Multivariate Analysis and Its Applications
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ON MULTIVARIATE MIXED MODEL ANALYSIS

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A general multivariate mixed effect linear model is introduced. Special cases of the model include the multivariate nested error covariance component regression and the random coefficient repeated measure model. Discussion is given on modeling the random effect structure and its effect on statistical inference. A procedure for testing certain class of hypotheses concerning the random effect structure is developed. The procedure is based on a statistic in a readily computable form, facilitating the use at the model building stage.

1. The Model. This paper is concerned with introducing a general multivariate mixed effect model, and with developing a procedure for testing hypotheses concerning the random effect structure in such a model. For simplicity we concentrate here on mixed models with the one-way random effect structure, i.e., with the random effect (other than the error term) involving one unknown covariance matrix. To introduce our general model, first consider the most widely used univariate mixed effect model with the one-way classification random effect or with the nested error structure. The response y_{ij} and the $k \times 1$ explanatory variable $\boldsymbol{x}_{ij}$ for the j-th individual in the i-th group are assumed to satisfy

$$y_{ij} = \boldsymbol{\beta}'\boldsymbol{x}_{ij} + u_i + e_{ij}, \qquad i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, r_i, \tag{1.1}$$

where $\boldsymbol{\beta}$ is a $k \times 1$ vector of unknown fixed parameters, u_i's and e_{ij}'s are independent random variables with mean zero, and $\mathrm{Var}\{u_i\} = \phi^2$ and Var $\{e_{ij}\} = \sigma^2$ are components of variance. This univariate model has been widely applied in animal breeding, small area estimation, and analyses of data arising in panel study and cluster sampling. See, e.g., Henderson (1973), Fuller and Battese (1974), and Prasad and Rao (1986). Harville (1977), Robinson (1991), and Searle et al. (1992) provide reviews of the variance component problems emphasizing the univariate models. Model (1.1) does not involve an unknown covariance matrix (of dimension at least 2×2) to be estimated. If more than one response variable are measured from each individual in the same setup, then we have a multivariate extension given by

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$$\boldsymbol{y}_{ij} = \boldsymbol{B}\boldsymbol{x}_{ij} + \boldsymbol{u}_i + \boldsymbol{e}_{ij}, \qquad i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, r_i, \tag{1.2}$$

where $\boldsymbol{y}_{ij}$ is the $p \times 1$ response vector, $\boldsymbol{B}$ is a $p \times k$ matrix of unknown fixed parameters, the $\boldsymbol{u}_i$'s and $\boldsymbol{e}_{ij}$'s are $p \times 1$ independent random vectors with mean zero, and $\mathrm{Var}\{\boldsymbol{u}_i\} = \boldsymbol{\Phi}$ and $\mathrm{Var}\ \{\boldsymbol{e}_{ij}\} = \boldsymbol{\Sigma}$ are $p \times p$ covariance components. This extension (1.2) is obtained by stacking p equations of the form (1.1), assuming that each response variable has the corresponding random group effect and error term. In this model, the $p \times p$ between group covariance matrix $\boldsymbol{\Phi}$ has to be estimated. Special cases of model (1.2) have been discussed and applied in Klotz and Putter (1969), Thompson (1973), Boch and Petersen (1975), Amemiya (1985), Meyer (1985), Amemiya et al. (1990), Anderson and Amemiya (1991), and Calvin and Dykstra (1991). Another widely used model with a covariance matrix to be estimated has a univariate response. The random coefficient regression or the random effect repeated measure model assumes that the univariate response y_{ij} and the explanatory variables $k \times 1\ \boldsymbol{x}_{ij}$ and $q \times 1\ \boldsymbol{z}_{ij}$ for the j-th measurement on the i-th individual satisfy

$$y_{ij} = \boldsymbol{\beta}'\boldsymbol{x}_{ij} + \alpha_i'\boldsymbol{z}_{ij} + e_{ij}, \qquad i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, r_i, \tag{1.3}$$

where $\boldsymbol{\beta}$ is a $k \times 1$ vector of unknown fixed parameters, $q \times 1\ \ \alpha_i$'s and scalar e_{ij}'s are independent random variables, the mean of α_i is a $q \times 1$ unknown vector, $E\{e_{ij}\} = 0$, $\mathrm{Var}\{\alpha_i\} = \boldsymbol{\Phi}$, and $\mathrm{Var}\{e_{ij}\} = \sigma^2$. This is a regression model with some coefficients (α_i) assumed to be random over the individuals who constitute, e.g., a random sample from some population. In social science panel data applications, model (1.3) is often used to explain intra-individual covariance structure not sufficiently explained by model (1.1). In the random effect growth curve analysis, the covariate $\boldsymbol{z}_{ij}$ corresponds to a low order polynomial in time, and the random coefficient α_i represents random individual differences in growth curve. See, e.g., Rao (1965), Swamy (1971), Laird and Ware (1982), Reinsel (1985), and Lange and Laird (1989). In some of these papers, model (1.3) with no fixed parameter $\boldsymbol{\beta}$ was given as an empirical Bayes model. Although some interpretation and inference may differ, the model itself is identical. Here we present models and related issues from a view point of classical fixed and random effects approach.

