Multivariate Analysis and Its Applications
IMS Lecture Notes - Monograph Series (1994) Volume 24

INVARIANT MEASURES ON STIEFEL MANIFOLDS WITH APPLICATIONS TO MULTIVARIATE ANALYSIS

BY YASUKO CHIKUSE
Kagawa University

Let $V_{k,m}$ denote the Stiefel manifold which consists of $m \times k(m \geq k)$ matrices X such that $X'X = I_k$. We present decompositions of a random matrix X and then of the invariant measure on $V_{k,m}$, relative to a fixed subspace ν in R^m, for all possible four cases to be considered according to the sizes of $k, m,$ and the dimension of ν. The results are utilized for deriving the distributions of the canonical correlation coefficients between two random matrices of "general" dimensions, and for discussing high dimensional limit theorems (as $m \to \infty$) on $V_{k,m}$.

1. Introduction. We consider the Stiefel manifold $V_{k,m}$ which consists of $m \times k(m \geq k)$ matrices X such that $X'X = I_k$, the $k \times k$ identity matrix. For $k = m$, the Stiefel manifold is the orthogonal group $O(m)$. An invariant measure (i.m.) on $V_{k,m}$ is given by the differential form (d.f.)

$$(X'dX) = \bigwedge_{i<j}{}_1^k \boldsymbol{x}_j' d\boldsymbol{x}_i \bigwedge_{j=1}^{m-k} \bigwedge_{i=1}^{k} \boldsymbol{b}_j' d\boldsymbol{x}_i, \tag{1.1}$$

in terms of the exterior products ($\wedge$), where we choose an $m \times (m-k)$ matrix B such that $[X \vdots B] = (\boldsymbol{x}_1 \cdots \boldsymbol{x}_k \vdots \boldsymbol{b}_1 \cdots \boldsymbol{b}_{m-k}) \in O(m)$ and $d\boldsymbol{x}$ is an $m \times 1$ vector of differentials. The volume of $V_{k,m}$ is given by $w(k,m) = 2^k \pi^{km/2}/\Gamma_k(m/2)$, where $\Gamma_k(a) = \pi^{k(k-1)/4} \prod_{i=1}^k \Gamma(a - (i-1)/2)$, and the normalized i.m. of unit mass on $V_{k,m}$ is denoted by $[dX](= (X'dX)/w(k,m))$.

The Grassmann manifold $G_{k,m-k}$ consists of k-planes, i.e., k-dimensional linear subspaces in R^m. For $X \in V_{k,m}$, we can write $X = GQ$; that is, X in $V_{\kappa,m}$ is determined uniquely by the specification of the k-plane, i.e., the "reference" matrix G in $G_{\kappa,m-k}$ and the orientation $Q \in O(k)$ of G. An i.m.

AMS 1991 Subject Classifications: Primary 15A23, 58C35, 62H11; secondary 62H10, 62H20.

Key words and phrases: Canonical correlation coefficients, decompositions of random matrices, Grassmann manifolds, matrix-variate normal distributions, Stiefel manifolds.

on $G_{k,m-k}$ is given by the d.f.

$$(B'dG) = \bigwedge_{j=1}^{m-k} \bigwedge_{i=1}^{k} \boldsymbol{b}_j' d\boldsymbol{g}_i, \tag{1.2}$$

where the column vectors of $G = (\boldsymbol{g}_1 \cdots \boldsymbol{g}_k)$ and those of $B = (\boldsymbol{b}_1 \cdots \boldsymbol{b}_{m-k})$ are orthonormal vectors spanning the k-plane and its orthogonal complement, respectively. The volume of $G_{k,m-k}$ is given by $v(k,m) = w(k,m)/w(k,k)$, and the normalized i.m. of unit mass on $G_{k,m-k}$ is denoted by $[dG](= (B'dG)/v(k,m))$. A detailed discussion of manifolds may be found in James (1954) and Farrell (1985).

Throughout this paper, probability density functions (pdf's) of distributions on the Stiefel and the Grassmann manifolds are expressed with respect to their normalized i.m.'s, while those on the spaces of all $m \times k$ matrices or of all $k \times k$ symmetric matrices are expressed with respect to their Lebesgue measures.

There exists an extensive literature of statistical analysis on Stiefel manifolds; of directional statistics on $V_{1,m}$ (see e.g., Mardia (1975), Watson (1983a), and many others), and of orientational statistics on $V_{k,m}$ (see e.g., Chikuse (1990, 1991a, b), Downs (1972), Jupp and Mardia (1976), Khatri and Mardia (1977), Prentice (1982), and Watson (1983b)). Chikuse (1993) and Chikuse and Watson (1992) investigate asymptotic distribution theory on Grassmann manifolds.

Let ν be a fixed subspace of R^m of dimension p and $\nu^{\perp}$ its orthogonal complement. In this paper, a random matrix $X \in V_{k,m}$ is decomposed as the sum of mutually orthogonal singular value decompositions (svd's) of $P_\nu X$ and $P_{\nu^\perp} X$, where P_ν and $P_{\nu^\perp}$ denote the orthogonal projection matrices onto ν and $\nu^{\perp}$, respectively. The matrix decomposition is presented for each of "all possible four cases" to be considered according to the sizes of k, m, and p, extending Chikuse (1991a), who considered only one special case of them. Based on each of these matrix decompositions of X, we shall express the i.m. $(X'dX)$ as the product of the i.m.'s on the component Stiefel and/or Grassmann manifolds and of a measure on the set of canonical correlation coefficients (ccc's) between the random subspace $\mathcal{M}(X)$ spanned by the columns of X and the subspace ν. The results are summarized in Section 2 and proved in Section 3. James (1954) considered this problem via an analytic approach, and this paper completes his result. Also, the results reduce to those in Watson (1983a, Sections 2 and 3.4), for $k = 1$.

The results presented in Section 2 are of interest and wide use in multivariate analysis. Various distributional results concerning the component matrices in the decompositions of X can be derived. Section 4 gives a method for deriving the distribution of the ccc's between two random matrices of "general" dimensions, with illustrations provided for the multivariate normal distribution and for a "conditionally matrix Langevin" distribution. Section 5 is concerned with high dimensional limit theorems for the distributions depending on $P_\nu X$ only.

