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CONCENTRATION INEQUALITIES FOR MULTIVARIATE DISTRIBUTIONS: II. ELLIPTICALLY CONTOURED DISTRIBUTIONS[1]

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In part I of this study it was shown that $\Sigma_1 \le \Sigma_2 \Rightarrow P_{\Sigma_1}(C) \ge P_{\Sigma_2}(C)$ under various convexity and symmetry assumptions on the set $C \subset \mathbb{R}^p$, where P_Σ denoted the p-variate normal distribution with mean vector 0 and positive definite covariance matrix Σ. In Part II extensions of these results to the family of elliptically contoured distributions are considered. The proof of the concentration inequality of Fefferman, Jodeit, and Perlman (1972) for convex centrally symmetric sets C is examined to determine whether it can be extended to sets C with other convexity and/or symmetry properties. Whereas it does not appear that this proof remains applicable, in the bivariate case ($p = 2$) an alternate geometric argument not only extends the concentration inequalities for convex G-invariant sets C and for G-decreasing sets C in Part I to elliptically contoured distributions, but also enlarges the class of groups G for which the concentration inequality for G-decreasing sets is valid. Also, sharpened forms of these concentration inequalities are presented for elliptically contoured distributions that are not absolutely continuous with respect to Lebesgue measure.

5. A Concentration Inequality for Convex Centrally Symmetric Sets

In Part I of this study[2] it was shown that

$$\Sigma_1 \le \Sigma_2 \Rightarrow P_{\Sigma_1}(C) \ge P_{\Sigma_2}(C) \tag{5.0}$$

under various convexity and symmetry assumptions on the set $C \in \mathbb{R}^p$, where P_Σ denoted the p-variate normal distribution with mean vector 0 and positive definite covariance matrix Σ. It is evident that such concentration

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[2]Eaton and Perlman (1991). Part I comprised Sections 1-4; Part II comprises Sections 5-7.

inequalities for multivariate normal distributions in Theorems 3.1, 3.2, and 3.3 of Part I remain valid when P_Σ is taken to be a scale mixture over $\lambda > 0$ of normal distributions on $\mathbb{R}^p$ with mean 0 and covariance matrix $\lambda\Sigma$, e.g., a multivariate Student-t distribution. Like the normal distribution itself, such a scale mixture is both unimodal and elliptically contoured. It is somewhat surprising that the first of these theorems, and possibly the other two as well, remain valid for *all* elliptically contoured distributions *without* assuming unimodality.

Fefferman, Jodeit, and Perlman (1972) substantially strengthened the concentration inequality in Theorem 3.1 for convex centrally symmetric sets $C \in \mathbb{R}^p$ by extending it from normal to elliptically contoured distributions (see also Das Gupta *et al* (1972), Theorem 3.3). Surprisingly, their proof is also based on Anderson's convolution theorem, Theorem 2.1, as was the proof of Theorem 3.1 in the normal case, although Anderson's theorem is now applied in a quite different way. In this section we review their proof in detail to determine whether or not it can be extended to sets C with other convexity and/or symmetry properties. Whereas it does not appear that their method of proof remains applicable, in the bivariate case ($p = 2$) an alternate geometric argument not only extends Theorem 3.2 (for convex G-invariant sets) and Theorem 3.3 (for G-decreasing sets) to elliptically contoured distributions but also enlarges the class of groups G to which Theorem 3.3 applies. These bivariate results are given in Theorems 6.1 and 6.2 of Section 6. In Section 7, sharpened forms of the concentration inequalities in Sections 5 and 6 are presented for elliptically contoured distributions that are not absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^p$ and which therefore may assign nonzero probability to the boundary of C.

DEFINITION 5.1 The random vector $X \in \mathbb{R}^p$ has an *elliptically contoured distribution*, denoted by $X \sim EC_p(\Sigma)$, if its characteristic function $\varphi(t) \equiv E\{\exp(it'X)\}$, $t \in \mathbb{R}^p$, has the form $\varphi(t) = \gamma(t'\Sigma t)$ for some function γ, where Σ is a $p \times p$ positive definite matrix. Equivalently,

$$X \sim EC_p(\Sigma) \Leftrightarrow X \stackrel{d}{=} \Sigma^{1/2} Z, \tag{5.1}$$

where $\Sigma^{1/2}$ is the $p \times p$ positive definite matrix such that $(\Sigma^{1/2})^2 = \Sigma$ and where Z is an orthogonally invariant random vector in $\mathbb{R}^p$. If X has a probability density function f on $\mathbb{R}^p$ then $X \sim EC_p(\Sigma)$ iff $f(x) = |\Sigma|^{-1/2} g(x'\Sigma^{-1}x)$ for some function g; in particular, the multivariate normal distribution $N_p(0, \Sigma)$ is $EC_p(\Sigma)$.

The following notation is used: B and S denote the unit ball and unit sphere in $\mathbb{R}^p$, ν denotes the uniform probability measure on S, and $D \equiv \text{Diag}(d_1, \ldots, d_p)$ denotes a $p \times p$ diagonal matrix with $0 < d_i \leq 1$ for $i =$

$1, \ldots, p$, so D is a *contraction*. The class of all convex centrally symmetric sets in $\mathbb{R}^p$ is denoted by $\mathcal{C}_1$.

THEOREM 5.1 (Fefferman, Jodeit, and Perlman (1972)). *Suppose that* $X \sim EC_p(\Sigma)$. *If* $C \in \mathcal{C}_1$ *and* C *is closed, then* $\Sigma_1 \leq \Sigma_2 \Rightarrow P_{\Sigma_1}(C) \geq P_{\Sigma_2}(C)$.

PROOF By (5.1),

$$X \sim EC_p(\Sigma) \Rightarrow P_\Sigma(C) \equiv P_\Sigma(X \in C) = P(Z \in \Sigma^{-1/2}C), \tag{5.2}$$

where $\Sigma^{-1/2} = (\Sigma^{1/2})^{-1} = (\Sigma^{-1})^{1/2}$. Since Z is orthogonally invariant, $Z \stackrel{d}{=} R \cdot U$, where R and U are independent, U is uniformly distributed on the sphere $S \equiv \{x \in \mathbb{R}^p : \|x\| = 1\}$, and $0 \leq R < \infty$. Therefore

$$P_\Sigma(C) = E\{P[U \in R^{-1}\Sigma^{-1/2}C|R]\} \equiv E\{\nu(R^{-1}\Sigma^{-1/2}C)\}. \tag{5.3}$$

Since $C \in \mathcal{C}_1 \Leftrightarrow R^{-1}C \in \mathcal{C}_1$ (provided $R > 0$) it therefore suffices to compare $\nu(\Sigma_1^{-1/2}C)$ and $\nu(\Sigma_2^{-1/2}C)$ for $C \in \mathcal{C}_1$.

By the Singular Value Decomposition

$$\Sigma_2^{-1/2}\Sigma_1^{1/2} = \psi' D\Gamma, \tag{5.4}$$

where ψ and Γ are $p \times p$ orthogonal matrices, $D = \mathrm{Diag}(d_1, \ldots, d_p)$, and $d_1, \ldots, d_p$ are the singular values of $\Sigma_2^{-1/2}\Sigma_1^{1/2}$. Since $0 < \Sigma_1 \leq \Sigma_2$, $0 < d_i \leq 1$ for $i = 1, \ldots, p$, so D is a contraction. Because ν is orthogonally invariant,

$$\begin{aligned} \nu(\Sigma_1^{-1/2}C) &= \nu(\Gamma\Sigma_1^{-1/2}C) = \nu(K) \\ \nu(\Sigma_2^{-1/2}C) &= \nu(\psi' D\Gamma\Sigma_1^{-1/2}C) = \nu(DK), \end{aligned} \tag{5.5}$$

where

$$K \equiv K(C; \Sigma_1, \Gamma) = \Gamma\Sigma_1^{-1/2}C \in \mathcal{C}_1. \tag{5.6}$$

Thus the desired result is equivalent to the following assertion: for every closed $K \in \mathcal{C}_1$ and every diagonal contraction mapping D,

$$\nu(K) \geq \nu(DK). \tag{5.7}$$

This inequality is nontrivial since DK need not be contained in K. By means of the Divergence Theorem, however, it can be shown that[3]

$$\begin{aligned} \frac{\partial}{\partial d_i}[\nu(DK)] &\equiv \frac{\partial}{\partial d_i}\int_S I_{DK}(x)d\nu(x) \\ &\doteq -d_i^{-1}\frac{\partial^2}{\partial \beta^2}[\int_B I_{DK}(x - \beta\theta_i)dx]_{\beta=0}, \end{aligned} \tag{5.8}$$

[3]The equality $\doteq$ in (5.8) may hold only for almost every d_i, so a more careful argument is needed which makes use of the assumption that C, and hence K, is closed. First, if K is not bounded, consider the bounded set $K^* \equiv K \cap (m^{-1}B)$, where $m \equiv min(d_1, \ldots, d_p) < 1$. Then $K^* \in \mathcal{C}_1, \nu(K^*) = \nu(K)$, and $\nu(DK^*) = \nu(DK)$, so it would suffice to establish

where I_E denotes the indicator function of the set E and θ_i is the unit vector with i-th component 1. Since both B and $DK \in \mathcal{C}_1$, Anderson's Theorem 2.1 implies that

$$\psi(\theta) \equiv \int_B I_{DK}(x-\theta)dx \tag{5.9}$$

is centrally symmetric and ray-decreasing in θ, hence has a local (in fact, global) maximum at 0, so the second derivative in (5.8) is nonpositive. Therefore $\nu(DK)$ is nondecreasing in each $d_i, i = 1, \ldots, p$, which establishes (5.7). □