The model considered in this paper is a general multivariate mixed effect repeated measure model which contains both (1.2) and (1.3) as special cases. Assume that the $p \times 1$ response $\boldsymbol{y}_{ij}$ and the explanatory variables $k \times 1\ \boldsymbol{x}_{ij}$ and $q \times 1\ \boldsymbol{z}_{ij}$ for the j-th measurement on the i-th individual satisfy

$$\boldsymbol{y}_{ij} = \boldsymbol{B}\boldsymbol{x}_{ij} + \boldsymbol{A}_i\boldsymbol{z}_{ij} + \boldsymbol{e}_{ij}, \qquad i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, r_i, \tag{1.4}$$

where $\boldsymbol{B}$ is a $p \times k$ matrix of unknown fixed parameters, $\boldsymbol{A}_i$'s are $p \times q$ random matrices with $\mathrm{Var}\{\mathrm{vec}\ \boldsymbol{A}_i\} = \boldsymbol{\Phi}$, $pq \times pq$, $\boldsymbol{e}_{ij}$'s are $p \times 1$ random vectors with $E\{\boldsymbol{e}_{ij}\} = \mathbf{0}$ and $\text{Var}\{\boldsymbol{e}_{ij}\} = \boldsymbol{\Sigma}$, and all $\boldsymbol{a}_i$'s and $\boldsymbol{e}_{ij}$'s are independent. Here, we used the vec notation such that, for any $a \times b$ $\boldsymbol{V} = (\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_b)$, $\text{vec}\,\boldsymbol{V} = (\boldsymbol{v}_1', \boldsymbol{v}_2', \cdots, \boldsymbol{v}_b')'$, $ab \times 1$. Model (1.4) is a multivariate extension of the univariate repeated measure model (1.3) in the sense that p characteristics are measured at each measurement period on each individual. The multivariate nested error mixed effect model (1.2) is also a special case of model (1.4) with $q = 1$, $E\{\boldsymbol{A}_i\} = 0$, and $\boldsymbol{z}_{ij} = 1$. If the n individuals or classes are assumed to be a random sample from a single population, then $E\{\boldsymbol{A}_i\}$ is assumed to be common for all $i = 1, 2, \cdots, n$. In general, the n individuals or classes may be grouped into several groups, or their means may depend on some explanatory covariate $s \times 1$ $\boldsymbol{w}_i$, A model for such a situation is

$$\boldsymbol{A}_i = \boldsymbol{\Gamma}(\boldsymbol{w}_i \otimes \boldsymbol{I}_q) + \boldsymbol{U}_i, \qquad i = 1, 2, \cdots, n, \tag{1.5}$$

where $\boldsymbol{\Gamma}$ is a $p \times qs$ matrix of unknown fixed parameters, $\otimes$ is the Kronecker product, and the $\boldsymbol{U}_i$'s are $p \times q$ independent random matrices with $E\{\boldsymbol{U}_i\} = \mathbf{0}$ and $\text{Var}\{\text{vec}\,\boldsymbol{U}_i\} = \boldsymbol{\Phi}$. Here, we assumed the covariance matrix $\boldsymbol{\Phi}$ is common for all $i = 1, 2, \cdots, n$. The single population or common mean case is obtained by setting $s = 1$ and $\boldsymbol{w}_i = 1$.

A special balanced case of model (1.4), with no $\boldsymbol{B}\boldsymbol{x}_{ij}$ term, $r_i = r$ for all i, and $\boldsymbol{z}_{ij}$ free of i, has been discussed in the literature. See Reinsel (1982, 1984). Such a balanced model is applicable in a repeated measure situation where the observations are taken in a complete rectangular panel design and the explanatory variables are common for all individuals. This balanced structure allows some exact inference procedures. Here we consider the general unbalanced model (1.4). The number of replicates r_i's differ over individuals, as r_i may be the size of progeny of a sire or may be the number of unbalanced repeated measures. The $\boldsymbol{x}_{ij}$'s and $\boldsymbol{z}_{ij}$'s are allowed to be any explanatory variables, as all variables may be measured at each time period in a panel or longitudinal study. For our general model, exact small sample inference procedures are not possible, and we need to appeal to some asymptotic approximation.