2. Decompositions of a Random Matrix and the Invariant Measure on the Stiefel Manifold. Let X be an $m \times k$ random matrix on $V_{k,m}$, and let ν and $\nu^\perp$ be a fixed subspace of R^m of dimension p and its orthogonal complement (of dimension $m-p$), respectively. There are four cases to be considered in this paper, putting $p_1 = m-p$;

$$\begin{aligned}&\text{Case (i)} \quad k \le p \quad \text{and} \quad k \le p_1,\\ &\text{Case (ii)} \quad p < k \le p_1,\\ &\text{Case (iii)} \quad p_1 < k \le p, \qquad \text{and}\\ &\text{Case (iv)} \quad p < k \quad \text{and} \quad p_1 < k.\end{aligned} \tag{2.1}$$

We need to introduce some notation. Let $\widetilde{V}_{k,m}$ denote the $2^{-k}th$ part of $V_{k,m}$ consisting of matrices X_1 whose elements of the first row positive with the normalized i.m. $[dX_1] = 2^k(X_1'dX_1)/w(k,m)$. $A^{1/2}$ is defined as the unique square root of a positive definite matrix A. $O_{q,r}$ and O_q are the $q \times r$ and $q \times q$ matrices of zero elements, respectively.

THEOREM 2.1. *Case (i). (Chikuse (1991a)) X may be uniquely decomposed as the sum of mutually orthogonal svd's in ν and $\nu^\perp$,*

$$X = C_1 H_1 T_k Q + C_2 U_1 (I_k - T_k^2)^{1/2} Q, \tag{2.2}$$

where C_1 and C_2 are $m \times p$ and $m \times p_1$ constant matrices in $V_{p,m}$ and $V_{p_1,m}$, respectively, such that $C_1'C_2 = 0, H_1 \in \widetilde{V}_{k,p}, U_1 \in V_{k,p_1}, Q \in O(k)$, and $T_k = \operatorname{diag}(t_1, \cdots, t_k), 0 < t_1 < \cdots < t_k < 1$. The decomposition (2.2) leads to

$$[dX] = [dH_1][dU_1][dQ][dT_k], \tag{2.3}$$

where $[dT_k]$ is the normalized measure of T_k (or $t_1, \cdots, t_k$) given by

$$[dT_k] = (dT_k)/K(m,k,p), \quad \text{with} \tag{2.4}$$

$$(dT_k) = \prod_{i=1}^{k}[t_i^{p-k}(1-t_i^2)^{(m-k-p-1)/2}]\prod_{i<j}{}_1^k(t_j^2 - t_i^2)\bigwedge_{i=1}^{k} dt_i, \tag{2.5}$$

and the normalizing constant $K(m,k,p) = \int (dT_k)$ being

$$K(m,k,p) = \Gamma_k((m-p)/2)\Gamma_k(p/2)\Gamma_k(k/2)/2^k\pi^{k^2/2}\Gamma_k(m/2). \tag{2.6}$$

Case (ii). X may be uniquely decomposed as

$$\begin{aligned}X &= C_1[H_2 \vdots O_{p,k-p}]\operatorname{diag}(T_p, O_{k-p})Q\\ &\quad + C_2 U_1 \operatorname{diag}((I_p - T_p^2)^{1/2}, I_{k-p})Q,\end{aligned} \tag{2.7}$$

where C_1, C_2, U_1 and Q have been defined in Case (i), $H_2 \in \tilde{O}(p)$, and $T_p = \text{diag }(t_1, \cdots, t_p), 0 < t_1 < \cdots < t_p < 1$. The decomposition (2.7) leads to

$$[dX] = [dH_2][dU_{11}][dU_{12*}][dQ][dT_p], \tag{2.8}$$

where we have the partition $U_1 = [U_{11} \vdots U_{12}]$ with $U_{11} \in V_{p,p_1}$, the $(p_1 - p) \times (k - p)$ matrix $U_{12*} \in G_{k-p,p_1-k}$ is defined such that $U_{12} = G(U_{11})U_{12*}$ for $G(U_{11})$ being chosen so that $[U_{11} \vdots G(U_{11})] \in O(p_1)$, and $[dT_p]$ is the normalized measure of T_p given by

$$[dT_p] = (dT_p)/K(m,p,k), \quad K \quad \text{being defined by (2.6) with} \tag{2.9}$$

$$(dT_p) = \prod_{i=1}^{p} [t_i^{k-p}(1 - t_i^2)^{(m-p-k-1)/2}] \prod_{i<j}^{p} {}_1 (t_j^2 - t_i^2) \bigwedge_{i=1}^{p} dt_i. \tag{2.10}$$

Case (iii). X may be uniquely decomposed as

$$\begin{aligned} X =& C_1 H_1 \text{diag}\,(T_{p_1}, I_{k-p_1})Q \\ &+ C_2[U_2 \vdots O_{p_1,k-p_1}]\text{diag}\,((I_{p_1} - T_{p_1}^2)^{1/2}, O_{k-p_1})Q, \end{aligned} \tag{2.11}$$

where C_1, C_2 and Q have been defined in Case (i), $H_1 \in V_{k,p}, U_2 \in \tilde{O}(p_1)$, and $T_{p_1} = diag(t_1, \cdots, t_{p_1}), 0 < t_1 < \cdots < t_{p_1} < 1$. The decomposition (2.11) leads to

$$[dX] = [dH_{11}][dH_{12*}][dU_2][dQ][dT_{p_1}], \tag{2.12}$$

where we have the partition $H_1 = [H_{11} \vdots H_{12}]$ with $H_{11} \in V_{p_1,p}$, the $(p - p_1) \times (k - p_1)$ matrix $H_{12*} \in G_{k-p_1,p-k}$ is defined such that $H_{12} = G(H_{11})H_{12*}$ for $G(H_{11})$ being chosen so that $[H_{11} \vdots G(H_{11})] \in O(p)$, and $[dT_{p_1}]$ is the normalized measure of T_{p_1} given by

$$[dT_{p_1}] = (dT_{p_1})/K(m,p_1,k), \quad \text{with} \tag{2.13}$$

$$(dT_{p_1}) = \prod_{i=1}^{p_1} [t_i^{p-k}(1 - t_i^2)^{(k+p-m-1)/2}] \prod_{i<j}^{p_1} {}_1 (t_j^2 - t_i^2) \bigwedge_{i=1}^{p_1} dt_i. \tag{2.14}$$

Case (iv). X may be uniquely decomposed as

$$\begin{aligned} X =& C_1 [H_2 \vdots O_{p,k-p}] \text{diag}\,(T_r, I_{p-r}, O_{k-p})Q \\ &+ C_2 [U_{21} \vdots O_{p_1,p-r} \vdots U_{22}] \\ &\times \text{diag}\,((I_r - T_r^2)^{1/2}, O_{p-r}, I_{k-p})Q, \quad \text{with} \quad r = m - k, \end{aligned} \tag{2.15}$$

where C_1, C_2 and Q have been defined in Case (i), $H_2 \in \tilde{O}(p), [U_{21} \vdots U_{22}] \in O(p_1)$ with $U_{21} \in V_{r,p_1}$, and $T_r = diag(t_1, \cdots, t_r), 0 < t_1 < \cdots < t_r < 1$. The decomposition (2.15) leads to

$$[dX] = [dH_{21}][dU_{21}][dQ][dT_r], \tag{2.16}$$

where we have the partition $H_2 = [H_{21} \vdots H_{22}]$ with $H_{21} \in \tilde{V}_{r,p}$, and $[dT_r]$ is the normalized measure of T_r given by

$$[dT_r] = (dT_r)/K(m, r, p), \quad \text{with} \tag{2.17}$$

$$(dT_r) = \prod_{i=1}^{r} [t_i^{k-p}(1 - t_i^2)^{(p+k-m-1)/2}] \prod_{i<j}{}_1^r (t_j^2 - t_i^2) \bigwedge_{i=1}^{r} dt_i. \tag{2.18}$$

COROLLARY 2.1. *In particular when ν is the subspace $\mathcal{M}(\Gamma)$ for an $m \times p$ constant matrix Γ in $V_{p,m}$, Theorem 2.1 holds with Γ replacing C_1.*

It is noted that there are singular situations such as when the projection of a column of X onto ν or $\nu^\perp$ is zero, or when some of the t_i are equal, but that such singular situations have zero i.m.. Hence, Theorem 2.1 and Corollary 2.1 hold with one i.m..

