# FOURIER INTEGRAL ESTIMATE OF THE FAILURE RATE FUNCTION AND ITS MEAN SQUARE ERROR PROPERTIES

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## 1. Introduction

The failure rate function h is important in reliability and biometry. Estimates of h using weighting functions or "kernels" are quite common in the literature (see Singpurwalla and Wong (1982b)). The kernels that have been considered so far are *nonnegative* and *absolutely integrable in* $(-\infty,\infty)$. (Kernels satisfying this latter condition are known as $L^1$ kernels.) Singpurwalla and Wong (1982a) -- abbreviated as SW (1982a) -- have shown that the mean square error (MSE) of a kernel estimator of h using a compact $L^1$ kernel restricted to be nonnegative has an optimal rate of convergence of at most $O(n^{-4/5})$, regardless of the smoothness of h; n is the sample size. If the nonnegativity condition of the compact $L^1$ kernel is relaxed, and if h is $(m+1)$ times continuously differentiable, then (for $m > 2$), the rate of convergence of the MSE (can be improved and) is at most $O(n^{-2m/(2m+1)})$. A method for producing kernel estimators having the above property is the generalized jackknife of Gray and Schucany (1972). Specifically, if we use the generalized jackknife on two kernel estimators of h, with each estimator being based upon a nonnegative compact $L^1$ kernel, then this is equivalent to directly producing a kernel estimator of h using a compact $L^1$ kernel which takes both positive and negative values. If we continue to apply the generalized jackknife method, then the rate of convergence of the MSE of the resulting estimator can be brought as close to $n^{-1}$ as is desired. This, plus the alternating behavior of the resulting kernel, has prompted us to conjecture that a repeated jackknifing of estimators based on compact $L^1$ kernels is equivalent to obtaining a kernel estimator using an alternating (wave-like) non $L^1$ kernel.

Motivated by the above considerations, our goal in this paper is to obtain an estimator of h whose MSE converges to 0 faster than $O(n^{-2m/(2m+1)})$ for any finite $m > 0$, and preferably is closer to the ideal $n^{-1}$. We achieve this goal by considering a kernel estimator of h based on the "sinc" kernel. In Section 3 we show that the sinc kernel, which is not an $L^1$ kernel and may not be a limiting case of jackknifing an $L^1$ kernel either, arises naturally when we estimate h via an estimate of the Fourier transform of h. The sinc kernel estimator of h is also referred to as the "Fourier integral estimate".

In Section 4 we show that the sinc kernel estimators of h are asymptotically unbiased and consistent. In Section 5, we discuss the rates of convergence of the bias and the MSE of these estimators. We show that for certain classes of failure rate functions, the sinc kernel estimators have a faster rate of convergence of the MSE than the corresponding $L^1$ kernel estimators. These rates are of the order (log n/n) or $(n^{(1/(2p-1))-1})$, depending upon whether the Fourier transform of h decreases "exponentially" or "algebraically with degree p" (see Definitions 5.1 and 5.2). Clearly, when $p > m+1$, both the above rates are faster than $n^{-2m/(2m+1)}$.

Sinc kernels have been considered before in the literature, first by Konakov (1972), and more recently by Davis (1975) on density estimation. Thus, the results of our paper complement those of Konakov and Davis.

## 2. Preliminaries: Kernel Estimates

Suppose that the time to failure of a device is a nonnegative random variable X, with an absolutely continuous distribution function F and a probability density function f. The *failure rate* at $x_0$, $h(x_0)$, for $F(x_0) \neq 1$, is defined as

$$h(x_0) = \frac{f(x_0)}{1 - F(x_0)} ;$$

note that $h(x) \geq 0$, for all $x \geq 0$.