To express the model given by (1.4) and (1.5) in a concise form, let $N = \sum_{i=1}^{n} r_i$,

$$\boldsymbol{Y} = (\boldsymbol{y}_{11}, \boldsymbol{y}_{12}, \cdots, \boldsymbol{y}_{1r_1}, \boldsymbol{y}_{21}, \boldsymbol{y}_{22}, \cdots, \boldsymbol{y}_{nr_n}), \qquad p \times N, \tag{1.6a}$$

$$\boldsymbol{X}_i = (\boldsymbol{x}_{i1}, \boldsymbol{x}_{i2}, \cdots, \boldsymbol{x}_{ir_i}), \qquad k \times r_i, \tag{1.6b}$$

$$\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n), \qquad k \times N, \tag{1.6c}$$

$$\boldsymbol{E}_i = (\boldsymbol{e}_{i1}, \boldsymbol{e}_{i2}, \cdots, \boldsymbol{e}_{ir_i}), \qquad p \times r_i, \tag{1.6d}$$

$$\boldsymbol{E} = (\boldsymbol{E}_1, \boldsymbol{E}_2, \cdots, \boldsymbol{E}_n), \qquad p \times N, \tag{1.6e}$$

$$Z_i = (z_{i1}, z_{i2}, \cdots, z_{ir_i}), \qquad q \times r_i, \tag{1.6f}$$

$$Z = \begin{pmatrix} Z_1 & 0 & \cdots & 0 \\ 0 & Z_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & Z_n \end{pmatrix}, \qquad nq \times N, \tag{1.6g}$$

$$A = (A_1, A_2, \cdots, A_n), \qquad p \times nq. \tag{1.6h}$$

Then, a single equation for all N observations can be written as

$$Y = BX + AZ + E. \tag{1.7}$$

For identification, assume that (X', Z') is of full column rank. The model for A_i given by (1.5) can be incorporated into (1.7) to obtain

$$Y = (B, \Gamma)\begin{pmatrix} X \\ Z^* \end{pmatrix} + UZ + E, \tag{1.8}$$

where

$$\begin{aligned} Z^* &= \left[(w_1 \otimes I_q)Z_1, (w_2 \otimes I_q)Z_2, \cdots, (w_n \otimes I_q)Z_n\right], \\ U &= (U_1, U_2, \cdots, U_n). \end{aligned}$$

Throughout the rest of this paper, our general multivariate mixed effect repeated measure model specified by (1.4) and (1.5), or alternatively (1.7) or (1.8), is assumed to hold with the associated assumptions mentioned in this section.

2. Discussion. A basic approach for inferences on the fixed regression coefficient B and the mean parameter Γ for the random coefficient is an extension of that used for the special cases (1.1) and (1.3) . The form (1.8) of our model can be transformed into a univariate linear model

$$\operatorname{vec} Y = \left[(X', Z^{*\prime}) \otimes I_p\right]\begin{pmatrix} \operatorname{vec} B \\ \operatorname{vec} \Gamma \end{pmatrix} + \eta, \tag{2.1}$$

where

$$\begin{aligned} \eta &= (Z' \otimes I_p) \operatorname{vec} U + \operatorname{vec} E, \\ \operatorname{Var}\{\eta\} &= \Omega(\Phi, \Sigma) \\ &= \text{block diag}\{(Z_1' \otimes I_p)\,\Phi\,(Z_1 \otimes I_p), \\ &\qquad \cdots, (Z_n' \otimes I_p)\,\Phi\,(Z_n \otimes I_p)\} + I_N \otimes \Sigma. \end{aligned}$$

Given some estimates $\widehat{\Phi}$ and $\widehat{\Sigma}$, the parameters B and Γ can be estimated by applying the generalized least squares to (2.1) with the estimated weight