The problem of deriving the decomposition of the i.m. on $V_{k,m}$ was considered by James (1954) via an analytical approach. In his method, those decompositions, except that of the i.m. for Case (i), are not explicitly given. Theorem 2.1, thus, may be a completion of his result.

We note the geometrical interpretation of Theorem 2.1, with the aid of James (1954, Section 7). The columns of C_1 and C_2, which are both not explicit in James (1954), are the orthonormal bases of the subspaces ν and $\nu^\perp$, respectively. The θ_i, where $\cos\theta_i = t_i$, are the critical angles between the random subspace $\mathcal{M}(X)$ in $G_{k,m-k}$ and the subspace ν in $G_{p,m-p}$, while the t_i are called the canonical correlation coefficients (ccc's).

3. Proofs of Theorem 2.1. The results for Case (i) have already been given and proved (Chikuse (1991a)). The proof for each case involves its own aspects, and we shall start giving the outline of the proof for Case (ii).

Proof for Case (ii). X is written as

$$X = P_\nu X + P_{\nu^\perp} X. \tag{3.1}$$

A svd of $P_\nu X$ in ν of dimension p is

$$P_\nu X = C_1\left[H_2 \vdots O_{p,k-p}\right]TQ, \tag{3.2}$$

where C_1 is an $m \times p$ constant matrix in $V_{p,m}, H_2 \in O(p), Q \in O(k)$, and $T = \text{diag}\,(T_p, O_{k-p})$, with $T_p = \underset{\sim}{diag}(t_1, \cdots, t_p), 0 < t_i < 1, i = 1, \cdots, p$. For uniqueness, we assume that $H_2 \in \widetilde{O}(p)$ and $0 < t_1 < \cdots < t_p < 1$.

Next, putting $Z = (P_{\nu\perp}X)Q'$ and substituting (3.2) into (3.1), we have $I_k = (XQ')'XQ' = T^2 + Z'Z$, and, hence, in view of the fact $Z \in \nu^\perp$,

$$Z = C_2 U_1 \text{diag}\,((I_p - T_p^2)^{1/2}, I_{k-p}), \quad \text{with} \quad U_1 \in V_{k,p_1}, \tag{3.3}$$

where C_2 is an $m \times p_1$ constant matrix in $V_{p_1,m}$ such that $C_1'C_2 = 0$. Combining (3.1) - (3.3) establishes the desired result (2.7).

Writing (2.7) as

$$X = GQ, \quad \text{with} \quad G = (\boldsymbol{g}_1 \cdots \boldsymbol{g}_k), \tag{3.4}$$

and differentiating (3.4), we obtain, in use of the notation in (1.1),

$$(X'dX) = (B'dG)(Q'dQ), \quad \text{with} \quad (B'dG) = \bigwedge_{j=1}^{m-k} \bigwedge_{i=1}^{k} \boldsymbol{b}_j' d\boldsymbol{g}_i, \tag{3.5}$$

(James (1954, (5.22))), and the i.m. $(B'dG)$ on $G_{k,m-k}$ is now to be decomposed further.

With G in (3.4), putting $H_2 = (\boldsymbol{h}_1 \cdots \boldsymbol{h}_p)$ and $U_1 = (\boldsymbol{u}_1 \cdots \boldsymbol{u}_k)$, we have

$$\left\{ \begin{array}{lll} \boldsymbol{g}_i = t_i C_1 \boldsymbol{h}_i + (1 - t_i^2)^{1/2} C_2 \boldsymbol{u}_i, & & i = 1, \cdots, p, \\ \boldsymbol{g}_i = C_2 \boldsymbol{u}_i, & & i = p+1, \cdots, k, \\ \boldsymbol{h}_i'\boldsymbol{h}_j = \delta_{ij}, \quad i,j = 1, \cdots, p, & \text{and} \quad \boldsymbol{u}_i'\boldsymbol{u}_j = \delta_{ij}, & i,j = 1, \cdots, k. \end{array} \right. \tag{3.6}$$

We now choose $\boldsymbol{b}_1, \cdots, \boldsymbol{b}_{m-k}$ as

$$\left\{ \begin{array}{ll} \boldsymbol{b}_j = -(1 - t_j^2)^{1/2} C_1 \boldsymbol{h}_j + t_j C_2 \boldsymbol{u}_j, & j = 1, \cdots, p \quad \text{and} \\ \boldsymbol{b}_j = C_2 \boldsymbol{u}_{k-p+j}, & j = p+1, \cdots, m-k, \end{array} \right. \tag{3.7}$$

where $U_{13} = (\boldsymbol{u}_{k+1} \cdots \boldsymbol{u}_{p_1})$ is chosen so that $[U_1 \vdots U_{13}] \in O(p_1)$.

A method similar to James (1954, Sections 5.3 and 7), with the conditions (3.6) and (3.7), leads to, after some differential calculation,

$$(B'dG) = (H_2'dH_2)(U_{11}'dU_{11})(dT_p) \cdot \alpha, \quad \text{with} \quad \alpha = \bigwedge_{j=k+1}^{p_1} \bigwedge_{i=p+1}^{k} \boldsymbol{u}_j' d\boldsymbol{u}_i, \tag{3.8}$$

where we recall $H_2 \in \widetilde{O}(p), U_{11} \in V_{p,p_1}$ is defined by $U_1 = [U_{11} \vdots U_{12}]$, and (dT_p) is given by (2.10). α is now to be evaluated, utilizing the argument of Muirhead (1982, Lemma 9.5.3). For fixed U_{11}, we can write the $p_1 \times (p_1 - p)$ matrix $V = [U_{12} \vdots U_{13}]$ as $V = G(U_{11})V_*$, where $G(U_{11})$ is any fixed matrix chosen so that $[U_{11} \vdots G(U_{11})] \in O(p_1)$, and $V_* = [U_{12*} \vdots U_{13*}] \in O(p_1 - p)$ with U_{12*} being $(p_1 - p) \times (k - p)$; the relationship between V and V_* is one-to-one. Putting $V_* = (\boldsymbol{v}_1 \cdots \boldsymbol{v}_{p_1-p}), \alpha$ is expressed as

$$\alpha = \bigwedge_{j=k-p+1}^{p_1-p} \bigwedge_{i=1}^{k-p} \boldsymbol{v}_j' d\boldsymbol{v}_i = (U_{13*}' dU_{12*}), \tag{3.9}$$

which is the i.m. on G_{k-p,p_1-k}.

