$.

To obtain a 95 percent lower confidence bound for θ, we first calculate $\ell n(0.6) = -0.5108$, $\ell n(0.7) = -0.3567$ and $\ell n(0.621) = -0.4764$. Then, interpolation in Table 1 yields for $\ell n(\theta_*/T)$, $\ell n(0.2803) + 0.2237(\ell n(0.3228) - \ell n(0.2803))$ $= -1.2719 + 0.2237(-1.1307 + 1.2719) = -1.2403$. Thus, θ_*/T is equal to 0.289, and 30.7 days is a 95 percent lower confidence bound for θ.

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