$\Omega^{-1}(\widehat{\Phi},\widehat{\Sigma})$, although the actual computation would use some simplification. (This form holds, even when all parameters are estimated jointly.) Another problem of practical interest is prediction of a linear function of the element of $\boldsymbol{B}$ and $\boldsymbol{A}$, i.e., of $\boldsymbol{B},\boldsymbol{\Gamma}$, and $\boldsymbol{U}$. A predictor can be obtained by evaluating the best linear unbiased predictor given $\boldsymbol{\Phi}$ and $\boldsymbol{\Sigma}$ at some estimators $\widehat{\boldsymbol{\Phi}}$ and $\widehat{\boldsymbol{\Sigma}}$. To make inferences, an approximate covariance matrix of the estimator or the prediction error needs to be estimated. A frequently used estimated covariance matrix is the sum of the covariance matrix for the case with known $\boldsymbol{\Phi}$ and $\boldsymbol{\Sigma}$ and an additional term representing variability due to estimation of $\boldsymbol{\Phi}$ and $\boldsymbol{\Sigma}$, both evaluated at the parameter estimates. Expressions for such covariance matrices have been derived for some special cases. See, e.g., Kacker and Harville (1984), Fuller and Harter (1987), and Prasad and Rao (1986).

An important issue associated with our general model (1.8) is inference for the structure of the random effect $\boldsymbol{U}_i$'s. Each $\boldsymbol{U}_i$ contains pq random variables corresponding to p response variables and q explanatory variables. Some of the p response variables may be constant over the individuals. More generally, the individual differences of the p variables may be explained by a smaller number of underlying random effect variables. On the other hand, the model assumes that the coefficient $\boldsymbol{B}$ for the k explanatory variables is fixed but $\boldsymbol{A}_i$'s for the q explanatory variables are random. There is a problem of determining which variables are assumed to have random coefficients. Thus, a model of the form (1.7) postulated based on subject matter knowledge and experience still needs to allow possible structure on the random coefficient covariance matrix $\boldsymbol{\Phi}$, or in general, a singular $\boldsymbol{\Phi}$. Such a possibility also has an effect on estimation of a fixed parameter and prediction of a random quantity mentioned earlier. For the estimated generalized least squares estimation of a fixed parameter, the effect of ignoring the possibility of a singular or structured $\boldsymbol{\Phi}$ may not be large, provided that an estimator $\widehat{\boldsymbol{\Phi}}$ has reasonable properties even under a singular $\boldsymbol{\Phi}$ condition and some measure of variability in $\widehat{\boldsymbol{\Phi}}$ is available. On the other hand, prediction of a quantity involving $\boldsymbol{U}_i$ may not be carried out properly without knowing the structure and singularity of $\boldsymbol{U}_i$. Inference for the structure of the random effect has not attracted much attention in the literature covering special cases of our general model. Development of a test procedure for such a purpose is the topic of the next section.

3. Testing the Structure of Random Coefficients. As described in the previous section, inference concerning possible singularity of the random coefficient covariance matrix $\boldsymbol{\Phi}$ is an integral part of analyzing models of the form (1.7). The assertion that some linear combinations of the q explanatory variables have coefficients with no random individual variability can be expressed in a hypotheses that, for a $q_1 \times q$ given matrix $\boldsymbol{L}_1$ of rank $q_1, \boldsymbol{U}_i\boldsymbol{L}_1'$ is constant, i.e.,

$$(\boldsymbol{L}_1 \otimes \boldsymbol{I}_p)\,\boldsymbol{\Phi}\,(\boldsymbol{L}_1' \otimes \boldsymbol{I}_p) = \mathbf{0}. \tag{3.1}$$

The possibility that the individual differences for the p response variables may be explained by a certain subset of the underlying random effect variables can be formulated by hypothesizing that, for a $p_1 \times p$ given matrix $\boldsymbol{L}_2$ of rank p_1, $\boldsymbol{L}_2\boldsymbol{U}_i$ is constant, i.e.,

$$(\boldsymbol{I}_q \otimes \boldsymbol{L}_2)\,\boldsymbol{\Phi}\,(\boldsymbol{I}_q \otimes \boldsymbol{L}_2') = \mathbf{0}. \tag{3.2}$$

Combining (3.1) and (3.2), we consider testing a hypothesis of the form

$$(\boldsymbol{L}_1 \otimes \boldsymbol{L}_2)\,\boldsymbol{\Phi}\,(\boldsymbol{L}_1' \otimes \boldsymbol{L}_2') = \mathbf{0}. \tag{3.3}$$

Because such testing is performed during the model building stage of the analysis, we develop a relatively quick test procedure not involving intensive iterative computation.

Except for very special cases, an exact small sample test is unavailable. We develop a large sample test, assuming that the number of individuals n is large while the numbers of replicates r_i's may not be. We also assume that the within-individual error $\boldsymbol{E}_i$'s are normally distributed, but that the random coefficient $\boldsymbol{A}_i$'s can have any distribution. Our technical regularity assumptions are:

(i) The $\boldsymbol{E}_i$'s are normally distributed.