From (3.5), (3.8) and (3.9), we obtain the decomposition of $(X'dX)$ as

$$(X'dX) = (H_2'dH_2)(U_{11}'dU_{11})(U_{13*}'dU_{12*})(Q'dQ)(dT_p). \tag{3.10}$$

Integrating both sides of (3.10) yields

$$\begin{aligned} \int (dT_p) &= 2^p w(k,m)/w(p,p)w(p,p_1)v(k-p,p_1-p)w(k,k) \\ &= K(m,p,k). \end{aligned} \tag{3.11}$$

Proof for Case (iii). The method for Case (ii) can be applied to this case, due to its symmetry with Case (ii). This fact leads to the decomposition of X in the form (2.11), with $(I_{p_1} - S_{p_1}^2)^{1/2}(S_{p_1} = \operatorname{diag}(s_1, \cdots, s_{p_1}), 0 < s_{p_1} < \cdots < s_1 < 1)$ replacing T_{p_1}, and, hence, putting $t_1 = (1 - s_i^2)^{1/2}, i = 1, \cdots, p_1$, establishes the desired result (2.11). The decomposition (2.12) of $[dX]$, thus, follows, where we have

$$\begin{aligned} \int (dT_{p_1}) &= 2^{p_1} w(k,m)/w(p_1,p)v(k-p_1,p-p_1)w(p_1,p_1)w(k,k) \\ &= K(m,p_1,k). \end{aligned} \tag{3.12}$$

Proof for Case (iv). We can write the unique svd of $P_\nu X$ in ν as

$$P_\nu X = C_1 [H_2 \vdots O_{p,k-p}] TQ, \tag{3.13}$$

where C_1 and Q follow the same conditions as in Case (i), $H_2 \in \widetilde{O}(p)$, and $T = diag(T_r, I_{p-r}, O_{k-p})$, with $T_r = diag(t_1, \cdots, t_r), 0 < t_1 < \cdots < t_r < 1$, since T is of rank p and we have $r(\leq p$ and $\leq p_1)$ ccc's; the value of r is yet to be known. From $I_k = T^2 + Z'Z$, with $Z = (P_{\nu^\perp}X)Q'$, we can write $Z'Z$, being of rank "p_1", as

$$Z'Z = \mathrm{diag}(I_r - T_r^2, O_{p-r}, I_{k-p}), \tag{3.14}$$

which is of rank "$r + k - p$", thus leading to "$r = m - k$".

From (3.14) and the fact $Z \in \nu^\perp$, we readily see that

$$Z = C_2 U \mathrm{diag}\big((I_r - T_r^2)^{1/2}, O_{p-r}, I_{k-p}\big), \tag{3.15}$$

where C_2 follows the same condition as in Case (i), and the $p_1 \times k$ matrix U is expressed as $U = [U_{21} \vdots O_{p_1,p-r} \vdots U_{22}]$, with $[U_{21} \vdots U_{22}] \in O(p_1)$, U_{21} being $p_1 \times r$. Combining the above results establishes the desired result (2.15).

Writing (2.15) in the same form $X = GQ$ as in (3.4), and putting $H_2 = [H_{21} \vdots H_{22}] = (\boldsymbol{h}_1 \cdots \boldsymbol{h}_r \vdots \boldsymbol{h}_{r+1} \cdots \boldsymbol{h}_p), U_{21} = (\boldsymbol{u}_1 \cdots \boldsymbol{u}_r)$, and $U_{22} = (\boldsymbol{u}_{p+1} \cdots \boldsymbol{u}_k)$, we have

$$\begin{cases} \boldsymbol{g}_i = t_i C_1 \boldsymbol{h}_i + (1 - t_i^2)^{1/2} C_2 \boldsymbol{u}_i, & i = 1, \cdots, r, \\ \boldsymbol{g}_i = C_1 \boldsymbol{h}_i, & i = r+1, \cdots, p, \\ \boldsymbol{g}_i = C_2 \boldsymbol{u}_i, & i = p+1, \cdots, k, \\ \boldsymbol{h}_i' \boldsymbol{h}_j = \delta_{ij}, & i, j = 1, \cdots, p, \qquad \text{and} \\ \boldsymbol{u}_i' \boldsymbol{u}_j = \delta_{ij}, & i, j = 1, \cdots, r, p+1, \cdots, k. \end{cases} \tag{3.16}$$

Now, we choose $\boldsymbol{b}_1, \cdots, \boldsymbol{b}_{m-k}$ as, noting $r = m - k$,

$$\boldsymbol{b}_j = -(1 - t_j^2)^{1/2} C_1 \boldsymbol{h}_j + t_j C_2 \boldsymbol{u}_j, \quad j = 1, \cdots, m-k. \tag{3.17}$$

Then, similarly to Case (ii), we obtain

$$(B'dG) = (H_{21}' dH_{21})(U_{21}' dU_{21})(dT_r), \tag{3.18}$$

where we recall $H_{21} \in \widetilde{V}_{r,p}$ and $U_{21} \in V_{r,p_1}$, and (dT_r) is given by (2.18). Integrating both sides of (3.5), with (3.18) being substituted, yields

$$\int (dT_r) = 2^r w(k, m)/w(r, p) w(r, p_1) w(k, k) = K(m, r, p). \tag{3.19}$$

4. Distributions of Canonical Correlation Coefficients of General Dimensions.

4.1. Canonical Correlation Coefficients of General Dimensions. It is already known that the angle θ between the two $m \times 1$ vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by

$$\cos\theta = \boldsymbol{y}_*' \boldsymbol{x}_*, \quad \text{i.e.,} \quad (\cos\theta)^2 = \boldsymbol{y}_*' P_{\boldsymbol{x}} y_*, \tag{4.1}$$

where $\boldsymbol{x}_* = \boldsymbol{x}/\|\boldsymbol{x}\|, \boldsymbol{y}_* = \boldsymbol{y}/\|\boldsymbol{y}\|$, with $\|\boldsymbol{x}\| = (\boldsymbol{x}'\boldsymbol{x})^{1/2}$ being the length of $\boldsymbol{x}$, and $P_{\boldsymbol{x}} = \boldsymbol{x}_*\boldsymbol{x}_*'$ is the orthogonal projection matrix onto the space $\mathcal{M}(\boldsymbol{x})$.