Given an ordered sample of n lifetimes from F, say $X_{(1)},\ldots,X_{(n)}$, a *kernel estimate* of $h(x_0)$, $h(n,x_0)$, is defined as

$$(1) \qquad h(n,x_0) = \sum_{j=1}^{n} \frac{1}{n-j+1} \frac{1}{b(n)} K\left(\frac{X_{(j)} - x_0}{b(n)}\right) ,$$

where the kernel K is a bounded, symmetric function of integral one; the scale parameter b(n) is a nonnegative decreasing function of n such that

$$(2) \qquad \text{(i)} \quad \lim b(n) = 0, \text{ and } n \to \infty , \quad \text{(ii)} \quad \lim n\, b(n) = \infty \cdot n \to \infty .$$

A motivation for considering the kernel estimates of the failure rate are given in Watson and Leadbetter (1964a).

Watson and Leadbetter (1964b) have shown that for a certain class of distribution functions, estimates based on $L^1$ kernels are asymptotically unbiased and consistent, at every point x at which h is continuous and $F(x) < 1$. The optimal rates of convergence of the bias and the MSE of $h(n,x_0)$ have been discussed by SW (1982b).

## 3. Kernel Estimates Based on the Fourier Integral

We shall confine our attention to the class of failure rate functions h for which the Fourier transform $\Phi_h$ exists; that is

$$\Phi_h(x) = \int e^{ixu}\, h(u)du < \infty \quad .$$

Let $x_0$ be a point of continuity of h(x), and assuming that $\Phi_h \varepsilon L^1$ (i.e., $\int |\Phi_h(x)| dx < \infty$, the following inversion formula gives us the basis for considering the Fourier integral estimate of the failure rate:

(3)
$$h(x_0) = \frac{1}{2\pi} \int e^{-ix_0 u}\, \Phi_h(u)du \quad .$$

Let $F_n$ be the *modified sample distribution function*; that is, the usual sample distribution function multiplied by n/(n+1). An estimate of h(x) at $x = X_{(j)}$, $h_n(x)$, is

$$h_n(x) = \frac{f_n(x)}{1 - F_n(x)} = \frac{dF_n(x)}{1 - F_n(x)} = \frac{1}{n - j+1} \quad .$$

Let $\Phi_{h_n}$ be the Fourier transform of $h_n$; that is

(4)
$$\phi_{h_n}(x) = \int e^{ixu}\, h_n(u)du = \sum_{j=1}^{n} \frac{1}{n - j+1}\, e^{ixX_{(j)}} \quad .$$

To obtain from (3) an estimate of $h(x_0)$, we replace $\Phi_h$ by $\Phi_{h_n}$, and to assure finiteness of the integral, we take it between the finite limits $(-\frac{1}{b(n)}, \frac{1}{b(n)})$, where the b(n) satisfy (2), we obtain the *Fourier integral estimator* of $h(x_0)$, $\tilde{h}(n,x_0)$, where

$$(5) \qquad \tilde{h}(n,x_0) = \frac{1}{2\pi} \int_{-\frac{1}{b(n)}}^{\frac{1}{b(n)}} e^{-ix_0 u} \Phi_{h_n}(u)du \quad .$$

A simple computation shows that

$$(6) \qquad \tilde{h}(n,x_0) = \sum_{j=1}^{n} \frac{1}{(n-j+1)b(n)} S\left(\frac{X_{(j)} - x_0}{b(n)}\right) \quad ,$$

where $S(x) = (\sin x)/\pi x$ is the "sinc" function.

Thus we see that the Fourier integral estimate of the failure rate is indeed a kernel estimate, with the sinc function S as the kernel.

Note that the kernel S is not an $L^1$ kernel, but that it is symmetric, bounded, and of integral one; also $\int S^2(x)dx = 1/\pi$.

## 4. Asymptotic Unbiasedness and Consistency

Since S is not an $L^1$ kernel, the asymptotic unbiasedness and consistency of $\tilde{h}(n,x_0)$ has to be established first. Once this is done, we will be able to discuss the rates of convergence of the bias and the MSE.