(ii) As $n \to \infty$,

$$\frac{1}{n}\sum_{i=1}^{n} r_i \longrightarrow r_0,$$

$$\frac{1}{n}\boldsymbol{X}\boldsymbol{X}' = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i\boldsymbol{X}_i' = O(1),$$

$$\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{w}_i\boldsymbol{w}_i' = O(1),$$

$$\left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{w}_i\boldsymbol{w}_i'\right)^{-1} = O(1),$$

$$\max_{i=1,2,\cdots,n} \|\boldsymbol{Z}_i\boldsymbol{Z}_i'\| = O(1),$$

$$\max_{i=1,2,\cdots,n} \|(\boldsymbol{Z}_i\boldsymbol{Z}_i')^{-1}\| = O(1),$$

$$\left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i\boldsymbol{M}_{\boldsymbol{Z}_i}\boldsymbol{X}_i'\right)^{-1} = O(1),$$

where $r_o > q$ is a constant, $\|\cdot\|$ is any matrix norm, and

$$\boldsymbol{M}_{\boldsymbol{Z}_i} = \boldsymbol{I}_{r_i} - \boldsymbol{Z}_i'(\boldsymbol{Z}_i\boldsymbol{Z}_i')^{-1}\boldsymbol{Z}_i.$$

Thus, our assumptions correspond to cases with a large number n of individuals and with the number r_i of replicates being small to moderate and being unbalanced. In the growth curve application, the elements of $\boldsymbol{z}_{ij}$ are polynomials in a time variable. Assumption (ii) applies to this case without worrying about different rates of increase of polynomials, if the r_i's are not very large.

Our test procedure is based on the following fitting constants technique. Consider applying the ordinary least squares to model (1.7), behaving as if $\boldsymbol{A}_i$'s are fixed. Then, we obtain

$$\begin{aligned}(\widehat{\boldsymbol{B}}, \widehat{\boldsymbol{A}}) &= (\widehat{\boldsymbol{B}}, \widehat{\boldsymbol{A}}_1, \cdots, \widehat{\boldsymbol{A}}_n) = \boldsymbol{Y}(\boldsymbol{X}', \boldsymbol{Z}')\left[\begin{pmatrix}\boldsymbol{X}\\ \boldsymbol{Z}\end{pmatrix}(\boldsymbol{X}', \boldsymbol{Z}')\right]^{-1}, \\ \widehat{\boldsymbol{\Sigma}} &= \frac{1}{d}\left[\boldsymbol{Y}\boldsymbol{Y}' - (\widehat{\boldsymbol{B}}, \widehat{\boldsymbol{A}})\begin{pmatrix}\boldsymbol{X}\\ \boldsymbol{Z}\end{pmatrix}\boldsymbol{Y}'\right],\end{aligned} \tag{3.4}$$

where $d = N - k - nq$. It follows that under the model

$$\begin{aligned}\widehat{\boldsymbol{A}}_i &= \boldsymbol{A}_i + \left[\boldsymbol{E}_i - (\widehat{\boldsymbol{B}} - \boldsymbol{B})\boldsymbol{X}_i\right]\boldsymbol{Z}_i'(\boldsymbol{Z}_i\boldsymbol{Z}_i')^{-1}, \\ d\,\widehat{\boldsymbol{\Sigma}} &\sim W_p(\boldsymbol{\Sigma}, d).\end{aligned} \tag{3.5}$$

Using model (1.5) for $\boldsymbol{A}_i$ and rearranging the elements of $p \times qs$ $\boldsymbol{\Gamma}$ in a $pq \times s$ matrix $\boldsymbol{\Gamma}_0$, we can write

$$\operatorname{vec}\widehat{\boldsymbol{A}}_i = \boldsymbol{\Gamma}_0\boldsymbol{w}_i + \operatorname{vec}\boldsymbol{U}_i + \operatorname{vec}\left\{\left[\boldsymbol{E}_i - (\widehat{\boldsymbol{B}} - \boldsymbol{B})\boldsymbol{X}_i\right]\boldsymbol{Z}_i'(\boldsymbol{Z}_i\boldsymbol{Z}_i')^{-1}\right\}. \tag{3.6}$$