We extend the definition (4.1) for vectors to that for matrices. Given $m \times k$ and $m \times p$ matrices X and $Y (m \geq k$ and $m \geq p)$, let us write the unique polar decomposition of X and Y as

$$\begin{cases} X = H_X T_X^{1/2}, & \text{with} \quad H_X = X(X'X)^{-1/2} \quad \text{and} \quad T_X = X'X, \quad \text{and} \\ Y = H_Y T_Y^{1/2}, & \text{with} \quad H_Y = Y(Y'Y)^{-1/2} \quad \text{and} \quad T_Y = Y'Y, \end{cases} \tag{4.2}$$

respectively. H_X indicates the "orientation" of the matrix X, extending the notion of direction for $k = 1$, and T_X the inner products of columns of X (see Downs (1972) and Chikuse (1990)). The "angles" between X and Y may be defined by the (nonzero) latent roots of

$$V = H_Y' H_X, \qquad \text{i.e.,} \quad W = VV' = H_Y' P_X H_Y, \tag{4.3}$$

where $P_X = H_X H_X'$ is the orthogonal projection matrix onto the space $\mathcal{M}(X)$.

The matrix W in (4.3) is rewritten as

$$W = (Y'Y)^{-1/2} Y'X(X'X)^{-1} X'Y(Y'Y)^{-1/2}. \tag{4.4}$$

Usually, in multivariate analysis, m is the sample size and $k+p$ is the dimension of the population, and the (nonzero) latent roots of W are well known to give the squares of the (sample) ccc's between the two observation matrices X and Y. Assuming $m \geq k + p$, the pdf of the ccc's has been obtained from the distribution of the sample covariance matrix $S = (S_{ij}), i,j = 1,2$, where $S_{11} = X'X, S_{12} = S_{21}' = X'Y$, and $S_{22} = Y'Y$, for the multivariate normal population (see e.g., James (1964, (76)) and also Muirhead (1982, Theorem 11.3.2)).

We give a method, based on the definition (4.3) utilizing the results in Section 2, for deriving the distributions of ccc's of "general" dimensions, relaxing the assumption $m \geq k + p$. For two random orientation matrices $X \in V_{k,m}$ and $Y \in V_{p,m}$, the ccc's given Y are the (nonzero) latent roots t_i of $Y'X$, utilizing Corollary 2.1, with Y replacing Γ. The conditional distribution of the ccc's given Y is obtained by integrating unnecessary variables out of the conditional distribution of X given Y, utilizing the decompositions of X and of the i.m. $[dX]$ for each of Cases (i)–(iv), and then the (marginal) distribution, by taking expectation over $Y \in V_{p,m}$. The method is now illustrated for two multivariate distributions.

4.2. Multivariate Normal Distribution. First an $m \times k$ (rectangular) random matrix Z is defined to have the $m \times k$ matrix- variate normal $N_{m\times k}(M, \Sigma_1 \otimes\Sigma_2)$ distribution, if its pdf is $(2\pi)^{-km/2}|\Sigma_1|^{-k/2}|\Sigma_2|^{-m/2}$ etr $[-\Sigma_1^{-1}(Z - M)$

$\Sigma_2^{-1}(Z-M)'/2]$, where etr $A = \exp(\operatorname{tr} A)$, M is an $m \times k$ matrix, and Σ_1 and Σ_2 are, respectively, $m \times m$ and $k \times k$ positive definite matrices.

Let $Z = [X \vdots Y]$, with X and Y being $m \times k$ and $m \times p$, respectively, be distributed as $N_{m \times q}(0, I_m \otimes \Sigma)(q = k + p)$. Then, from the conditional pdf of X given Y (e.g., Muirhead (1982, Theorem 1.2.11)), utilizing Herz (1955, Lemma 1.4) (see also James (1954, (8.19))), the pdf of H_X given Y is obtained as

$$c(Y) \int_{T_X > 0} \operatorname{etr}\left[-\frac{1}{2}\Sigma_{11\cdot 2}^{-1} T_X + M'(Y) H_X T_X^{1/2}\right] |T_X|^{(m-k-1)/2} (dT_X), \quad (4.5)$$

where $c(Y)$ is the normalizing constant, depending on Y, we have the partition $\Sigma = (\Sigma_{ij}), i, j = 1, 2$, with Σ_{11} being $k \times k, \Sigma_{11\cdot 2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$, and $M(Y) = Y\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11\cdot 2}^{-1}$.

Noting the decomposition $[dH_X] = [dG_X][dQ_X]$ based on the matrix decomposition $H_X = G_X Q_X$, where $G_X \in G_{k,m-k}$ and $Q_X \in O(k)$ (see the Introduction), we integrate (4.5) over $Q_X \in O(k)$, utilizing the integral definition (e.g., James (1964, (27))) of the ${}_0F_1$ hypergeometric function of matrix argument (see the Appendix for the hypergeometric function of matrix argument). Then, making the transformation $S = \Sigma_{11\cdot 2}^{-1/2} T_X \Sigma_{11\cdot 2}^{-1/2}/2$, utilizing the Laplace transform (e.g., James (1964, (28))) of the hypergeometric function of matrix argument, and evaluating the normalizing constant, we obtain the pdf of G_X given Y as

$$\operatorname{etr}(-\Theta T_Y/2) {}_1F_1(m/2; k/2; Y\Theta Y' G_X G_X'/2), \quad (4.6)$$

where

$$\Theta = \Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11\cdot 2}^{-1}\Sigma_{12}\Sigma_{22}^{-1}. \quad (4.7)$$

For given Y, we recall Corollary 2.1, with $H_X \in V_{k,m}$ and $H_Y \in V_{p,m}$ replacing X and Γ, respectively, and utilize the decompositions of H_X (and, hence, of G_X) and of $[dG_X]$, for each of Cases (i)–(iv).

Case (i). It is seen from (4.6) that the pdf of T_k given Y, with respect to the normalized measure $[dT_k]$ given by (2.4), is

$$f_1(T_k|Y) = \operatorname{etr}\Big(-\frac{1}{2}\Theta T_Y\Big) \int_{\widetilde{V}_{k,p}} {}_1F_1\Big(\frac{1}{2}m; \frac{1}{2}k; \frac{1}{2} T_Y^{1/2} \Theta T_Y^{1/2} H_1 T_k^2 H_1'\Big)[dH_1], \quad (4.8)$$

which depends only on T_Y; we may write $f_1(T_k \mid T_Y)$ instead of $f_1(T_k \mid Y)$. Making the transformation $H_1 \to HH_1, H \in O(p)$, and then integrating over $H \in O(p)$ yields

$$f_1(T_k \mid T_Y) = \operatorname{etr}(-\Theta T_Y/2) {}_1F_1^{(p)}(m/2; k/2; \Theta T_Y/2, T_k^2), \quad (4.9)$$

where ${}_1F_1^{(p)}$ is the hypergeometric function of two matrix arguments (see the Appendix).