In Section 3 of Part I we saw that for multivariate normal distributions, the method of proof of Theorem 3.1 could be used to establish Theorems 3.2 and 3.3 simply by replacing Theorem 2.1 by Theorems 2.2 and 2.4 respectively. Unfortunately this is not so for elliptically contoured distributions. In the proof of Theorem 5.1, Anderson's Theorem 2.1 was applied to show that ψ in (5.9) has a maximum at $\theta = 0$ when C (*and therefore* DK!!) $\in \mathcal{C}_1$. In order to extend Theorem 3.2 to elliptically contoured distributions by this method, it would be necessary to apply Theorem 2.2 to show that ψ has a maximum at $\theta = 0$ when $C \in \mathcal{C}_G$, where G is a compact subgroup of the orthogonal group $\mathcal{O}_p$ that acts effectively on $\mathbb{R}^p$, $\mathcal{C}_G$ is the class of all convex G-invariant subsets of $\mathbb{R}^p$, and Σ_1 is G-invariant (for detailed definitions, see Section 3; recall from (5.6) that K, and hence ψ, depends on Σ_1). Now it can be shown[4] that

$$\begin{aligned} \Sigma_1 \text{ is } G-\text{invariant} \ &\Rightarrow \ \Sigma_1^{1/2} \text{ is } G-\text{invariant} \\ &\Rightarrow \ \Sigma_1^{-1/2} \text{ is } G-\text{invariant}, \end{aligned} \tag{5.10}$$

so

$$C \in \mathcal{C}_G \Rightarrow \Sigma_1^{-1/2} C \in \mathcal{C}_G \Rightarrow K \in \mathcal{C}_{\tilde{G}}, \tag{5.11}$$

where K is defined in (5.6) and

$$\tilde{G} = \Gamma G \Gamma' = \{\Gamma g \Gamma' | g \in G\} \tag{5.12}$$

(5.7) with K replaced by K^*. Thus we may assume that K is in fact compact. Since K is compact, convex, and centrally symmetric, by considering its supporting hyperplanes we see that it is the decreasing limit of a sequence of compact convex centrally symmetric polyhedra in $\mathbb{R}^p$, so we may assume that K is such a polyhedron. Then we may construct a sequence of smooth centrally symmetric unimodal functions u_ϵ which converges to I_K everywhere in $\mathbb{R}^p$ except possibly on ∂K as $\epsilon \to 0$, but $\nu(\partial K) = 0$ since K is a polyhedron. If we now replace $I_{DK}(x) \equiv I_K(D^{-1}x)$ by $u_\epsilon(D^{-1}x)$ in (5.8) then $\doteq$ becomes $=$ for every d_i, so $\int_S u_\epsilon(D^{-1}x)d\nu(x)$ is nondecreasing in each d_i, hence so is $\nu(DK)$. (See Fefferman *et al* (1972) for further details.)

[4]If Σ_1 is G-invariant then so is $f(\Sigma_1)$, where f is any polynomial with real coefficients. But $(\Sigma_1)^{1/2} = f(\Sigma_1)$ where f is any real polynomial such that $f(\lambda_i) = (\lambda_i)^{1/2}$ for $i = i, \ldots, p$, where $\lambda_1, \ldots, \lambda_p$ are the eigenvalues of Σ_1. (The coefficients of f may depend on $\lambda_1, \ldots, \lambda_p$ and hence on Σ_1.) Similarly, $(\Sigma_1)^{-1}$ and $(\Sigma_1)^{-1/2}$ are G-invariant. (We thank Steen A. Andersson for this observation.)

is also a compact effective subgroup of $\mathcal{O}_p$. Unfortunately, although the linear transformation $K \to DK$ preserves convexity *it need not preserve $\tilde{G}$-invariance* (unlike Theorem 5.1 where $G = \tilde{G} = \{\pm I\}$ with I the $p \times p$ identity matrix), so we cannot conclude that $DK \in \mathcal{C}_{\tilde{G}}$ and thus are unable to apply Theorem 2.2 to ψ in (5.9). Similarly, when $C \in \mathcal{M}_G$ (the class of all G-decreasing subsets of $\mathbb{R}^p$) with G a compact effective reflection group, then $\tilde{G}$ is also a compact effective reflection group but we cannot conclude that $DK \in \mathcal{M}_{\tilde{G}}$, hence cannot apply Theorem 2.4 to extend Theorem 3.3 to elliptically contoured distributions.

Despite these difficulties, we conjecture that the concentration inequalities for the classes $\mathcal{C}_G$ and $\mathcal{M}_G$ in Theorems 3.2 and 3.3 remain valid for elliptically contoured distributions. To support this conjecture, in Section 6 we present an alternate geometric argument, similar to that in Section 1 of Fefferman, Jodeit, and Perlman (1972), which establishes these results in the bivariate case, i.e., when $p = 2$. In fact Theorem 6.2, the extension of Theorem 3.3 thus obtained, is *strictly stronger* than Theorem 3.3 in the bivariate case in that it applies to a *larger* class of groups G (acting on $\mathbb{R}^2$) than the class of effective reflection groups.

REMARK 5.1 If $-I \in G$ then $\mathcal{C}_G \subseteq \mathcal{C}_1$, so in this case the extension of Theorem 3.2 to elliptically contoured distributions is implied by Theorem 5.1 *without* the assumption that Σ_1 is G-invariant (also see Remark 3.1 of Part I). □

REMARK 5.2 Because it suffices to show only that ψ in (5.9) has a *local* maximum at $\theta = 0$, the method of proof in this section may succeed in extending Theorems 3.2 and 3.3 to elliptically contoured distributions provided that suitable *local* versions of Theorems 2.2 and 2.4 can be found. Note too that one of the sets in (5.9), namely B, is a ball, so the full generality of these latter theorems would not be needed. Furthermore, even the existence of a local maximum at $\theta = 0$ is not necessary; it would suffice to show that C is locally concave at $\theta = 0$. □

6. Bivariate Concentration Inequalities for Elliptically Contoured Distributions

In this section unless otherwise noted, $p = 2$, B and S denote the closed unit disk and unit circle in $\mathbb{R}^2$, respectively, ν denotes the uniform measure on S with $\nu(S) = 1$, and $D \equiv \mathrm{Diag}(d_1, d_2)$ is a contraction ($0 < d_1, d_2 \leq 1$). Lemma 6.1 presents the basic geometric construction by means of which we shall extend Theorems 3.2 and 3.3 to elliptically contoured distributions in the bivariate case. This argument, based on that on pp. 114-5 in Fefferman *et al* (1972), is an alternative to that used to derive (5.7) in the proof of Theorem 5.1 above (but see Remark 6.3).

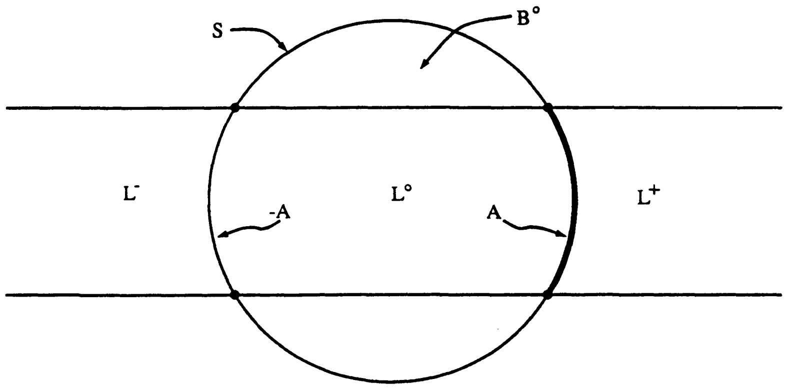

Figure 6.1. The arc A, the strip $L \equiv L(A)$, and the ball $B \equiv B^\circ \cup S$.

DEFINITION 6.1 For any closed arc $A \subset S$ with $0 \leq \text{arclength}(A) < \pi$, define $L \equiv L(A)$ to be the closed centrally symmetric strip such that $L \cap S = (-A) \cup A$ (see Figure 6.1.) Then L can be expressed as the disjoint union

$$(6.1) \qquad L = L^- \cup L^\circ \cup L^+,$$

where $L^\circ \equiv L^\circ(A) = L(A) \cap B^\circ$ with B° the *open* unit disk, $L^- \equiv L^-(A) \supset (-A)$, and $L^+ \equiv L^+(A) \supset A$; note that L^- and L^+ are both *closed* sets. Note too that if $A_1, \ldots, A_m$ are disjoint then $L^+(A_1), \ldots, L^+(A_m)$ (hence also $D[L^+(A_1)], \ldots, D[L^+(A_m)]$) are disjoint.

LEMMA 6.1 *Let K be a closed subset of $\mathbb{R}^2$ with $K \cap S = \cup\{A_j | j = 1, \ldots, m\}$, a disjoint union of closed arcs such that $0 \leq \text{arclength}(A_j) < \pi$. Define $L_j = L(A_j)$, $L_j^- = L^-(A_j)$, and $L_j^+ = L^+(A_j)$. If*

$$(6.2) \qquad K \backslash B^\circ \subseteq \cup L_j^+$$

then

$$(6.3) \qquad \nu(K) \geq \nu(DK).$$

PROOF If we define

$$(6.4) \qquad K(j) = (K \backslash B^\circ) \cap L_j^+,$$

(see Figure 6.2) then $K(1), \ldots, K(m)$ are disjoint and (6.2) implies that

$$(6.5) \qquad K \backslash B^\circ = \cup K(j).$$

Express K as the disjoint union $K = (K \backslash B^\circ) \cup (K \cap B^\circ)$. Since D is a contraction, $DB^\circ \cap S = \emptyset$, hence $DK \cap S = D(K \backslash B^\circ) \cap S$, so $\nu(DK) =$

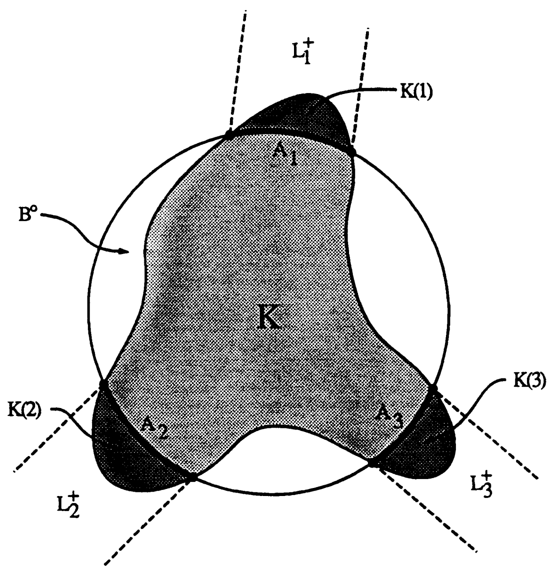

Figure 6.2. A set K (shaded) that satisfies (6.2).