The idea behind our test procedure is to form a quadratic form in vec $\widehat{\boldsymbol{A}}_i$, $i = 1, 2, \cdots, n$, which does not depend on the $\boldsymbol{U}_i$'s under the null (3.3), and to compare it to some function of $\widehat{\boldsymbol{\Sigma}}$. We remove the fixed mean part $\boldsymbol{\Gamma}_0\boldsymbol{w}_i$ by estimating $\boldsymbol{\Gamma}_0$ with a simple least squares estimator

$$\widehat{\boldsymbol{\Gamma}}_0 = \sum_{i=1}^{n}(\operatorname{vec}\widehat{\boldsymbol{A}}_i)\boldsymbol{w}_i'\left(\sum_{i=1}^{n}\boldsymbol{w}_i\boldsymbol{w}_i'\right)^{-1}. \tag{3.7}$$

For this $\widehat{\boldsymbol{\Gamma}}_0$ and $\widehat{\boldsymbol{B}}$ defined in (3.4), we have the following lemma.

LEMMA 1. *As $n \to \infty$,*

$$\begin{aligned}\widehat{\boldsymbol{B}} - \boldsymbol{B} &= O_p(n^{-1/2}), \\ \widehat{\boldsymbol{\Gamma}}_0 - \boldsymbol{\Gamma}_0 &= O_p(n^{-1/2}).\end{aligned}$$

PROOF. The result for $\widehat{\boldsymbol{B}}$ follows from assumption (ii), because $E\{\widehat{\boldsymbol{B}}\} = \boldsymbol{B}$ and

$$\operatorname{Var}\{\operatorname{vec}\widehat{\boldsymbol{B}}\} = \left(\sum_{i=1}^{n} \boldsymbol{X}_i \boldsymbol{M}_{\boldsymbol{Z}_i} \boldsymbol{X}_i'\right)^{-1} \otimes \boldsymbol{\Sigma} = O(n^{-1}).$$

For $\widehat{\boldsymbol{\Gamma}}_0$, we can write

$$\operatorname{vec}\left(\widehat{\boldsymbol{\Gamma}}_0 - \boldsymbol{\Gamma}_0\right)\left(\sum_{i=1}^{n} \boldsymbol{w}_i \boldsymbol{w}_i'\right) = \boldsymbol{R}_1 + \boldsymbol{R}_2 + \boldsymbol{R}_3 \operatorname{vec}(\widehat{\boldsymbol{B}} - \boldsymbol{B}),$$

where

$$\begin{aligned}
\boldsymbol{R}_1 &= \sum_{i=1}^{n} (\boldsymbol{w}_i \otimes \boldsymbol{I}_{pq}) \operatorname{vec} \boldsymbol{U}_i, \\
\boldsymbol{R}_2 &= \sum_{i=1}^{n} \left[\boldsymbol{w}_i \otimes (\boldsymbol{Z}_i \boldsymbol{Z}_i')^{-1} \boldsymbol{Z}_i \otimes \boldsymbol{I}_p\right](\operatorname{vec} \boldsymbol{E}_i), \\
\boldsymbol{R}_3 &= \sum_{i=1}^{n} \left[\boldsymbol{w}_i \otimes (\boldsymbol{Z}_i \boldsymbol{Z}_i')^{-1} \boldsymbol{Z}_i \boldsymbol{X}_i' \otimes \boldsymbol{I}_p\right].
\end{aligned}$$

Assumption (ii) and the Cauchy-Schwarz inequality can be used to show that $\operatorname{Var}\{\boldsymbol{R}_i\} = O(n)$, $i = 1, 2$, and $\boldsymbol{R}_3 = O(n)$. Thus, the result for $\widehat{\boldsymbol{\Gamma}}_0$ follows from that for $\widehat{\boldsymbol{B}}$ and assumption (ii). ∎

It follows from (3.6) and Lemma 1 that $\operatorname{vec}\widehat{\boldsymbol{A}}_i - \widehat{\boldsymbol{\Gamma}}_0 \boldsymbol{w}_i$ is approximately equal to $\operatorname{vec} \boldsymbol{U}_i + \varepsilon_i$, where

$$\varepsilon_i = \left[(\boldsymbol{Z}_i \boldsymbol{Z}_i')^{-1} \boldsymbol{Z}_i \otimes \boldsymbol{I}_p\right] \operatorname{vec} \boldsymbol{E}_i \sim N(\boldsymbol{0}, (\boldsymbol{Z}_i \boldsymbol{Z}_i')^{-1} \otimes \boldsymbol{\Sigma}).$$