Now, the (marginal) $N_{m\times p}(0, I_m \otimes \Sigma_{22})$ distribution of Y, in view of Herz (1955, Lemma 1.4), yields the pdf of T_Y

$$g(T_Y) = \mathrm{etr}(-\Sigma_{22}^{-1}T_Y/2)\,|T_Y|^{(m-p-1)/2}/2^{pm/2}\Gamma_p(m/2)\,|\Sigma_{22}|^{m/2}. \qquad (4.10)$$

The (marginal) pdf of T_k, $f_1(T_k) = \int_{T_Y>0} f_1(T_k \mid T_Y)g(T_Y)(dT_Y)$, with respect to $[dT_k]$, is obtained by making the transformation $S = (\Theta + \Sigma_{22}^{-1})^{1/2}T_Y(\Theta + \Sigma_{22}^{-1})^{1/2}/2$ and utilizing the Laplace transform (James (1964, (31))) of the hypergeometric function of two matrix arguments, as

$$f_1(T_k) = |I_p - P^2|^{m/2}{}_2F_1^{(p)}(m/2, m/2; k/2; P^2, T_k^2),$$
$$\text{with} \quad P^2 = \Sigma_{22}^{-1/2}\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1/2}. \qquad (4.11)$$

Noting that ${}_2F_1^{(p)}(a_1, a_2; k/2; A, B) = {}_2F_1^{(k)}(a_1, a_2; p/2; A, B)$, we establish the distribution of $t_1^2, \cdots, t_k^2 (= T_k^2)$ as

$$2^{-k}K^{-1}(m,k,p)|I_p - P^2|^{m/2}{}_2F_1^{(k)}(m/2, m/2; p/2; P^2, T_k^2)$$
$$\times |T_k^2|^{(p-k-1)/2}|I_k - T_k^2|^{(m-k-p-1)/2}\prod_{i<j}^{k}{}_1(t_j^2 - t_i^2)\bigwedge_{i=1}^{k} dt_i^2, \qquad (4.12)$$

where $K(m,k,p)$ is given by (2.6).

Noting that replacing $I_p - P^2$ and P^2 by $I_k - P_1^2$ and $P_1^2 = \Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1/2}$, respectively, does not change the result in (4.12), it is seen that (4.12) is exactly the same as the known result (James (1964, (76))).

Case (ii). Integrating (4.6) over $H_2 \in \widetilde{O}(p)$ yields the pdf of T_p given T_Y, with respect to $[dT_p]$ given by (2.9), as $\mathrm{etr}(-\Theta T_Y/2)_1F_1^{(p)}(m/2; k/2; \Theta T_Y/2, T_p^2)$. Thus, the similar method leads to the distribution of $t_1^2, \cdots, t_p^2$ as

$$2^{-p}K^{-1}(m,p,k)|I_p - P^2|^{m/2}{}_2F_1^{(p)}(m/2, m/2; k/2; P^2, T_p^2)$$
$$\times |T_p^2|^{(k-p-1)/2}|I_p - T_p^2|^{(m-p-k-1)/2}\prod_{i<j}^{p}{}_1(t_j^2 - t_i^2)\bigwedge_{i=1}^{p} dt_i^2, \qquad (4.13)$$

which is exactly the same as (4.12) interchanging p and k; this is the known result.

Case (iii). The pdf of T_{p_1} given T_Y, with respect to $[dT_{p_1}]$ given by (2.13), is

$$f_3(T_{p_1}|T_Y) = \mathrm{etr}(-\Theta T_Y/2)\int_{V_{p_1,p}}\int_{G_{k-p_1,p-k}} {}_1F_1(m/2; k/2;$$
$$T_Y^{1/2}\Theta T_Y^{1/2}H_1\mathrm{diag}(T_{p_1}^2, I_{k-p_1})H_1'/2)[dH_{12*}][dH_{11}],$$
$$\text{where} \quad H_1 = [H_{11} \vdots G(H_{11})H_{12*}] \in V_{k,p}. \qquad (4.14)$$

From the invariance of $[dH_{11}]$ and $[dH_{12*}]$, making the transformation $H_1 \to HH_1, H \in O(p)$, and then integrating over $H \in O(p)$ yields

$$f_3(T_{p_1}|T_Y) = \operatorname{etr}\left(-\frac{1}{2}\Theta T_Y\right){}_1F_1^{(p)}\left(\frac{1}{2}m; \frac{1}{2}k; \frac{1}{2}\Theta T_Y, \operatorname{diag}(T_{p_1}^2, I_{k-p_1})\right). \quad (4.15)$$

The similar method leads to the distribution of $t_1^2, \cdots, t_{p_1}^2$ as

$$2^{-p_1}K^{-1}(m, p_1, k)|I_p - P^2|^{m/2}{}_2F_1^{(k)}\left(\frac{1}{2}m, \frac{1}{2}m; \frac{1}{2}p; P^2, \operatorname{diag}(T_{p_1}^2, I_{k-p_1})\right)$$
$$\times |T_{p_1}^2|^{(p-k-1)/2}|I_{p_1} - T_{p_1}^2|^{(k+p-m-1)/2}\prod_{i<j}^{p_1}(t_j^2 - t_i^2)\bigwedge_{i=1}^{p_1} dt_i^2. \quad (4.16)$$

Case (iv). The integration of (4.6) over $H_{21} \in \widetilde{V}_{r,p}$ is evaluated by making the transformation $H_2(= [H_{21} \vdots H_{22}]) \to HH_2,\ H \in O(p)$, and then integrating over $H \in O(p)$, so that the pdf of T_r given T_Y, with respect to $[dT_r]$ given by (2.17), is obtained as $\operatorname{etr}(-\Theta T_Y/2){}_1F_1^{(p)}(m/2; k/2; \Theta T_Y/2, \operatorname{diag}(T_r^2, I_{p-r}))$. Thus, the similar method leads to the distribution of $t_1^2, \cdots, t_r^2$ as

$$2^{-r}K^{-1}(m, r, p)\,|I_p - P^2|^{m/2}{}_2F_1^{(p)}\left(\frac{1}{2}m, \frac{1}{2}m; \frac{1}{2}k; P^2, \operatorname{diag}(T_r^2, I_{p-r})\right)$$
$$\times |T_r^2|^{(k-p-1)/2}|I_r - T_r^2|^{(k+p-m-1)/2}\prod_{i<j}^{r}(t_j^2 - t_i^2)\bigwedge_{i=1}^{r} dt_i^2. \quad (4.17)$$

4.3. Conditionally Matrix Langevin Distribution. For random matrices X and Y on $V_{k,m}$ and $V_{p,m}$, respectively, let the conditional distribution of X given Y be the matrix Langevin $L(m, k; YA)$, (see (A.4)), with A being a $p \times k$ constant matrix, while Y has "any" marginal distribution. Hence, the mode of the conditional distribution of X given Y is in the orientation of Y.