$\nu[D(K\backslash B^\circ)] = \nu\{\cup D[K(j)]\}$ by (6.5). By (6.4), however, $D[K(j)] \subseteq D(L_j^+)$, hence

$$\begin{aligned} \nu(DK) &\le \nu\{\cup[D(L_j^+)]\} &&= \textstyle\sum \nu[D(L_j^+)] \\ &= \tfrac{1}{2}\textstyle\sum \nu(DL_j) &&\le \tfrac{1}{2}\textstyle\sum \nu(L_j) \\ &= \textstyle\sum \nu(L_j^+) &&= \textstyle\sum \nu(A_j) \\ &= \nu(K). \end{aligned} \tag{6.6}$$

The second equality in (6.6) follows from (6.1), the inclusion $D(L_j^\circ) \subset B^\circ$, and the relation $L_j^- = -L_j^+$ (implied by the central symmetry of L_j):

$$\nu(DL_j) = \nu[D(L_j^-)] + \nu[D(L_j^+)] = 2\nu[D(L_j^+)]. \tag{6.7}$$

The second inequality in (6.6) follows since the width of the strip DL_j cannot exceed that of L_j as D is a contraction. Thus (6.3) is established. □

REMARK 6.1 In Lemma 6.1 suppose in addition that K is star-shaped with respect to the origin. Then for each $x \in K\backslash B^\circ$ the closed line segment $[0, x]$ intersects the unit circle S at a unique point $y(x) \in K \cap S \equiv \cup A_j$. Thus if we define

$$K_j = \{x \in K\backslash B^\circ \mid y(x) \in A_j\} \tag{6.8}$$

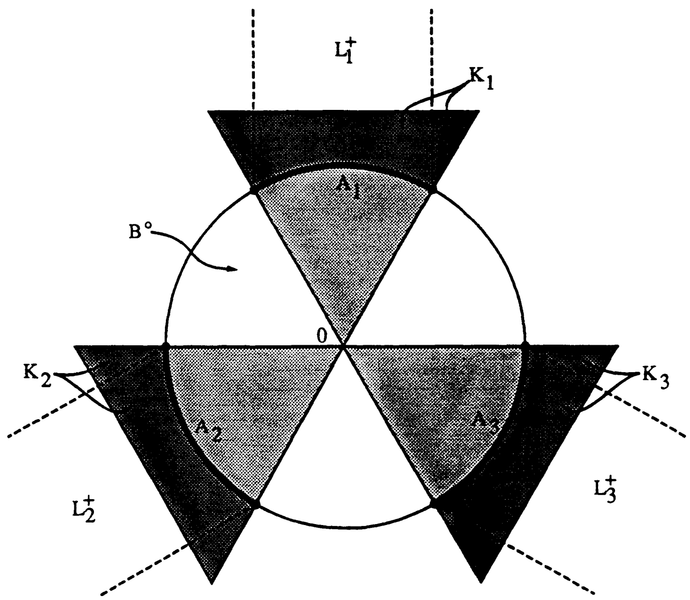

Figure 6.3. A star-shaped set K (shaded) that does not satisfy (6.9).

(see Figure 6.3), then $K_1, \ldots, K_m$ are disjoint and $\cup K_j = K \backslash B^\circ$. It is readily verified that $K(j) \subseteq K_j$ and that (6.2) is equivalent to the condition that for each $j = 1, \ldots, m$

$$K_j \subseteq L_j^+. \tag{6.9}$$ □

REMARK 6.2 For the validity of Lemma 6.1 it is not necessary that the strips $L_j \equiv L(A_j)$ be centrally symmetric, only that $L_j \cap S = (A_j^-) \cup A_j$ where A_j^- is any closed arc such that the relative interiors of A_j^- and A_j do not intersect. Note that this condition still implies that $\nu(A_j^-) = \nu(A_j)$ since L_j is a strip. Again L_j can be decomposed as in (6.1), where now $L_j^- \equiv L^-(A_j) \supset A_j^-$ and $L_j^+ \equiv L^+(A_j) \supset A_j$. Similarly, decompose DL_j as $(DL_j)^- \cup (DL_j)^\circ \cup (DL_j)^+$ where $(DL)^\circ = DL \cap B^\circ$. Then the proof of Lemma 6.1 remains valid with the following three modifications: (i) the first equality in (6.6) must be replaced by the inequality $\leq$, for now $L_1^+, \ldots, L_m^+$ (hence $D(L_1^+), \ldots, D(L_m^+)$) need not be disjoint; (ii) although $L_j^- \neq -L_j^+$ if L_j is not centrally symmetric, it follows from the fact that D is a contraction that $D(L_j^-) \cap S = (DL_j)^- \cap S$ and $D(L_j^+) \cap S = (DL_j)^+ \cap S$, hence $\nu[D(L_j^-)] = \nu[(DL_j)^-] = \nu[(DL_j)^+] = \nu[D(L_j^+)]$; (iii) $\nu(DL_j) \leq \nu(L_j)$ since the strip DL_j is both narrower and closer to the origin than L_j . □

Theorems 6.1 and 6.2 below extend Theorems 3.2 and 3.3 from normal distributions to elliptically contoured distributions in the bivariate case. To prove these extensions we shall apply Lemma 6.1 and Remark 6.1 to the set $K \equiv K(C; \Sigma_1, \Gamma)$ defined in (5.6). If G is a compact subgroup of the orthogonal group $\mathcal{O}_2$ that acts effectively on $\mathbb{R}^2$ and $C \in \mathcal{C}_G$ (or $\mathcal{M}_G$) then $K \in \mathcal{C}_{\widetilde{G}}$ (or $\mathcal{M}_{\widetilde{G}}$) (see (5.11) or (6.14)) so K is star-shaped (apply Lemma 3.1 of Part I), hence to apply these results it must be verified that K satisfies (6.9). This will be accomplished in the proofs of Theorems 6.1 and 6.2 by means of the convexity (or monotonicity) and $\widetilde{G}$-invariance of K. ($\widetilde{G}$ is defined in (5.12).)

Before proceeding with the statements and proofs of Theorems 6.1 and 6.2 we describe the compact subgroups $G \subseteq \mathcal{O}_2$ acting on $\mathbb{R}^2$. It is well known (e.g., see Grove and Benson (1985), Theorem 2.2.1) that if G is finite then either G is the cyclic group $\mathcal{C}_2^n$ of order n generated by the rotation through angle $2\pi/n$ or else G is the dihedral group $\mathcal{H}_2^n$ of order $2n$ generated by $\mathcal{C}_2^n$ and a single reflection in $\mathbb{R}^2$, where $n = 1, 2, \ldots$. The group $\mathcal{C}_2^n(\mathcal{H}_2^n)$ is the group of all rotations (all rotations and reflections) that leave a regular n-gon invariant, and G is a finite reflection group iff $G = \mathcal{H}_2^n$ for some $n \geq 1$. Thus

$$\mathcal{C}_2^1 = \{I\}, \qquad \mathcal{C}_2^2 = \{\pm I\},$$

$$\mathcal{H}_2^1 \cong \left\{\begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}, \quad \mathcal{H}_2^2 \cong \left\{\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}\right\} = \mathcal{D}_2,$$

where $\mathcal{D}_2$ is the group of sign changes of coordinates in $\mathbb{R}^2$ (recall Section 2 of Part I). Thus $\mathcal{C}_2^1$ and $\mathcal{H}_2^1$ do not act effectively on $\mathbb{R}^2$, $\mathcal{C}_2^2$ and $\mathcal{H}_2^2$ act effectively but not irreducibly, while $\mathcal{C}_2^n$ and $\mathcal{H}_2^n$ act effectively and irreducibly for $n \geq 3$ (see Section 3 of Part I for definitions). Finally, the only infinite compact subgroups of $\mathcal{O}_2$ are $\mathcal{O}_2$ itself and $\mathcal{SO}_2$, the subgroup of all proper rotations of $\mathbb{R}^2$, both of which act effectively and irreducibly.

THEOREM 6.1 *Suppose that $X \sim EC_2(\Sigma)$. If $C \in \mathcal{C}_G$ and C is closed, then $\Sigma_1 \leq \Sigma_2 \Rightarrow P_{\Sigma_1}(C) \geq P_{\Sigma_2}(C)$ provided that Σ_1 is G-invariant and G acts effectively on $\mathbb{R}^2$ (i.e., $G \neq \mathcal{C}_2^1$ or $\mathcal{H}_2^1$).*

PROOF As in the proof of Theorem 5.1, the desired result is equivalent to the assertion that (5.7) holds for every closed $K \in \mathcal{C}_{\widetilde{G}}$ and every contraction D, where $\widetilde{G}$ is defined in (5.12).

If $G = \mathcal{O}_2$ or $\mathcal{SO}_2$ then $\widetilde{G} = G$ and $\mathcal{C}_{\widetilde{G}}$ is simply the class of all open or closed disks centered at 0, so $DK \subseteq K$ and (5.7) is trivially valid. Thus we may assume that $G = \mathcal{C}_2^n$ or $\mathcal{H}_2^n$, $n \geq 2$. If n is *even*, however, then $-I \in G$ and the desired result is already a consequence of Theorem 5.1 (see Remark 5.1). Since $\mathcal{H}_2^n \supset \mathcal{C}_2^n$ it therefore suffices to establish (5.7) when $G = \mathcal{C}_2^n$ for $n \geq 3$ and n *odd* (for, $G \supset G' \Rightarrow \mathcal{C}_G \subset \mathcal{C}_{G'}$).

In fact, the following argument establishes (5.7) when G is the rotation group $\mathcal{C}_2^n$ for *any* $n \geq 2$. First, note that $\widetilde{G} = G$ and that we may assume that $K \in \mathcal{C}_G$ is compact, convex and G-invariant, hence is the limit of a

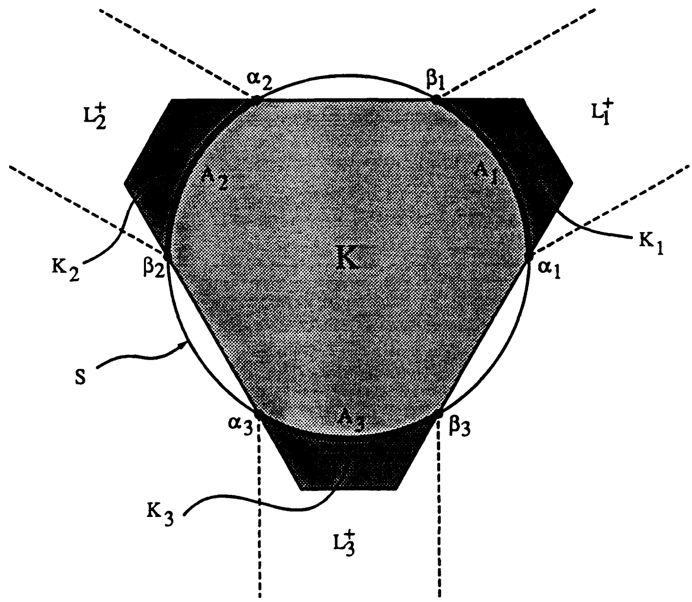

Figure 6.4. The set $K \in C_G$ (shaded); $G = C_2^n$, $n = m = 3$.

decreasing sequence of closed convex G-invariant polygons (recall Footnote 3). Thus it suffices to establish (5.7) when K is such a polygon.