THEOREM 1: Let $X_{(1)} \leqq X_{(2)} \leqq \cdots \leqq X_{(n)}$ be an ordered sample of lifetimes from an absolutely continuous distribution function F. Suppose that:

(i) the failure rate function h is absolutely integrable;

(ii) h satisfies Dirichlet's conditions in any finite interval; that is, h has at most a finite number of finite discontinuities, and no infinite discontinuities in any finite interval, and, furthermore, h has only a finite number of maxima and minima in any finite interval;

(iii) $h(x)$ is continuous at $x_0$;

(iv) $F(x_0) < 1$; and

(v) $F$ is such that for any fixed $x'$, and every fixed $\lambda > 0$, there exists a $G_\lambda > 0$, such that

$$\frac{1}{\pi(1-F(x))} \left| \frac{\sin((x-x')/b(n))}{x-x'} \right| \leqq G_\lambda$$

for all sufficiently large $n$ and for all

$$\left| x - x' \right| \geqq \lambda ,$$

then $\tilde{h}(n,x_0)$ defined by (6) is an asymptotically unbiased and consistent estimator of $h(x_0)$.

Furthermore, an asymptotic expression for the expected value of $\tilde{h}(n,x_0)$ is*

$$(7) \qquad E[\tilde{h}(n,x_0)] \sim \int \frac{1}{\pi} \frac{\sin((u-x_0)/b(n))}{u-x_0} h(u)du \quad ,$$

and the variance $Var[\tilde{h}(n,x_0)]$ converges to zero at the rate $1/nb(n)$.

PROOF: $E[\tilde{h}(n,x_0)]$

$$(8) \quad = \sum_{j=1}^{n} \int \frac{1}{n-j+1} \frac{\sin((u-x_0)/b(n))}{\pi(u-x_0)} f_{X_{(j)}}(u)du$$

$$= \frac{1}{\pi} \int \frac{\sin((u-x_0)/b(n))}{u-x_0} h(u)du - \frac{1}{\pi} \int \frac{\sin((u-x_0)/b(n))}{u-x_0} h(u)\, F^n(u)du \quad .$$

*The notation "$a_n \sim b_n$" denotes the fact that the ratio of $a_n$ to $b_n$ has limit one.

Consider the limit of the first term on the right-hand side of (8):

$$(9) \qquad \lim_{n\to\infty} \frac{1}{\pi}\int \frac{\sin((u-x_0)/b(n))}{u-x_0}\,h(u)du = \lim_{b(n)\to 0} \frac{1}{\pi}\int \frac{\sin((u-x_0)/b(n))}{u-x_0}\,h(u)du$$

$$= h(x_0) \quad .$$

The last equality follows by the Fourier integral formula (see Titchmarsh (1962, pp. 3,25)).

Next we show that the second term on the right-hand side of (8) tends to zero, as $n \to \infty$. Since $F(x_0) < 1$, we can choose a $\lambda > 0$ so that $F(x_0 + \lambda) < 1$, and such that $h(u)$ is bounded in $|u-x_0| \leqq \lambda$. We split the interval of integration $(-\infty,\infty)$ into two parts, $|u-x_0| \leqq \lambda$ and $|u-x_0| > \lambda$, and note that

$$(10) \qquad \int_{|u-x_0| \leqq \lambda} \frac{1}{\pi}\,\frac{\sin\left(\frac{u-x_0}{b(n)}\right)}{u-x_0}\,h(u)\,F^n(u)du \leqq (\text{const})\;F^n(x_0+\lambda) \to 0 \;,$$

as $n \to \infty$, and

$$(11) \qquad \int_{|u-x_0| > \lambda} \frac{1}{\pi}\,\frac{\sin\left(\frac{u-x_0}{b(n)}\right)}{u-x_0}\,h(u)\,F^n(u)du \leqq G_\lambda \int_0^1 F^n dF = \frac{G_\lambda}{n+1} \to 0 \;,$$

as $n \to \infty$.