Case (i). The distribution of T_k given Y, with $[dT_k]$ given by (2.4), is

$$c\int_{\widetilde{V}_{k,p}}\int_{O(k)} \operatorname{etr}(A'H_1T_kQ)[dQ][dH_1][dT_k], \quad \text{with} \quad c = {}_0F_1^{-1}(m/2; A'A/4),$$
$$= c\int_{\widetilde{V}_{k,p}} {}_0F_1(k/2; H_1T_k^2H_1'AA'/4)[dH_1][dT_k]$$
$$= c\,{}_0F_1^{(p)}(k/2; A'A/4, T_k^2)[dT_k] = c\,{}_0F_1^{(k)}(p/2; A'A/4, T_k^2)[dT_k], \quad (4.18)$$

which is independent of Y, and, hence, gives the (marginal) distribution of $t_1, \cdots, t_k$.

Similarly, integrating the conditional pdf of X given Y over the common $Q \in O(k)$, and (ii) $H_2 \in \widetilde{O}(p)$, (iii) $H_{12*} \in G_{k-p_1,p-k}$ and $H_{11} \in V_{p_1,p}$, and (iv) $H_{21} \in \widetilde{V}_{r,p}$, respectively, yield

Case (ii). the distribution of $t_1, \cdots, t_p$, with $[dT_p]$ given by (2.9), as

$$c_0 F_1^{(p)}(k/2; AA'/4, T_p^2)[dT_p], \tag{4.19}$$

Case (iii). the distribution of $t_1, \cdots, t_{p_1}$, with $[dT_{p_1}]$ given by (2.13), as

$$c_0 F_1^{(k)}(p/2; A'A/4, \operatorname{diag}(T_{p_1}^2, I_{k-p_1}))[dT_{p_1}], \tag{4.20}$$

and

Case (iv). the distribution of $t_1, \cdots, t_r$, with $[dT_r]$ given by (2.17), as

$$c_0 F_1^{(p)}(k/2; AA'/4, \operatorname{diag}(T_r^2, I_{p-r}))[dT_r]. \tag{4.21}$$

5. High Dimensional Limit Theorems. Let X be a random matrix on $V_{k,m}$ and ν a subspace of R^m of dimension p. In this section, we utilize the results in Theorem 2.1 for Cases (i) and (ii) (i.e., for $k + p \leq m$), since our interest is in the limiting behaviours, as $m \to \infty$, of random matrices. Theorems 5.1 and 5.2 which follow can be proved by comparing the limiting joint distributions of $H_1, m^{1/2}T_k, Q$ and of $H_2, m^{1/2}T_p, Q$ for Cases (i) and (ii), respectively, with the joint distribution of the svd of a random $N_{p\times k}(M, \Sigma_1 \otimes I_k)$ matrix, extending Chikuse (1991a). The results extend Watson (1983b) for $k = 1$, and are useful for inferential problems on $V_{k,m}$ for large m.

5.1. The Distribution Depending on $P_\nu X$ Only. We consider the case when X has the distribution whose pdf is of the form $f(P_\nu X)$, for a suitable function $f(\cdot)$ being continuous at the origin 0.

THEOREM 5.1. *The random matrix $m^{1/2}C_1'X (= m^{1/2}C_1'P_\nu X)$ is $N_{p\times k}(0, I_p \otimes I_k)$, and, hence, $m^{1/2}P_\nu X$ is "degenerate" $N_{m\times k}(0, P_\nu \otimes I_k)$, in the limit as $m \to \infty$.*

EXAMPLES. Let us consider the two cases when X is distributed as the matrix Langevin $L(m, k; F)$ and the matrix Bingham $B(m, k; D)$, having the svd's, with Γ, (A.5) of F and (A.7) of D, respectively. Then, the $m^{1/2}\Gamma'X$ are $N_{p\times k}(0, I_p \otimes I_k)$ in the limit as $m \to \infty$. Since the case F (or D) $= 0$ reduces to the uniform distribution $[dX]$ on $V_{k,m}$, the limiting normality of $m^{1/2}\Gamma'X$ holds, for a uniform X on $V_{k,m}$ and any $m \times p$ constant matrix $\Gamma \in V_{p,m}$

5.2. The Distribution Depending on $m^{1/2}P_\nu X$ Only. When we investigate high dimensional behaviours for the distribution depending on $P_\nu X$, it would be natural to let the pdf be of the form $f(m^{1/2}P_\nu X)$; otherwise, the distribution seems to become flat as $m \to \infty$. Two cases are considered.

THEOREM 5.2. *For the matrix* $g - L(m,k;p,\nu;m^{1/2}F)$ *distribution (see (A.8)),* $m^{1/2}C_1'X$ *is* $N_{p\times k}(C_1'F, I_p \otimes I_k)$, *and, hence,* $m^{1/2}P_\nu X$ *is degenerate* $N_{m\times k}(F, P_\nu \otimes I_k)$, *in the limit as* $m \to \infty$.

THEOREM 5.3. *For the matrix* $g - SW(m,k;p,\nu;mD)$ *distribution (see (A.9)), where* $I_p - 2C_1'DC_1$ *is positive definite,* $m^{1/2}C_1'X$ *is* $N_{p\times k}(0,(I_p - 2C_1'DC_1)^{-1} \otimes I_k)$, *and, hence,* $m^{1/2}P_\nu X$ *is degenerate* $N_{m\times k}(0, C_1(I_p - 2C_1'DC_1)^{-1}C_1' \otimes I_k)$, *in the limit as* $m \to \infty$.

APPENDIX

A.1. Hypergeometric Functions of Matrix Arguments. The hypergeometric function ${}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;A)$ with a $k \times k$ symmetric matrix A has a series representation

$$\sum_{l=0}^{\infty}\sum_{\lambda}[(a_1)_\lambda\cdots(a_p)_\lambda/(b_1)_\lambda\cdots(b_q)_\lambda l!]C_\lambda(A). \tag{A.1}$$

Here, λ ranges over all ordered partitions of l, i.e., $\lambda = (l_1,\cdots,l_k), l_1 \geq \cdots \geq l_k \geq 0, \sum_{i=1}^k l_i = l, a_1,\cdots,a_p,b_1,\cdots,b_q$ are real or complex constants, $(a)_\lambda = \prod_{i=1}^k(a-(i-1)/2)_{l_i}$, with $(a)_l = a(a+1)\cdots(a+l-1)$, and $C_\lambda(A)$ is a zonal polynomial. The zonal polynomials $C_\lambda(A)$ were defined by the theory of group representations of the real linear group on the vector space of homogeneous polynomials of degree l on the space of symmetric matrices, and constitute a basis of the space of all homogeneous symmetric polynomials in the latent roots of A. The hypergeometric function with two symmetric matrices A and B is defined by

$$\begin{aligned}&{}_pF_q^{(k)}(a_1,\cdots,a_p;b_1,\cdots,b_q;A,B)\\ &= \int_{O(k)} {}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;AHBH')[dH],\end{aligned} \tag{A.2}$$

and has a series representation

$$\sum_{l=0}^{\infty}\sum_{\lambda}[(a_1)_\lambda\cdots(a_p)_\lambda/(b_1)_\lambda\cdots(b_q)_\lambda l!C_\lambda(I_k)]C_\lambda(A)C_\lambda(B). \tag{A.3}$$

See James (1964) and Constantine (1963) for a more detailed discussion of the zonal polynomials and hypergeometric functions of matrix arguments.