In this case either $K \cap S = S$ and (5.7) is trivial, or $K \cap S = \emptyset$ so $DK \cap S = \emptyset$ and (5.7) is also trivial, or else $K \cap S = \cup A_j$, the union of $m \geq 2$ disjoint closed arcs $A_1, \ldots, A_m$, some possibly degenerate at single points (see Figure 6.4; note that $-A_j$ does not necessarily appear in $\{A_1, \ldots, A_m\}$ if n is odd). For $j = 1, \ldots, m$, let $\alpha_j \equiv \exp(i\theta_j)$ and $\beta_j \equiv \exp(i\varphi_j)(i = \sqrt{-1})$ denote the endpoints of arc A_j in counterclockwise order. Without loss of generality assume that

$$(6.10) \qquad 0 \leq \theta_1 \leq \varphi_1 < \theta_2 \leq \varphi_2 < \cdots < \theta_m \leq \varphi_m < 2\pi,$$

i.e., $A_1, \ldots, A_m$ are arranged in consecutive counterclockwise order on the unit circle S. Since K is C_2^n-invariant so is $\cup A_j$, hence m is a multiple of n and for each $j = 1, \ldots, m$,

$$(6.11) \qquad \begin{aligned} 0 < \theta_{j+1} - \theta_j \leq 2\pi/n \leq \pi \\ 0 < \varphi_{j+1} - \varphi_j \leq 2\pi/n \leq \pi, \end{aligned}$$

where $\theta_{m+1} \equiv \theta_1 + 2\pi$, $\varphi_{m+1} \equiv \varphi_1 + 2\pi$. By (6.10) this implies that for each $j = 1, \ldots, m$,

$$(6.12) \qquad \begin{aligned} 0 \leq \varphi_j - \theta_j < 2\pi/n \leq \pi \\ 0 < \theta_{j+1} - \varphi_j \leq 2\pi/n \leq \pi. \end{aligned}$$

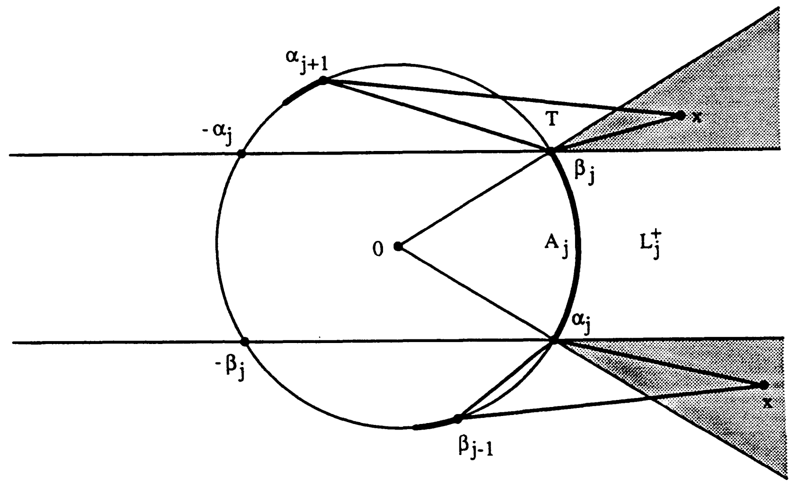

Figure 6.5.

In particular, $0 \leq \text{arclength}(A_j) < \pi$ for $j = 1, \ldots, m$. Also, because $K \in \mathcal{C}_G \subset \mathcal{M}_G$ and G acts effectively on $\mathbb{R}^2$, the line segment $[0, x] \subset K$ whenever $x \in K$ (apply Lemma 3.1 of Part I), hence K is star-shaped with respect to the origin. By Lemma 6.1 and Remark 6.1, therefore, in order to establish (5.7) it suffices to show that K satisfies (6.9), where $L_j^+ = L^+(A_j)$ and K_j is defined in (6.8).

The inclusion (6.9) is trivial if $\alpha_j = \beta_j$ so assume that $\alpha_j \neq \beta_j$, i.e., that $\theta_j < \varphi_j$. If (6.9) fails, consider $x \in K_j \backslash L_j^+$. By definition of K_j, x must lie in one of the two shaded wedge-shaped regions indicated in Figure 6.5. If x lies in the upper shaded region, consider the triangle T with vertices x, β_j, and α_{j+1}.[5] Clearly $T \subset K$, since K is convex and $x, \beta_j, \alpha_{j+1} \in K$. But T must intersect the open arc $(\beta_j,\ \alpha_{j+1})$, hence this arc has a nonempty intersection with K, which contradicts the fact that this arc is contained in $S \backslash K$. If it is assumed that x lies in the lower shaded region in Figure 6.5, replace β_j, α_{j+1} by α_j, β_{j-1} to get a similar contradiction. Thus (6.9) is confirmed. □

REMARK 6.3 This method of proof does not extend in any obvious way to the multivariate case $p \geq 3$. To see this, suppose that K is a closed convex G-invariant polyhedron in $\mathbb{R}^3$; let B (or S) denote the closed unit ball (or sphere) in $\mathbb{R}^3$ and ν denote the uniform measure on S. As in the above proof it suffices to consider the case where $K \cap S = \cup A_j$, the union of disjoint closed subsets $A_1, \ldots, A_m$ of S. Since K is star-shaped, again the sets K_j are well-defined by (6.8) and $\cup K_j = K \backslash B^\circ$. Unlike the case $p = 2$, however, the

[5] It is essential to verify that Figure 6.5 accurately depicts the location of α_{j+1}, i.e., that α_{j+1} lies in the half-open arc $(\beta_j, -\alpha_j]$. But this is equivalent to the condition $\varphi_j < \theta_{j+1} \leq \theta_j + \pi$, which follows from (6.12). (Note that we define $\alpha_{m+1} = \alpha_1, \theta_{m+1} = \theta_1 + 2\pi$). Similarly, β_{j-1} lies in the half-open arc $(\alpha_j,\ -\beta_j]$. (Define $\beta_0 = \beta_m, \varphi_0 = \varphi_m - 2\pi$.)

sets A_j no longer need have a simple form, so it is not apparent how to define sets $L_1^+, \ldots, L_m^+$ such that $L_j^+ \cap B^\circ = \emptyset$, $L_j^+ \cap S = A_j$, $\nu[D(L_j^+)] \leq \nu(L_j^+)$, and such that (6.9) holds. (If it is possible to find such sets L_j^+ then (5.7) would follow as in (6.6).) Nonetheless we conjecture that Theorem 6.1, like Theorem 5.1, is valid for $p \geq 3$. (Also see Remark 5.1.) □

REMARK 6.4 If Σ_1 is not assumed G-invariant in Theorem 6.1 then $K \equiv K(C; \Sigma_1, \Gamma)$, although still star-shaped, need not be $\widetilde{G}$-invariant. In this case it is easy to find examples where the sets K_j constructed from the arcs A_j as in (6.8) do not satisfy (6.9) – for example, take $G = \mathcal{C}_2^3$ and C an equilateral triangle centered at 0, then choose Σ_1 such that K is an isosceles triangle with altitude$(K) >> 1 >>$ base(K). Nonetheless, we conjecture that Theorem 6.1 remains valid (when $p = 2$) even if Σ_1 is not G-invariant. By Remark 5.1, this is true (in fact, true for all $p \geq 2$) if $-I \in G$. However, Example 3.1 of Part I shows that Theorem 6.1 may fail when $p \geq 3$ if Σ_1 is not G-invariant and $-I \notin G$, even if the probability distribution is normal. □

For the bivariate case ($p = 2$), Theorem 6.2 below not only extends Theorem 3.3 from normal distributions to elliptically contoured distributions but also applies to almost every effective subgroup G of $\mathcal{O}_2$, including the rotation groups $\mathcal{C}_2^n$, $n \geq 4$, whereas Theorem 3.3 applies only to the reflection groups $\mathcal{H}_2^n$, $n \geq 2$ (as well as to $\mathcal{O}_2$ itself).

Recall (Section 3, Part I) that the class $\mathcal{M}_G$ of all G-decreasing subsets of $\mathbb{R}^p$ is closed under unions, whereas $\mathcal{C}_G$ is not, although both are closed under intersections. In fact,

$$C \in \mathcal{M}_G \Leftrightarrow C = \cup\{C_G(x) | x \in C\}, \tag{6.13}$$

where $C_G(x)$ denotes the convex hull of the G-orbit of x; note that $C_G(x) \in \mathcal{C}_G$ is a closed convex G-invariant polygon for every $x \in \mathbb{R}^p$. Recall also that every $C \in \mathcal{M}_G$ is G-invariant. It is readily verified from (6.13) (recall (5.6) and (5.10)-(5.12)) that if Σ_1 is G-invariant, then

$$C \in \mathcal{M}_G \Rightarrow \Sigma_1^{-1/2} C \in \mathcal{M}_G \Rightarrow K \in \mathcal{M}_{\widetilde{G}}. \tag{6.14}$$

THEOREM 6.2 *Suppose that $X \sim EC_2(\Sigma)$. If $C \in \mathcal{M}_G$ and C is closed, then $\Sigma_1 \leq \Sigma_2 \Rightarrow P_{\Sigma_1}(C) \geq P_{\Sigma_2}(C)$ provided that Σ_1 is G-invariant and G acts effectively on $\mathbb{R}^2$ (i.e., $G \neq \mathcal{C}_2^1$ or $\mathcal{H}_2^1$), but also $G \neq \mathcal{C}_2^2$ or $\mathcal{C}_2^3$.*

PROOF If $G = \mathcal{O}_2$ or $S\mathcal{O}_2$ then $\mathcal{M}_G = \mathcal{C}_G$ and the result is trivial. Two cases remain.

(i) $G = \mathcal{C}_2^n$, $n \geq 4$. Again $\widetilde{G} = G$. As in the proof of Theorem 6.1, the desired result is equivalent to the assertion that (5.7) holds for every compact $K \in \mathcal{M}_G$ and every contraction $D = \mathrm{Diag}(d_1, d_2)$ $(0 < d_1, d_2 \leq 1)$. Such a set K is the limit of a sequence $\{K_\lambda\}$ of finite unions of closed convex

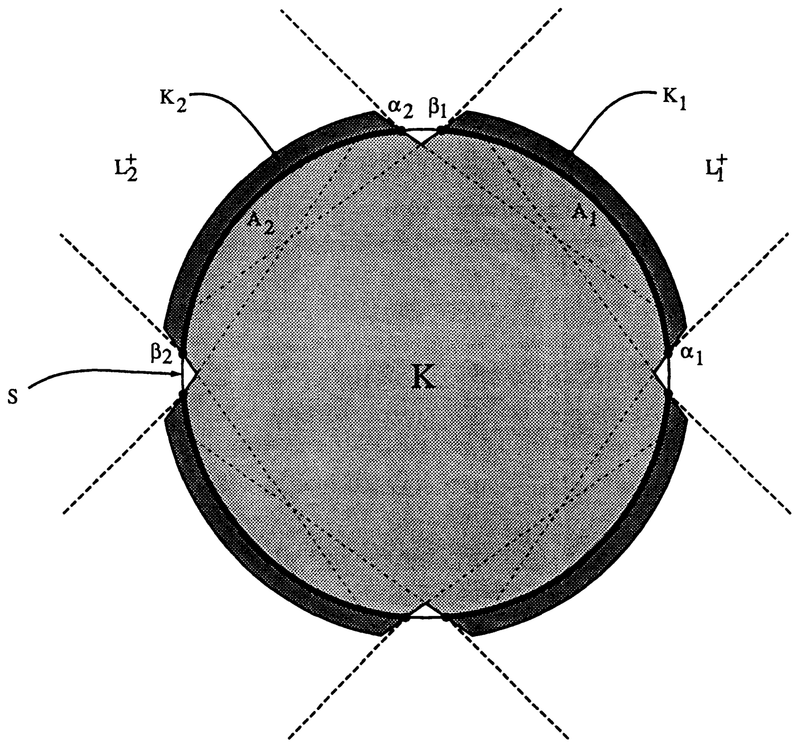

Figure 6.6. The set $K \in \mathcal{M}_G$ (shaded); $G = \mathcal{C}_2^n$, $n = m = 4$.