To construct a quantity free of $\boldsymbol{U}_i$ under the null (3.3), let

$$\begin{aligned}
\boldsymbol{h}_i &= \left(\boldsymbol{L}_{1i}^* \otimes \boldsymbol{L}_2\right)\left(\operatorname{vec} \widehat{\boldsymbol{A}}_i - \widehat{\boldsymbol{\Gamma}}_0 \boldsymbol{w}_i\right), \\
\boldsymbol{H} &= \frac{1}{n-s} \sum_{i=1}^{n} \boldsymbol{h}_i \boldsymbol{h}_i',
\end{aligned} \tag{3.8}$$

where $\widehat{\boldsymbol{\Gamma}}_0$ is defined in (3.7), and

$$\boldsymbol{L}_{1i}^* = \left[\boldsymbol{L}_1 (\boldsymbol{Z}_i \boldsymbol{Z}_i')^{-1} \boldsymbol{L}_1'\right]^{-1/2} \boldsymbol{L}_1.$$

Under (3.3), $(\boldsymbol{L}_1 \otimes \boldsymbol{L}_2) \operatorname{vec} \boldsymbol{U}_i = \boldsymbol{0}$ with probability one, and $\boldsymbol{h}_i$ does not contain the term $\boldsymbol{U}_i$ (except in part of $\widehat{\boldsymbol{\Gamma}}_0$). If (3.3) is not true, $\boldsymbol{h}_i$ has additional variability due to the $\boldsymbol{U}_i$ term. The normalizing transformation in $\boldsymbol{L}_{1i}^*$ was chosen so that

$$\varepsilon_i^* = (\boldsymbol{L}_{1i}^* \otimes \boldsymbol{L}_2)\varepsilon_i \sim N(\boldsymbol{0}, \boldsymbol{I}_{q_1} \otimes \boldsymbol{\Sigma}^*), \tag{3.9}$$

where $\Sigma^* = L_2 \Sigma L_2'$. This motivates the use of H in (3.8) for our testing. We now derive an asymptotic expansion of H under the null (3.3).

LEMMA 2. *Under* (3.3), *as* $n \to \infty$,

$$H = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i^* \varepsilon_i^{*\prime} + O_p(n^{-1}),$$

where ε_i^* *is given in* (3.9).

PROOF. Under (3.3), we can write, with probability one,

$$h_i = \varepsilon_i^* + (L_{1i}^* \otimes L_2)(\Gamma_0 - \widehat{\Gamma}_0) w_i + [I_{q_1} \otimes L_2(B - \widehat{B})] \operatorname{vec}[X_i Z_i'(Z_i Z_i')^{-1} L_{1i}^{*\prime}].$$

The result follows by using assumption (ii) and Lemma 1 to evaluate the terms in $\sum_{i=1}^n h_i h_i'$. ∎

Hence, the $p_1 q_1 \times p_1 q_1, H$ is approximately a Wishart matrix with covariance matrix $I_{q_1} \otimes \Sigma^*$ when n is large and (3.3) holds. If (3.3) does not hold, H includes additional variability due to U_i. We compare H to another $p_1 q_1 \times p_1 q_1$ matrix $I_{q_1} \otimes S$, where $S = L_2 \widehat{\Sigma} L_2'$ and $\widehat{\Sigma}$ is defined in (3.4). We consider a test statistic which is a function of the roots of a determinantal equation

$$\left| H - \lambda (I_{q_1} \otimes S) \right| = 0. \tag{3.10}$$

All the roots of (3.10) approach one as $n \to \infty$ under the null (3.3), and tend to be larger under the alternative. Except for the Kronecker product form, the equation (3.10) is similar to that appearing in multivariate regression. See Anderson (1984). Based on this similarity, various functions of the roots can be proposed. We consider a function which provides a relatively straightforward derivation of the approximate null distribution. As an analogy to the Lawley-Hotelling statistic in multivariate regression, we consider the sum of the $p_1 q_1$ roots of (3.10), which is also equal to

$$T = \operatorname{tr}\left\{ H(I_{q_1} \otimes S^{-1}) \right\}. \tag{3.11}$$

If we let H_{ll} denote the l-th $p_1 \times p_1$ diagonal block of H in dividing H into q_1^2 blocks of size $p_1 \times p_1$, then we can write

$$T = \operatorname{tr}\left\{ \sum_{l=1}^{q_1} H_{ll} S^{-1} \right\}. \tag{3.12}$$

To derive the limiting null distribution of T, we rewrite T as

$$T = \operatorname{tr}\left\{ H^* S^{*-1} \right\}, \tag{3.13}$$

where

$$H^* = \Sigma^{*-1/2} \sum_{l=1}^{q_1} H_{ll} \Sigma^{*-1/2},$$
$$S^* = \Sigma^{*-1/2} S \Sigma^{*-1/2}.$$

We next present the limiting distribution of H^* and S^*. Note that the independence of H^* and S^* follows from the independence of $\widehat{A}$ and $\widehat{\Sigma}$ in (3.4).