From (8) through (11), we conclude that

$$E[\tilde{h}(n,x_0)] \sim \int \frac{1}{\pi}\,\frac{\sin\left(\frac{u-x_0}{b(n)}\right)}{u-x_0}\,h(u)du \to h(x_0) \text{ as } n \to \infty \quad .$$

To prove consistency of $\tilde{h}(n,x_0)$, we follow the detailed steps given in Watson and Leadbetter (1964b). From equation (4) of Watson and Leadbetter, we write

$$\begin{aligned}
&\mathrm{Var}[\tilde{h}(n,x_0)] \\
&= \int \frac{1}{b^2(n)} S^2\left(\frac{u-x_0}{b(n)}\right) h(u)\, I_n(F(u))\, dF(u) \\
(12) \qquad &+ 2 \iint_{0 \leq u \leq v} \frac{\frac{1}{b(n)} S\left(\frac{u-x_0}{b(n)}\right) \frac{1}{b(n)} S\left(\frac{v-x_0}{b(n)}\right)}{1 - F(v)} \left\{ \frac{1 - F^n(u)}{1-F(u)} F^n(v) - \frac{F^n(v) - F^n(u)}{F(v) - F(u)} \right\} dF(u)\, dF(v) ,
\end{aligned}$$

where $I_n(F) = \int_0^{1-F} \frac{(F+B)^n - F^n}{B} dB$ .

If we multiply both sides of (12) by $n/\alpha_n$, where

$$\alpha_n = \int \frac{1}{b^2(n)} S^2\left(\frac{u-x_0}{b(n)}\right) du = \frac{1}{\pi b(n)}$$

and take the limit as $n \to \infty$, we note that the first term on the right-hand side of (12) equals $h(x_0)/(1-F(x_0))$ whereas the second term is 0. Thus

$$\lim_{n\to\infty} \frac{n}{\alpha_n} \mathrm{Var}[\tilde{h}(n,x_0)] = \frac{h(x_0)}{1 - F(x_0)}$$

or that

$$\mathrm{Var}[\tilde{h}(n,x_0)] \sim \frac{\alpha_n}{n} \frac{h(x_0)}{1 - F(x_0)} .$$

Since $\alpha_n/n = (1/\pi)(1/nb(n)) \to 0$ by (2), it follows that $\mathrm{Var}[\tilde{h}(n,x_0)] \to 0$. Thus, $\tilde{h}(n,x_0)$ is a consistent estimator of $h(x_0)$, and the variance of $\tilde{h}(n,x_0)$ goes to zero at the rate $1/nb(n)$.

## 4.1 An Alternate Expression for the Bias

We shall find it useful to express the asymptotic bias of $\tilde{h}(n,x_0)$ in terms of the Fourier transform $\Phi_h$ of h. We first note that if w(t) is the indicator of the interval [-1,1], then

$$S(x) = \frac{1}{2\pi} \int e^{-ixu} w(u)du \quad . \tag{13}$$

In view of the above, the Fourier transform of S(x) is w(x), $|x| \neq 1$. Recall, from (7), that

$$E[\tilde{h}(n,x_0)] \sim \frac{1}{b(n)} \int h(u) \; S((x_0-u)/b(n))du$$

$$= \frac{1}{2\pi} \int e^{-ix_0 t} w(b(n)t) \; \phi_h(t)dt \quad .$$

The asymptotic bias of $\tilde{h}(n,x_0)$ is therefore given by

$$\mathrm{Bias}[\tilde{h}(n,x_0)] \sim \frac{1}{2\pi} \int e^{-ix_0 t} w(tb(n)) \; \phi_h(t) \; dt - h(x_0)$$

$$= \frac{1}{2\pi} \int e^{-ix_0 t} \{w(tb(n)) - 1\} \; \phi_h(t) \; dt \tag{14}$$

$$= -\frac{1}{2\pi} \int_{|t| > \frac{1}{b(n)}} e^{-ix_0 t} \; \phi_h(t) \; dt \quad .$$

## 5. Rates of Convergence of the Bias and the MSE

We are able to investigate and optimize the rates of convergence of the bias and the MSE of $\tilde{h}(n,x_0)$ when the Fourier transform of the failure rate decreases exponentially or algebraically.