A.2. Some Population Distributions on $V_{k,m}$.

A.2.1. The Matrix Langevin Distribution. A random matrix $X \in V_{k,m}$ is said to have the matrix Langevin (or von Mises-Fisher) $L(m, k; F)$ distribution, if its pdf is given by (Downs (1972))

$$\mathrm{etr}\,(F'X)/{}_0F_1(m/2; F'F/4), \tag{A.4}$$

Where F is an $m \times k$ matrix (see also Chikuse (1990) and Khatri and Mardia (1977)). Writing the svd of F, being of rank $p(0 \leq p \leq k)$, as

$$F = \Gamma\Delta\Theta', \text{ where } \Gamma \in V_{p,m}, \Theta \in V_{p,k}, \text{ and } \Delta = \mathrm{diag}(\lambda_1, \cdots, \lambda_p), \lambda_i > 0, \tag{A.5}$$

Γ and Θ indicate "orientations", extending the notion of directions for $k = 1$, and the λ_i are "concentration" parameters in the p directions determined by Γ and Θ. $M = \Gamma\Theta'$ gives the "modal orientation" of the distribution (Chikuse (1991b)).

A.2.2. The Matrix Bingham Distribution. The matrix Bingham $B(m, k; D)$ distribution has the pdf

$$\mathrm{etr}\,(X'DX)/{}_1F_1(k/2; m/2; D), \tag{A.6}$$

where D is an $m \times m$ symmetric matrix, with a restriction, e.g., tr $B = 0$ (see Bingham (1974) (for $k = 1$), Jupp and Mardia (1976), and Prentice (1982)). We write the svd of D, being of rank $p\,(0 \leq p \leq m)$, as

$$D = \Gamma\Delta\Gamma', \quad \text{where} \quad \Gamma \in V_{p,m}, \quad \text{and} \quad \Delta \quad \text{is a} \quad p \times p \quad \text{diagonal matrix.} \tag{A.7}$$

A.2.3. Generalized Distributions. Letting ν be a given subspace of R^m of dimension p, we generalize the above two distributions. It is noticed that putting $\nu = \mathcal{M}(\Gamma)$ yields the distributions (A.4) and (A.6).

The matrix generalized Langevin distribution $(g - L(m, k; p, \nu; F))$ has the pdf, with F being an $m \times k$ matrix,

$$\mathrm{etr}\,(F'P_\nu X)/{}_0F_1(m/2; P_\nu FF'/4). \tag{A.8}$$

The matrix generalized Scheiddegger-Watson distribution $(g{-}SW(m, k; p, \nu; D))$ has the pdf, with D being an $m \times m$ symmetric matrix,

$$etr[(P_\nu X)'DP_\nu X]/{}_1F_1(k/2; m/2; P_\nu D), \tag{A.9}$$

generalizing that of Watson (1983a, Section 3.4) for $k = 1$.

Acknowledgment. The author is grateful to the referees for suggesting a number of improvements.

REFERENCES

BINGHAM, C. (1974). An antipodally symmetric distribution on the sphere. *Ann. Statist.* **2**, 1201–1225.

CHIKUSE, Y. (1990). Distributions of orientations on Stiefel manifolds. *J. Multivariate Analysis* **33**, 247–264.

CHIKUSE, Y. (1991a). High dimensional limit theorems and matrix decompositions on the Stiefel manifold. *J. Multivariate Analysis* **34**, 145–162.

CHIKUSE, Y. (1991b). Asymptotic expansions for distributions of the large sample matrix resultant and related statistics on the Stiefel manifold. *J. Multivariate Analysis* **39**, 270–283.

CHIKUSE, Y. (1993). Asymptotic theory for the concentrated matrix Langevin distributions on the Grassmann manifold. To appear in *the Proceedings of the 3rd Pacific Area Statistical Conference.*

CHIKUSE, Y. and WATSON, G. S. (1992). Large sample asymptotic expansions for distributions on the Grassmann manifold. *Research Paper* No.43, Kagawa University.

CONSTANTINE, A. G. (1963). Some non-central distribution problems in multivariate analysis. *Ann. Math. Statist.* **34**, 1270–1285.

DOWNS, T. D. (1972). Orientation statistics. *Biometrika*, **59**, 665–676.

FARRELL, R. H. (1985). *Multivariate Calculation.* Springer-Verlag, Berlin.

HERZ, C. S. (1955). Bessel functions of matrix argument. *Ann. Math.* **61**, 474–523.

JAMES, A. T. (1954). Normal multivariate analysis and the orthogonal group. *Ann. Math. Statist.* **25**, 40–75.

JAMES, A. T. (1964). Distributions of matrix variates and latent roots derived from normal samples. *Ann. Math. Statist.* **35**, 475–501.

JUPP, P. E. and MARDIA, K. V. (1976). Maximum likelihood estimators for the matrix von Mises-Fisher and Bingham distributions. *Ann. Statist.* **7**, 599–606.

KHATRI, C. G. and MARDIA, K. V. (1977). The von Mises-Fisher matrix distribution in orientation statistics. *J. Roy. Statist. Soc. Ser. B* **39**, 95–106.

MARDIA, K. V. (1975). *Statistics of Directional Data.* Academic, New York.

MUIRHEAD, R. J. (1982). *Aspects of Multivariate Statistical Theory.* Wiley, New York.

PRENTICE, M. J. (1982). Antipodally symmetric distributions for orientation statistics. *J. Statist. Plann. Inference* **6**, 205–214.

WATSON, G. S. (1983a). *Statistics on Spheres.* Lecture Notes in Mathematics **6**, Wiley, New York.

WATSON, G. S. (1983b). Limit theorems on high dimensional spheres and Stiefel manifolds. *In Studies in Econometrics, Time Series, and Multivariate Statistics* (S. Karlin, T. Amemiya and L.A. Goodman eds.) 559–570. Academic, New York.

DEPARTMENT OF INFORMATION SCIENCE
COLLEGE OF ECONOMICS
KAGAWA UNIVERSITY
2-1 SAIWAI-CHO, TAKAMATSU-SHI
KAGAWA 760, JAPAN