G-invariant polygons[6], so it suffices to establish (5.7) when K itself is a finite union of such polygons.

Again we need consider only the case where $K \cap S = \cup A_j$, a finite disjoint union of closed arcs. Define α_j, θ_j, β_j, φ_j, L_j^+, and K_j as in the proof of Theorem 6.1 but replace Figures 6.4 and 6.5 by Figures 6.6 and 6.7, respectively. If the inclusion (6.9) can be established then (5.7) again follows from Lemma 6.1 and Remark 6.1. All arguments in the previous proof continue to hold with the following two exceptions: (a) since $n \geq 4$, replace π by $\pi/2$ as the upper bound in (6.11) and (6.12); (b) since $K \in \mathcal{M}_G$ need not be convex, the verification of (6.9) in the final paragraph of the proof of Theorem 6.1 must be modified as follows.

[6]Since $G \equiv \mathcal{C}_2^n$ is irreducible if $n \geq 3$, Lemma 3.2 of Part I implies that $[C_G(x)]^\circ \neq \emptyset$ if $x \neq 0$, where $^\circ$ denotes "interior". It follows from (6.9) that for each $\lambda > 1$, $\cup\{\lambda[C_G(x)]^\circ | x \in K\}$ is an open covering of the compact set K, hence there exists a finite subcovering $\cup\{\lambda[C_G(x_i)]^\circ | i = 1, \ldots, n\}$. Then $K_\lambda \equiv \cup\{\lambda C_G(x_i) | i = 1, \ldots, n\}$ is a finite union of closed convex G-invariant polygons such that $K \subset K_\lambda \subset \lambda K$. Thus $K_\lambda \to K$ as $\lambda \downarrow 1$. [In case (ii) below, $G \equiv \mathcal{H}_2^n$ is again irreducible if $n \geq 3$ so this argument remains valid. If $n = 2$, then G is not irreducible but again $[C_G(x)]^\circ \neq \emptyset$ unless x lies in the wall of a fundamental region (see Footnote 8) in which case we define $[C_G(x)]^\circ$ to be the relative interior of $C_G(x)$.]

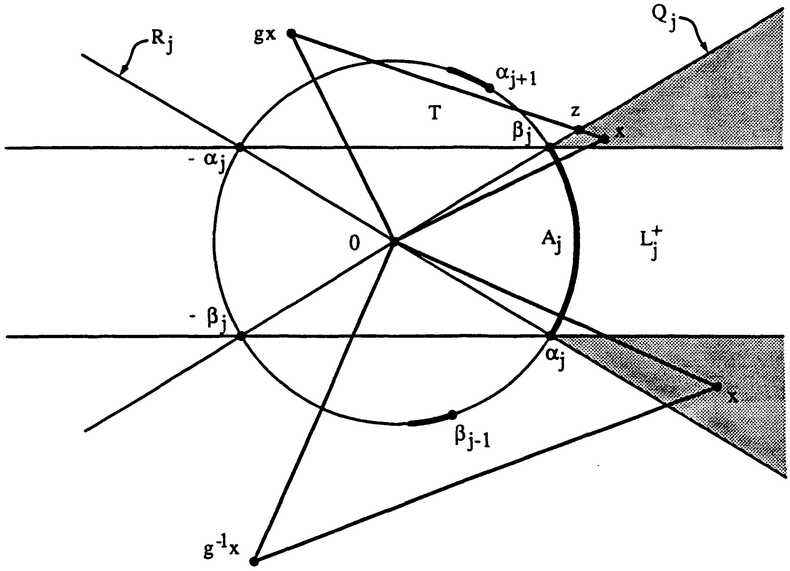

Figure 6.7.

First note that $G \equiv C_2^n = \{I, g, \ldots, g^{n-1}\}$, where g is the rotation through angle $2\pi/n$ about 0 in $\mathbb{R}^2$. Again we may assume that $\alpha_j \neq \beta_j$, i.e., that $\theta_j < \varphi_j$. If (6.9) fails, consider $x \in K_j \backslash L_j^+$. By definition of K_j, x must lie in one of the two shaded wedge-shaped regions indicated in Figure 6.7. If x lies in the upper shaded region, then its image gx must lie in the open region outside B and strictly between the infinite rays Q_j and R_j emanating from 0 and passing through β_j and $-\alpha_j$ respectively[7]. Thus the half-open line segment $[x,\ gx)$ must intersect the ray Q_j at some point z outside B. This implies that the triangle T with vertices 0, x, and gx intersects the *open* arc $(\beta_j, \alpha_{j+1}) \subset S \backslash K$. Since G acts effectively on $\mathbb{R}^2$, however, $0 \in C_G(x)$ (see Lemma 3.1 of Part I), hence $T \subset C_G(x) \subset K$ (recall that $K \in \mathcal{M}_G$), a contradiction. If it is assumed that x lies in the lower shaded region in Figure 6.7, simply replace gx by $g^{-1}x$ and $(\beta_j,\ \alpha_{j+1})$ by $(\alpha_j,\ \beta_{j-1})$ to reach a similar contradiction. Thus (6.9) is again verified.

(ii) $G = \mathcal{H}_2^n$, $n \geq 2$. Since $C_2^n \subset \mathcal{H}_2^n$ this case is covered by (i) when $n \geq 4$, but the following argument is valid for all $n \geq 2$. Note that if $F_1, \ldots, F_{2n}$

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[7]It is again essential to verify that Figure 6.7 accurately depicts the location of gx, i.e., that gx lies strictly between the rays Q_j and R_j. If we write $x = |x|\exp(i\eta)$ with $\theta_j \leq \eta \leq \varphi_j$ then $gx = |x|\exp i[\eta + (2\pi/n)]$, so it must be verified that $\varphi_j < \eta + (2\pi/n) < \theta_j + \pi$. But this follows from (6.12) with π replaced by $\pi/2$. Similarly, $g^{-1}x$ lies in the open region outside B and strictly between the infinite rays $-R_j$ and $-Q_j$.

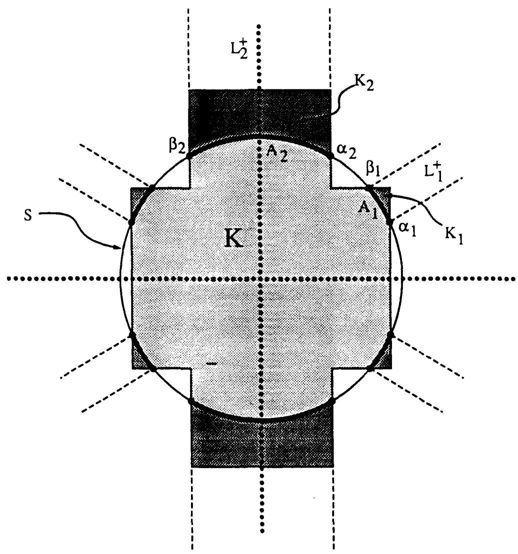

Figure 6.8. The set $K \in \mathcal{M}_G$ (shaded); $G = \mathcal{H}_2^n$, $n = 2$, $m = 6$.

are the fundamental regions[8] for the finite reflection group $G \equiv \mathcal{H}_2^n$, then $\Gamma F_1, \ldots, \Gamma F_{2n}$ are the fundamental regions for the finite reflection group $\tilde{G} \equiv \Gamma G \Gamma'$, where $\Gamma \in \mathcal{O}_2$. Thus, by means of an orthogonal change of basis we may assume that $\tilde{G} = G$.

As in (i), it must be shown that (5.7) holds for every compact $K \in \mathcal{M}_G$ and every contraction D. Again we may assume that K is a finite union of closed convex G-invariant polygons (see Footnote 6).

As before we need consider only the case where $K \cap S = \cup A_j$, a finite disjoint union of closed arcs. Define α_j, θ_j, β_j, φ_j, L_j^+, and K_j as in the proof of Theorem 6.1, but now replace Figures 6.4 and 6.5 by Figures 6.8 and 6.9, respectively[9]. To establish (5.7) it again suffices to verify (6.9). All arguments in the proof of Theorem 6.1 continue to hold (including (6.11) and (6.12) since $\mathcal{H}_2^n \supset C_2^n$) with the exception of the verification of (6.9), which must be modified as follows.

Again we may assume that $\alpha_j \neq \beta_j$. Neither α_j nor β_j can lie in the wall

[8]The reader may review the elementary geometric structure of the reflection groups $\mathcal{H}_2^n$ in Grove and Benson (1985, pp. 8-9), in particular the representation $\mathbb{R}^2 = \cup\{g\bar{F} | g \in \mathcal{H}_2^n\}$, where $\bar{F}$ is the closure of any fixed *fundamental region* F for $\mathcal{H}_2^n$. Such a region is an open convex cone in $\mathbb{R}^2$ that subtends an angle of π/n at 0 and which is oriented such that the reflections across its two boundary rays, or *walls*, generate the group $\mathcal{H}_2^n$. There are exactly $2n$ disjoint fundamental regions $F_1, \ldots, F_{2n}$, and for each $g \in \mathcal{H}_2^n$, $\{gF_1, \ldots, gF_{2n}\}$ is some permutation of $\{F_1, \ldots, F_{2n}\}$. Additional properties of finite reflections groups utilized in the present paper may be found in Chapter 4 of Grove and Benson (1985) and in Section 3 of Eaton and Perlman (1977).

[9]In Figure 6.8, $n = 2$ and the $2n \equiv 4$ fundamental regions (whose walls are indicated by heavily dotted lines) coincide with the four (open) quadrants of $\mathbb{R}^2$.