DEFINITION 1: (Parzen (1962)): A function $g(x)$ is said to *decrease exponentially* with degree $0 < r \leq 2$, and coefficient $\rho > 0$, if

$$(15) \qquad |g(x)| \leq Ae^{-\rho|x|^r} \quad \text{for some constant } A > 0 \quad ,$$

and

$$(16) \qquad \lim_{x \to \infty} \int_0^1 [1 + \exp(2\rho x^r) \; |g(xu)|^2]^{-1} \, du = 0 \; .$$

We shall first need to prove the following lemmas. The first is a simple application of L'Hôpital's rule.

LEMMA 1:

$$(17) \qquad \lim_{n \to \infty} b(n) \; e^{\frac{1}{b^r(n)}} \int_{\frac{1}{b(n)}}^{\infty} e^{-t^r} \, dt = 0, \; (r > 0).$$

LEMMA 2: If the Fourier transform of h, $\phi_h$, decreases exponentially with degree r and coefficient $\rho$, then, for sufficiently large n,

$$(18) \qquad |\text{Bias}[\tilde{h}(n,x_0)]| \leq \frac{1}{\pi} \int_{-\frac{1}{b(n)}} Ae^{-\rho t^r} \, dt \; .$$

PROOF: From (14), we note that for n large

$$\mathrm{Bias}[\tilde{h}(n,x_0)] \sim \left| -\frac{1}{2\pi} \int_{|t| > \frac{1}{b(n)}} e^{-ix_0 t} \phi_h(t)dt \right|$$

$$\leqq \frac{1}{2\pi} \int_{|t| > \frac{1}{b(n)}} Ae^{-\rho|t|^r} dt = \frac{1}{\pi} \int_{t > \frac{1}{b(n)}} Ae^{-\rho t^r} dt;$$

the statement of the lemma now follows.

LEMMA 3: Suppose that the Fourier transform $\phi_h$ of h decreases exponentially with degree r and coefficient $\rho$. Then

$$\lim_{n\to\infty} b(n)\, e^{\rho/b^r(n)}\, |\mathrm{Bias}[\tilde{h}(n,x_0)]| = 0 \,. \tag{19}$$

PROOF: The result follows if we make a change of variable $u^r = \rho t^r$, and use Lemmas 1 and 2.

The following theorem establishes the choice of b(n) which enables us to obtain the optimal rate of convergence of the mean square error of $\tilde{h}(n,x_0)$, when $\phi_h$ decreases exponentially. It follows from Lemma 3, Theorem 1, and Davis (1975).

THEOREM 2: Suppose that the Fourier transform $\phi_h$ of the unknown failure rate h exists and decreases exponentially with degree $0 < r \leqq 2$ and coefficient $\rho > 0$. Then, if b(n) in the Fourier integral estimator of h, $\tilde{h}(n,x_0)$, given by (6), is chosen such that $b(n) = 0(\log n/2\rho)^{-(1/r)}$, the optimal rate of convergence of the MSE of $\tilde{h}(n,x_0)$ is of the order log n/n.

We shall now consider the class of failure rate functions h whose Fourier transforms $\phi_h$ decrease albegraically.

DEFINITION 2: (Parzen (1962)): A function g(x) is said to *decrease algebraically* with degree $p > 0$, if

$$(20) \qquad \lim_{|x|\to\infty} |x|^p \, |g(x)| = \alpha^{\frac{1}{2}} > 0, \text{ for some } \alpha > 0 \; .$$

LEMMA 4: Suppose that the Fourier transform $\phi_h$ of h decreases algebraically with degree $p > 1$. Then

$$(21) \qquad \lim_{n\to\infty} b^{1-p}(n) \int_{|t| > \frac{1}{b(n)}} |\phi_h(t)|\,dt = 2\alpha^{\frac{1}{2}}(p-1)^{-1} \; .$$

PROOF: From (20), we note that for $\varepsilon > o$, there exists an $M > 0$ such that for $|t| > M$

$$|t|^{-p}(\alpha^{\frac{1}{2}} - \varepsilon) < |\phi_h(t)| < |t|^{-p}(\alpha^{\frac{1}{2}} + \varepsilon) \; .$$

The proof is completed by integrating both sides of the above for $|t| > 1/b(n)$, and noting that when n is sufficiently large, $1/b(n) > M$, and $|t| > 1/b(n)$ implies that $|t| > M$ .