LEMMA 3. *Under* (3.3), *as* $n \to \infty$

$$\sqrt{q_1 n}\left(\frac{1}{q_1} H^* - I_{p_1}\right) \xrightarrow{L} F,$$
$$\sqrt{d}(S^* - I_{p_1}) \xrightarrow{L} G,$$

where the distinct elements of $p_1 \times p_1$ *symmetric* F *and* G *are independent normal random variables with mean zero and variance 2 for diagonal and 1 for off-diagonal elements, and* $\xrightarrow{L}$ *denotes the convergence in distribution.*

PROOF. The result for S^* follows, because $dS^* \sim W_{p_1}(I_{p_1}, d)$ by (3.5), and because $d = N - k - nq \to \infty$ as $n \to \infty$ by assumption (ii). By Lemma 2 and (3.9),

$$H^* = \frac{1}{n} H^0 + O_p(n^{-1}), \tag{3.14}$$

where $H^0 \sim W_{p_1}(I_{p_1}, q_1 n)$. Thus, the result follows from the standard property of a Wishart matrix. ∎

The following theorem gives the limiting null distribution of T as $n \to \infty$.

THEOREM. *Under* (3.3), *as* $n \to \infty$

$$\sqrt{n}\left(\frac{1}{q_1} T - p_1\right) \xrightarrow{L} N\left(0, 2p_1\left[q_1^{-1} + (r_0 - q)^{-1}\right]\right),$$

where r_0 *is defined in assumption* (ii).

PROOF. By assumption (ii), as $n \to \infty$,

$$\frac{d}{n} \longrightarrow r_0 - q. \tag{3.15}$$

Thus, by Lemma 3, $S^* = I_{p_1} + O_p(n^{-1/2})$, and

$$\begin{aligned}
\sqrt{n}\left(\frac{1}{q_1} T - p_1\right) &= \sqrt{n}\,\mathrm{tr}\left\{\frac{1}{q_1} H^* [I_{p_1} - (S^* - I_{p_1})] - I_{p_1}\right\} + O_p(n^{-1/2}) \\
&= \mathrm{tr}\left\{\sqrt{n}\left(\frac{1}{q_1} H^* - I_{p_1}\right) - \sqrt{n}(S^* - I_{p_1})\right\} + O_p(n^{-1/2}) \\
&\xrightarrow{L} \mathrm{tr}\left\{q_1^{-1/2} F - (r_0 - q)^{-1/2} G\right\},
\end{aligned}$$

where $\boldsymbol{F}$ and $\boldsymbol{G}$ are as given in Lemma 3. Hence, the result follows. ∎

Using this theorem and (3.15), we can construct an asymptotic test of the null hypothesis (3.3) when n is large. We reject (3.3) if

$$\left[2p_1\left(n^{-1}q_1^{-1} + d^{-1}\right)\right]^{-1/2}\left(\frac{1}{q_1}T - p_1\right) \tag{3.16}$$

exceeds a standard normal percentile.

Based on the form (3.13) of T and the expansion (3.14), one might argue for the use of the percentiles of the Lawley-Hotelling trace distribution with dimension p_1 and degrees of freedom $q_1 n$ and d as approximate cut-off points for T. Such an argument might make sense only if n is large. But, the percentiles of the Lawley-Hotelling distribution have been tabulated only for small $q_1 n$, e.g., $q_1 n \leq 15$ (in multivariate regression this degree of freedom is typically small), and are unavailable for cases of our interest. One might also consider using the existing χ^2 or F approximation to the Lawley-Hotelling distribution. See, e.g., Anderson (1984). However, such an approximation was derived under the assumption that the first of the two degrees of freedom is fixed and the second tends to infinity (as in multivariate regression). For our problem, both $q_1 n$ and d tend to infinity as $n \to \infty$, and such an approximation does not apply. Of course, the use of the Lawley-Hotelling distribution based on the expansion (3.14) is not quite valid, because (3.14) is an expansion as $n \to \infty$. The proper limiting distribution of T as $n \to \infty$ was derived in the above theorem. For the general multivariate mixed effect model (1.7) with large n, we recommend the use of the asymptotic test procedure (3.16) for testing the random effect structure of type (3.3).

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