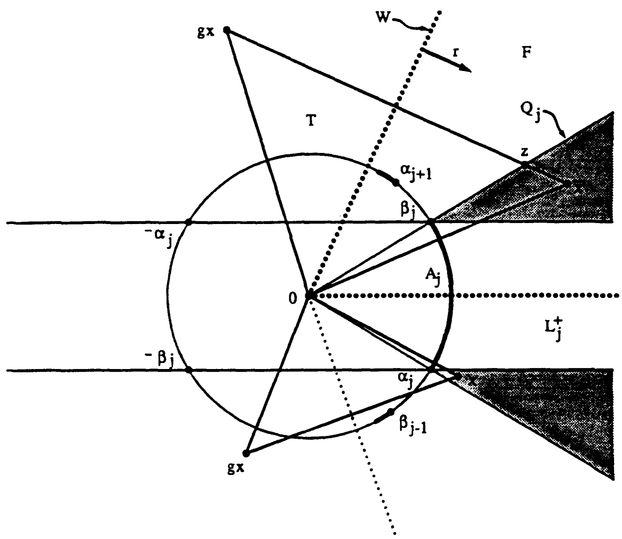

Figure 6.9.

of a fundamental region[10]. Either (a) α_j and β_j lie in the same fundamental region or (b) α_j and β_j lie in adjacent fundamental regions, for otherwise the union of the arc A_j and all its G-images would completely cover S. (Cases (a) and (b) both occur in Figure 6.8.) In case (b), α_j and β_j must be equidistant from the common wall between them (see Figure 6.9).

If (6.9) fails, consider $x \in K_j \backslash L_j^+$. By the definition of K_j, x must lie in one of the two shaded wedge-shaped regions indicated in Figure 6.9. If x lies in the upper shaded region, then

$$(6.15) \qquad (x - \beta_j)'(\beta_j - \alpha_j) > 0$$

and x lies in the same fundamental region (call it F) as β_j. Let W denote the first wall of F encountered when traversing S in a counterclockwise direction starting at β_j and let r denote the unit vector normal to W that points into F. Define $g = I - 2rr'$, i.e., g is the reflection across the wall W, hence $g \in G$. Then

$$(6.16) \qquad \begin{aligned}(gx - \beta_j)'(\beta_j - \alpha_j) &= \\ &(x - \beta_j)'(\beta_j - \alpha_j) - 2(r'x)(r'\beta_j - r'\alpha_j)) > 0\end{aligned}$$

[10]Suppose that α_j lies in the wall of some fundamental region. Since K is G-invariant, the reflection of the closed arc A_j across that wall is contained in K, hence the closed arc consisting of the union of A_j and its reflection is contained in $K \cap S$. But β_j lies in the interior of this closed arc, which contradicts the fact that β_j is not an interior point of $K \cap S$. Similarly, β_j cannot lie in the wall of a fundamental region.

by (6.15) and the two inequalities $r'\beta_j \leq r'\alpha_j$, $r'x > 0$.[11] Thus, by (6.15) and (6.16) both x and its reflected image gx lie strictly on the same side of the strip L_j, and both lie outside B. Because $gx \in gF$ which is disjoint from F, gx cannot lie in the upper shaded region that contains x (see Figure 6.9). Therefore the half-open line segment $[x, gx)$ must intersect the ray Q_j at some point z outside B. As before, this implies that the triangle T with vertices 0, x, and gx intersects the *open* arc $(\beta_j, \alpha_{j+1}) \subset S\backslash K$. Since G acts effectively on $\mathbb{R}^2$, however, $0 \in C_G(x)$ (by Lemma 3.1 of Part I), hence $T \subset C_G(x) \subset K$ (since $K \in \mathcal{M}_G$), a contradiction. If x lies in the lower shaded region in Figure 6.9, replace F by the fundamental region containing α_j and replace (β_j, α_{j+1}) by (α_j, β_{j-1}) to reach a similar contradiction. Thus (6.9) is again verified. □

REMARK 6.5 Examples 3.2 and 3.3 in Part I show that the assumption that Σ_1 is G-invariant cannot be discarded in Theorem 6.1. Example 3.4 shows that the conclusion of Theorem 6.2 is false if $G = \mathcal{C}_2^2 \equiv \{\pm I\}$, in which case $\mathcal{M}_G \equiv \mathcal{M}_1$ is the class of centrally symmetric sets that are star-shaped with respect to the origin in $\mathbb{R}^2$. (This counterexample easily may be extended to $G \equiv \{\pm I\}$ acting on $\mathbb{R}^p$ with $p \geq 3$). □

REMARK 6.6 If $G = \mathcal{C}_2^3$ then the crucial inclusion (6.9) fails for some (but not all) sets $K \in \mathcal{M}_G \backslash \mathcal{C}_G$, hence the above proof fails to establish the inequality (5.7) for such sets. It is uncertain, however, whether or not (5.7) (and hence the conclusion of Theorem 6.2) is true for such sets. To see this, consider the three sets K in Figures 6.10-6.12. For the first two sets (6.9) does hold so (5.7) is true, while for the third set (6.9) *fails* but (5.7) is uncertain. We conjecture that (5.7) is true for *every* $K \in \mathcal{M}_G$, hence that Theorem 6.2 is valid also for $G = \mathcal{C}_2^3$. If this is true then Theorem 6.2 *would be valid for every effective subgroup G of* $\mathcal{O}_2$ *except* $\{\pm I\}$. With somewhat less confidence we conjecture that when $p \geq 3$, Theorem 6.2 *is valid for every effective subgroup G of* $\mathcal{O}_p$ *except those G for which there exists a G-invariant subspace* $V \subseteq \mathbb{R}^p$ *of dimension* ≥ 2 *such that the restriction of the action of G to V is* $\{\pm I\}$. As with Theorem 6.1, however, the method of proof used above to establish Theorem 6.2 in the bivariate case does not extend in any obvious way to the multivariate case $p \geq 3$ (recall Remark 6.3). □

[11] The scalar product $r'v$ is the (signed) distance from the vector v to the wall W. Because $\beta_j \in F$ and since the angle subtended by F at 0 is $\leq \pi/2$, β_j is closer to W than α_j (consider the cases (a) and (b) separately), so the first inequality holds. The second is immediate since $x \in F$.

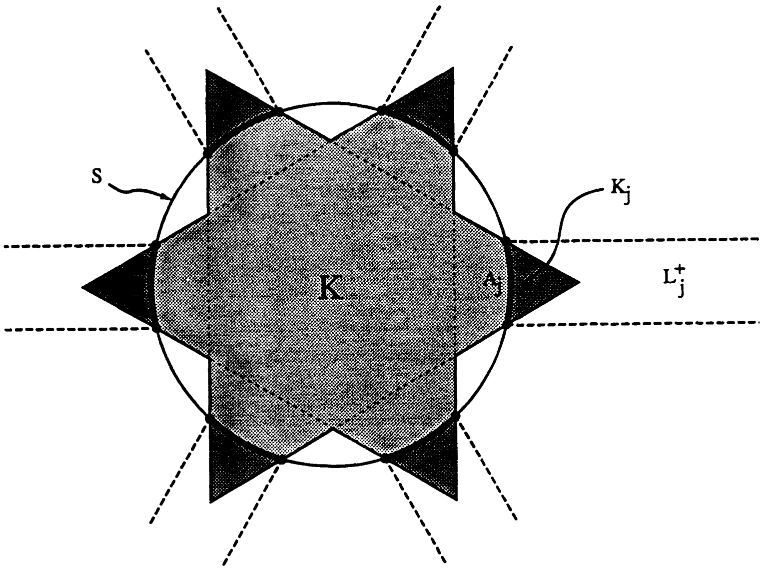

Figure 6.10. $K \in \mathcal{M}_G \backslash \mathcal{C}_G$ (shaded), $G = \mathcal{C}_2^3$; (6.9) holds, (5.7) true.

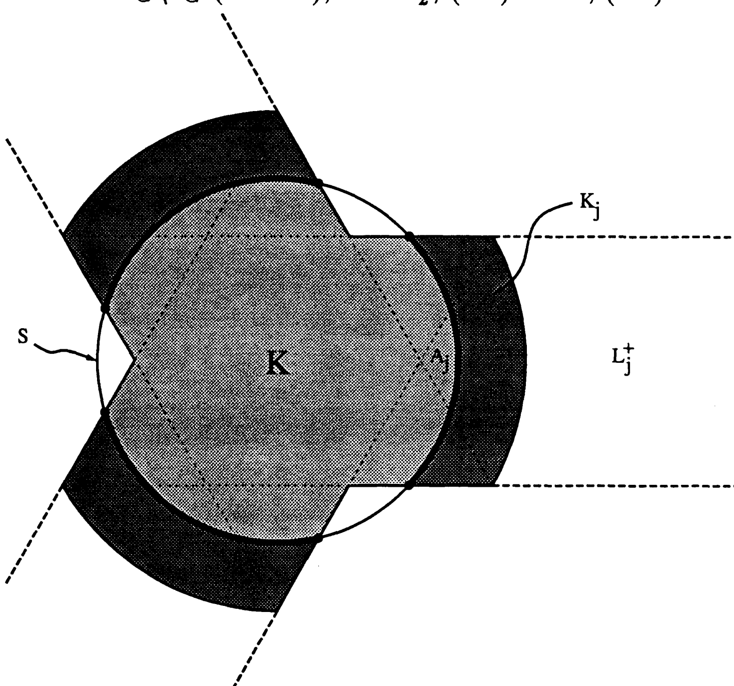

Figure 6.11. $K \in \mathcal{M}_G \backslash \mathcal{C}_G$ (shaded), $G = \mathcal{C}_2^3$; (6.9) holds, (5.7) true.

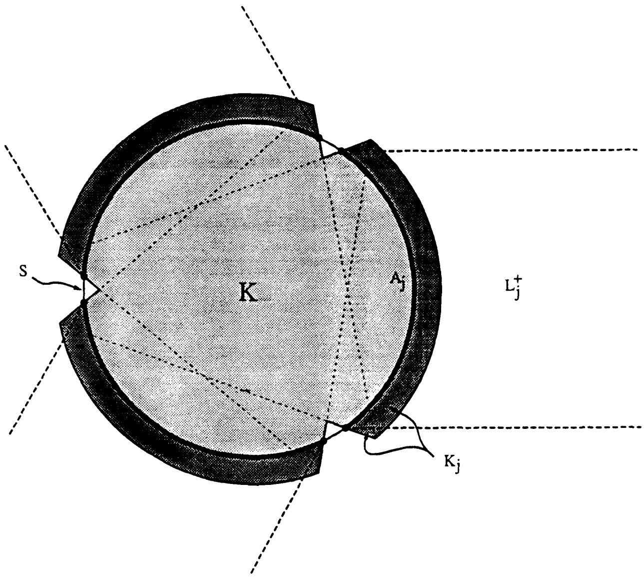

Figure 6.12. $K \in \mathcal{M}_G \backslash \mathcal{C}_G$ (shaded), $G = \mathcal{C}_2^3$; (6.9) fails, (5.7) uncertain.