LEMMA 5: Suppose that the Fourier transform $\phi_h$ of h decreases algebraically with degree $p > 1$. Then the bias of $\tilde{h}(n,x_0)$, $\text{Bias}[\tilde{h}(n,x_0)]$, satisfies

$$(22) \qquad \lim_{n\to\infty} b^{1-p}(n) \, |\text{Bias}[\tilde{h}(n,x_0)]| \leqq \alpha^{\frac{1}{2}} \pi^{-1} (p-1)^{-1} \; ;$$

thus the bias decreases at the rate $b^{p-1}(n)$ .

PROOF: From (14), we have

$$\left| \frac{1}{2\pi} \int_{|t| > \frac{1}{b(n)}} e^{-ix_0 t} \phi_h(t)dt \right| \leqq \frac{1}{2\pi} \int_{|t| > \frac{1}{b(n)}} |\phi_h(t)|dt .$$

We obtain from (14) and (21)

$$\lim_{n\to\infty} b^{1-p}(n)|\text{Bias}[\tilde{h},(n,x_0)]| \leqq \frac{1}{2\pi} 2\alpha^{\frac{1}{2}}(p-1)^{-1} = \alpha^{\frac{1}{2}}\pi^{-1} (p-1)^{-1} .$$

The following theorem is analogous to Theorem 2.

THEOREM 3: Suppose that the Fourier transform $\phi_h$ of the unknown failure rate h exists and decreases algebraically with degree $p > 1$. Then, if $b(n)$ in the Fourier integral estimator of h, $\tilde{h}(n,x_0)$, given by (6), is chosen such that $b(n) = O(n^{-1/(2p-1)})$, the optimal rate of convergence of the MSE of $\tilde{h}(n,x_0)$ is of the order $n^{1/(2p-1)-1}$.

## 6. A Comparison of the Rates of Convergence of the MSE's

We can now compare the optimal rates of convergence of the MSE's for estimates of h based on $L^1$ kernels and the sinc function kernel which is not an $L^1$ kernel.

In general, for $L^1$ kernels which belong to the class $A_m$ (i.e., an $L^1$ kernel K which satisfies the condition that $\int x^j K(x)dx = 0$, for $j = 1,2,\ldots,m-1$), and if $h^{(m+1)}$ exists (that is, if h is m times continuously differentiable), then we have shown in SW (1982a) that the optimal rate of convergence of the MSE of the kernel estimator of h is of the order $n^{-2m/(2m+1)}$. The following results are immediate consequences of this and Theorems 2 and 3 of this paper.

THEOREM 4: For the class of failure rate functions whose Fourier transform exists, and decreases exponentially, the Fourier integral estimate (based on the sinc function) is better in terms of the rate of convergence of the MSE than a kernel estimate based on any $L^1$ kernel.

THEOREM 5: For the class of failure rate functions whose Fourier transform exists and decreases albegraically with degree p, the Fourier integral estimate (based on the sinc function) is better in terms of the MSE than a kernel estimate based on any $L^1$ kernel belonging to the class $A_m$, if $p > m+1$ .

## 7. Concluding Remarks

It is evident from Theorems 2 and 3 that one would consider using the Fourier integral estimate of h only when one had some prior knowledge about h. A disadvantage of the sinc kernel estimator stems from the fact that the estimator of h can be negative at some points, a result unacceptable to practitioners. One may argue that this is the price that must be paid for obtaining an estimator which has good bias and MSE properties. On the other hand, a Bayesian may view this as another situation wherein unbiased estimation and MSE minimization lead us to unacceptable answers.

ACKNOWLEDGEMENTS

The work reported here was begun when the authors were visitors at the Department of Statistics of Stanford University (1978-1979). It has been supported by Contract N00014-77-C-0263, Project NR-042-372, Office of Naval Research, and Grant DAAG-29-80-C-0067, U.S. Army Research Office, Durham, North Carolina.

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