7. A Sharper Inequality

We return to the general case $p \geq 2$ and let B, S, ν, and D be as defined in Section 5. Theorem 5.1 implies that for every (not necessarily closed) $K \in \mathcal{C}_1$,

$$\nu(\bar{K}) \equiv \nu(K^\circ) + \nu(\partial K) \geq \nu(D\bar{K}), \tag{7.1}$$

where $\bar{K}$, K°, and ∂K denote the closure, interior, and boundary of K, respectively (see (5.7)). Fefferman, Jodeit, and Perlman (1972, Section 3) sharpened this inequality by showing that if $D \neq I$ then (7.1) remains valid with the term $\nu(\partial K)$ deleted, even though $\nu(\partial K)$ may be positive and/or $\nu(K^\circ)$ may be 0. Therefore, when $C \in \mathcal{C}_1$ the contribution of the boundary of C plays no role in the concentration inequality (5.0) for elliptically contoured distributions even though such distributions need not be absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^p$, hence may assign nonzero probability to the boundary of K.

In this section we extend this sharpened result from $\mathcal{C}_1$ to the classes $\mathcal{C}_G$ and $\mathcal{M}_G$ in the bivariate case and show further that if Theorems 6.1 and 6.2 can be extended from $\mathbb{R}^2$ to $\mathbb{R}^p$ for $p \geq 3$ then for many groups G the sharper forms of their concentration inequalities will follow as corollaries. This requires a non-trivial modification of the argument of Fefferman, Jodeit, and Perlman (1972, Theorem 2), again because the transformation $K \to DK$ need not preserve the $\tilde{G}$-invariance of K unless $G = \tilde{G} = \{\pm I\}$ (see the paragraph containing (5.12)).

The following four lemmas contain the technical core of the argument.

Recall that $\overline{AC} = A\bar{C}$, $(AC)^\circ = AC^\circ$, and $\partial(AC) = A(\partial C)$ for any set $C \subseteq \mathbb{R}^p$ and any nonsingular linear transformation $A: \mathbb{R}^p \to \mathbb{R}^p$. If $\{C_t\}$ is a family of subsets of $\mathbb{R}^p$ indexed by a real parameter $t \geq 0$, we write $C_t \uparrow C$ to indicate pointwise monotone convergence of the indicator function of C_t to that of C and $C_t \to C$ a.e. $[\nu]$ to indicate pointwise convergence of the indicator function of $C_t \cap S$ to that of $C \cap S$ a.e. $[\nu]$. If D is a contraction, note that $D_t \equiv D + t(I - D)$ is also a contraction for every $0 < t < 1$ and that $D_t \downarrow D$ as $t \downarrow 0$.

LEMMA 7.1 *Let $\mathcal{C}$ be a family of subsets $C \subseteq \mathbb{R}^p$ with the following four properties:*

(i) $tC \in \mathcal{C}\ \forall t > 0$.

(ii) If $C^\circ \neq \emptyset$ then $t\bar{C} \uparrow C^\circ$ a.e. $[\nu]$ as $t \uparrow 1$.

(iii) If $C^\circ = \emptyset$ then $\nu(D\bar{C}) = 0$ for every contraction $D \neq I$.

(iv) If $C^\circ \neq \emptyset$ then for each contraction $D \neq I$, $D_tC^\circ \to D\bar{C}$ a.e. $[\nu]$ as $t \downarrow 0$.

Then the following three conditions are equivalent:

(a) $\nu(\bar{C}) \geq \nu(D\bar{C})\ \forall C \in \mathcal{C}$ and $\forall$ contractions D.

(b) $\nu(C^\circ) \geq \nu(DC^\circ)\ \forall C \in \mathcal{C}$ and $\forall$ contractions D.

(c) $\nu(C^\circ) \geq \nu(D\bar{C})\ \forall C \in \mathcal{C}$ and $\forall$ contractions $D \neq I$.

If in addition, (v): $C \in \mathcal{C} \Rightarrow \bar{C} \in \mathcal{C}$, then (a) $\Leftrightarrow$ (a′): $\nu(C) \geq \nu(DC)\ \forall$ closed $C \in \mathcal{C}$ and $\forall$ contractions D.

PROOF The implications $(c) \Rightarrow (a)$ and $(c) \Rightarrow (b)$ are immediate.

(a) $\Rightarrow$ (b): If $C^\circ = \emptyset$ then (b) is trivial. If $C^\circ \neq \emptyset$ then $\nu(t\bar{C}) \geq \nu(tD\bar{C})$ by (i) and (a). Now let $t \uparrow 1$ and apply (ii) to obtain (b).

(b) $\Rightarrow$ (c): If $C^\circ = \emptyset$ then (c) is trivial by (iii). If $C^\circ \neq \emptyset$ then $\nu(C^\circ) \geq \nu(D_tC^\circ) \to \nu(D\bar{C})$ as $t \downarrow 0$ by (b) and (iv).

(a) $\Leftrightarrow$ (a′): obviously (a) $\Rightarrow$ (a′); under assumption (v), clearly (a′) $\Rightarrow$ (a). □

LEMMA 7.2 *For any compact subgroup $G \subseteq \mathcal{O}_p$ that acts effectively on $\mathbb{R}^p$, the class $\mathcal{C}_G$ satisfies conditions (i)-(v) of Lemma 7.1.*

PROOF Suppose that $C \in \mathcal{C}_G$. The convexity of C implies the convexity of $\bar{C}$ and tC (cf. Eggleston (1966), p. 9), while the G-invariance of C implies the G-invariance of $\bar{C}$ and tC, so conditions (i) and (v) are satisfied. If $C^\circ = \emptyset$ then C convex $\Rightarrow$ $\bar{C}$ lies in a proper subspace of $\mathbb{R}^p$, hence $D\bar{C}$ is ν-null for every D, so (iii) holds.

To verify (iii), assume that $C^\circ \neq \emptyset$. Since G is effective, $0 \in C$ [12], hence C is star-shaped with respect to 0. Thus tC, and therefore $t\bar{C}$, increases as

[12] For any $x \in C$ define $x_G = \int_G gx\, d\mu(g)$, where μ is the Haar probability measure on G. Clearly $gx_G = x_G\ \forall g \in G$, so $x_G = 0$ as G is effective. But $x_G \in C_G(x) \subseteq C$ since $C \in \mathcal{C}_G$, so $0 \in C$. (Note that this also provides a proof of Lemma 3.1.)

$t \uparrow 1$. Also, if $x \in C^\circ$ then $\tau x \in C^\circ$ for some $\tau > 1$, hence $x \in \tau^{-1}C^\circ \subseteq \tau^{-1}\bar{C} \subseteq \cup\{t\bar{C}|t \uparrow 1\}$, so $C^\circ \subseteq \cup\{t\bar{C}|t \uparrow 1\}$. Next we show that $0 \in C^\circ$. If not then $0 \in \partial C$, so the convex set C must be supported at 0 by some $(p-1)$-dimensional subspace, i.e., $C \subseteq \{y \in \mathbb{R}^p | a'y \geq 0\}$ for some $a \neq 0$. Thus, since C is G-invariant, $a'(gx) \geq 0$ $\forall x \in C$ and $\forall g \in G$, so $a'x_G = \int_G a'(gx)d\mu(g) \geq 0$ (see Footnote 12). But $x_G = 0$ as G is effective, hence $a'(gx) = 0$ $\forall x \in C$ and $\forall g \in G$. In particular $a'x = 0$ $\forall x \in C$, contradicting the assumption that $C^\circ \neq \emptyset$. Thus it must hold that $0 \in C^\circ$. Therefore $t\bar{C} \equiv t\bar{C} + (1-t)0 \subset C^\circ$ if $0 < t < 1$ (cf. Eggleston (1966, Corollary 2, p. 10)), so $\cup\{t\bar{C}|t \uparrow 1\} \subseteq C^\circ$ and (ii) is verified.

To verify (iv), assume that $C^\circ \neq \emptyset$ and $D \neq I$. Let χ_t and χ denote the indicator functions of the sets D_tC° and $D\bar{C}$, respectively. If $x \in DC^\circ$ then $x \in D_tC^\circ$ for all t near 0, while if $x \notin D\bar{C}$ then $x \notin D_t\bar{C}$ for all t near 0, so in both cases $\chi_t(x) \to \chi(x)$ as $t \downarrow 0$. If $x \in \partial(DC)(\subseteq D\bar{C})$ then either $x \in D_tC^\circ$ for all t near 0, in which case again $\chi_t(x) \to \chi(x)$ as $t \downarrow 0$, or else there exists a sequence $t_n \downarrow 0$ such that $x \notin D_{t_n}C^\circ$ for every n, in which case $\chi_t(x) \not\to \chi(x)$ as $t \downarrow 0$. Therefore, in order to complete the verification of (iv) it must be shown that $\nu(\Delta) = 0$, where

$$\Delta = \{x \in \gamma(DC) |\ \exists t_n \downarrow 0 \text{ such that } x \notin D_{t_n}C^\circ \text{ for every } n\}. \tag{7.2}$$

Since $\nu(\{x|x_i = 0 \text{ for some } i = 1,\ldots,p\}) = 0$, it suffices to show that $\nu(\Delta \cap \{x|x_i \neq 0,\ i = 1,\ldots,p\}) = 0$. We shall show that $\nu(\Delta \cap \mathbb{R}^+) = 0$ where $\mathbb{R}^+ = \{x \in \mathbb{R}^p | x_i > 0, i = 1,\ldots,p\}$; the other $2^p - 1$ cases follow similarly. Set $K = DC$ and $x_n = D(D_{t_n})^{-1}x$ in (7.2). Since $0 < x_n \uparrow x$ when $x \in \mathbb{R}^+$ (note that $x_n \neq x$ since $D \neq I$) we have that

$$\begin{aligned} \Delta \cap \mathbb{R}^+ \subseteq \{x \in \partial K \cap \mathbb{R}^+ \mid \exists x_n \uparrow x \text{ such that} \\ x_n \notin K^\circ \text{ for every } n\} \equiv \Delta^+ \end{aligned} \tag{7.3}$$

and shall show that $\nu(\Delta^+) = 0$.

Let $Q \equiv \{x \equiv (x_1,\ldots,x_p) |\ 0 \leq x_i \leq 1,\ i = 1,\ldots,p\}$ denote the closed unit cube in $\mathbb{R}^p$, let θ_i be the unit vector with i-th component 1, and for $\epsilon > 0$ let $L_i(x,\epsilon) \equiv [x - \epsilon\theta_i, x)$ denote the half-open line segment connecting $x - \epsilon\theta_i$ and x. For $\epsilon > 0$ define $Q(x,\epsilon) = (x - \epsilon Q)\backslash\{x\}$ and $L(x,\epsilon) = \cup\{L_i(x,\epsilon)|i = 1,\ldots,p\}$; then

$$L(x,\epsilon) \subset Q(x,\epsilon) \subset \text{ convex hull}[L(x,\epsilon\sqrt{p})]. \tag{7.4}$$

Fix $x \in \Delta^+$. By (7.3), $Q(x,\epsilon) \not\subset K^\circ$ $\forall \epsilon > 0$, hence by (7.4) and the convexity of K°, $L(x,\epsilon\sqrt{p}) \not\subset K^\circ$ $\forall \epsilon > 0$. Therefore, there exist $i \in \{1,\ldots,p\}$ and a sequence $\{\delta_n\} \downarrow 0$ such that $x - \delta_n\theta_i \notin K^\circ$ $\forall n$, hence there exists $i \in \{1,\ldots,p\}$ such that $L_i(x,x_i) \cap K^\circ = \emptyset$ (since $x \in \partial K$ and $K^\circ \notin \emptyset$ – apply Eggleston (1966, Corollary 2, p. 10)). Thus

$$\Delta^+ \subseteq \cup\{\Delta_i | i = 1,\ldots,p\}, \tag{7.5}$$

where $\triangle_i = \{x \in \partial K \cap \mathbb{R}^+ | L_i(x, x_i) \cap K^\circ = \emptyset\}$. (In fact, equality holds in (7.5).) In order to show that $\nu(\triangle^+) = 0$, it therefore suffices to show that $\nu(\triangle_i) = 0$, $i = 1, \ldots, p$.

The remainder of the proof now parallels the treatment of cases (iii) and (iv) in the proof of Theorem 2, Fefferman, Jodeit, and Perlman (1972). First consider $\triangle_1$. Since $\bar{K}$ is convex and $x \in \partial K$, $\triangle_1 = A \cup B$ where

$$A = \{x \in \partial K \cap \mathbb{R}^+ | \ L_1(x, \epsilon) \subset \partial K \text{ for some } \epsilon > 0\}$$

$$B = \{x \in \partial K \cap \mathbb{R}^+ | \ L_1(x, x_1) \cap \bar{K} = \emptyset\}$$

(note that $A \cap B = \emptyset$). Because the projection of A onto $\{x | x_1 = 0\}$ lies in the boundary of the projection of $\bar{K}$ onto $H_0 \equiv \{x | x_1 = 0\}$[13], which boundary has $(p-1)$-dimensional Lebesgue measure 0, it follows that $\nu(A) = 0$. Finally, B is contained in the graph of a positive convex function (the "lower boundary" of $\bar{K} \cap \mathbb{R}^+$) so $\nu(B) = 0$ (apply the Lemma following Theorem 2 in Fefferman, Jodeit, and Perlman (1972)). Similarly, $\nu(\triangle_i) = 0$ for $i = 2, \ldots, p$. □

By (6.13), $C \in \mathcal{M}_G$ iff C is an (arbitrary) union of sets in $\mathcal{C}_G$. Since the boundary of such a set may be irregular, in order to extend Lemma 7.2 to $C \in \mathcal{M}_G$ it is necessary to impose an additional smoothness assumption on C. One such condition, which covers most sets occurring in applications, is the following: define $\hat{\mathcal{M}}_G$ to be the collection of all $C \in \mathcal{M}_G$ such that $\partial C = \partial(\bar{C}) = \cup M_j$, a finite or countable disjoint union of smooth $(p-1)$-dimensional manifolds M_j (hence ∂C is piecewise smooth). Furthermore, it is necessary to impose a stronger assumption on the group G itself.

LEMMA 7.3 *For any compact subgroup $G \subseteq \mathcal{O}_p$ that acts irreducibly on $\mathbb{R}^p$, the class $\hat{\mathcal{M}}_G$ satisfies conditions (i)-(v) of Lemma 7.1.*

PROOF Suppose that $C \in \hat{\mathcal{M}}_G$. By (6.13), C satisfies (i) since $C_G(tx) = tC_G(x)$. To verify (v), consider $x \in \bar{C}$. Then there exists a sequence $\{x_n\} \subset C$ such that $x_n \to x$. Since

$$C_G(x) = \{\alpha_1 g_1 x + \cdots + \alpha_k g_k x | k \geq 1, \ g_i \in G, \ \alpha_i \geq 0, \ \textstyle\sum \alpha_i = 1\}$$

and $\|gx_n - gx\| = \|x_n - x\|$ for each $g \in G$, it follows that $\delta(C_G(x_n), C_G(x)) \leq \|x_n - x\| \to 0$, where δ denotes the Hausdorff metric (cf. Valentine (1976),

[13]If not, then there would exist $x \in A$ such that the projection $x - x_1\theta_1$ of x onto H_0 lies in the interior of the projection of $\bar{K}$ onto H_0. This would imply that there exist $y \in \bar{K}$ and $\delta > 0$ such that the projection $y - y_1\theta_1$ of y onto H_0 satisfies $y - y_1\theta_1 = (1+\delta)(x - x_1\theta_1)$. Also, since $x \in A$ there would exist $\epsilon > 0$ such that the closed triangle with vertices x, $x - \epsilon\theta_1$, and y is contained in $\bar{K}$. But since $0 \in K^\circ$, this would imply that the open line segment $(x - \epsilon\theta_1, x) \subset K^\circ$, contradicting the fact that $x \in A$.

p. 36). But $C_G(x_n) \subseteq C$ for every n, hence $C_G(x) \subseteq \bar{C}$. Thus $\bar{C} \in \mathcal{M}_G$ and (v) is satisfied.

Since G is irreducible, $C^\circ = \emptyset$ implies that $C = \{0\}$ (apply Lemma 3.2), so (iii) is trivial. To verify (ii), assume that $C^\circ \neq \emptyset$. It follows as in the second paragraph of the proof of Lemma 7.2 that $t\bar{C}$ increases as $t \uparrow 1$ and $C^\circ \subseteq \cup\{t\bar{C} | t \uparrow 1\}$. To show that $\cup\{t\bar{C} | t \uparrow 1\} \subseteq C^\circ$ it suffices to show that $t\bar{C} \subseteq C^\circ$ if $0 < t < 1$. For $x \in \bar{C}\backslash\{0\}$ choose a sequence $\{x_n\} \subset C$ such that $x_n \to x$; then as above, $\delta(C_G(x_n), C_G(x)) \to 0$. Because $C_G(x_n)$ and $C_G(x)$ are bounded convex sets with non-empty interiors (since G is irreducible), it follows that $\delta([C_G(x_n)]^\circ, [C_G(x)]^\circ) = \delta(C_G(x_n), C_G(x)) \to 0$. But $tx \in [C_G(x)]^\circ$ because $0 \in [C_G(x)]^\circ$ (since G is irreducible), so $\exists\ n$ such that[14] $tx \in [C_G(x_n)]^\circ \subseteq C^\circ$, as claimed.

To verify (iv), as in the proof of Lemma 7.2 it suffices to show that $\nu(\Delta^+) = 0$, where Δ^+ is given by (7.3) and $K = DC$. Since $\partial K = D(\partial C) = \cup(DM_j) \equiv \cup N_j$, it is enough to to show that $\nu(\Delta_j^+) = 0$, where Δ_j^+ is defined as Δ^+ in (7.3) but with ∂K replaced by the relative interior of N_j, a smooth $(p-1)$-dimensional open manifold. Since for every $x \in \Delta_j^+$ it holds that $Q(x, \epsilon) \not\subseteq K^\circ\ \forall \epsilon > 0$, it can be shown that

$$\Delta_j^+ \subseteq \{x \in [\text{rel int}(N_j)] \cap \mathbb{R}^+ |\ N(x) \notin \mathbb{R}^+\},$$

where $N(x)$ is the outward normal vector to N_j at x. But $S(x) \in \mathbb{R}^+$ for each $x \in \mathbb{R}^+$, where $S(x)$ denotes the outward normal vector to the sphere S at x. Therefore the sphere S and the manifold Δ_j^+ intersect *transversely*, so their intersection must be a manifold of dimension $\leq p-2$ (cf. Guilleman and Pollack (1974), Theorem, p. 30; Do Carmo (1976), Ex. 17, p. 90), hence $\nu(\Delta_j^+) = 0$ as required. □

LEMMA 7.4 *Suppose that $G = G_1 \times \cdots \times G_t$, a direct product of compact irreducible groups acting on $\mathbb{R}_1 \times \cdots \times \mathbb{R}_t$, where $\sum \dim(\mathbb{R}_i) = p$. Then the class $\hat{\mathcal{M}}_G$ satisfies conditions (i)-(v) of Lemma 7.1.*

PROOF The first and third paragraphs of the proof of Lemma 7.3 carry over to this case without change, while the second paragraph must be modified as follows:

For each $i = 1, \ldots, t$ define $\tilde{\mathbb{R}}_i = \{0\} \times \cdots \times \{0\} \times \mathbb{R}_i \times \{0\} \times \cdots \times \{0\}$ and note that $\nu(\tilde{\mathbb{R}}_i) = 0$. Since each G_i acts irreducibly on $\mathbb{R}_i$, $C^\circ = \emptyset$ implies that $\bar{C} \subseteq \cup\tilde{\mathbb{R}}_i$, so (iii) is immediate. To verify (ii), assume that $C^\circ \neq \emptyset$. As in the second paragraph of the proof of Lemma 7.2, $t\bar{C}$ increases as $t \uparrow 1$ and

[14]This requires the following fact: if A_n, A are nonempty, convex, open sets in $\mathbb{R}^p$ such that $\delta(A_n, A) \to 0$, then $A \subseteq \cup A_n$. (For $y \in A$, choose n such that $\delta(A_n, A) < \|y - \partial A\|$. If $y \notin A_n$ then $\exists$ a hyperplane H that separates A_n and y. This would imply that $\exists z \in A \cap N$, where N is the line normal to H through y, such that $\|z - A_n\| > \|z - y\| \geq \delta(A_n, A)$, contradicting the definition of $\delta(A_n, A)$. Therefore $y \in A_n$.)

$C^\circ \subseteq \cup\{t\bar{C}|t \uparrow 1\}$. To show that $\nu(\cup\{t\bar{C}|t \uparrow 1\}\backslash C^\circ) = 0$ it suffices to show that $t\bar{C} \subseteq C^\circ \cup (\cup\tilde{\mathbb{R}}_i)$ if $0 < t < 1$. For $x \in \bar{C}\backslash(\cup\tilde{\mathbb{R}}_i)$ choose a sequence $\{x_n\} \subset C$ such that $x_n \to x$; then as above, $\delta(C_G(x_n), C_G(x)) \to 0$. Because $C_G(x_n)$ and $C_G(x)$ are bounded convex sets with non-empty interiors (since $x_n, x \notin \cup\tilde{\mathbb{R}}_i$), it follows that $\delta([C_G(x_n)]^\circ, [C_G(x)]^\circ) = \delta(C_G(x_n), C_G(x)) \to 0$. But $tx \in [C_G(x)]^\circ$ because $0 \in [C_G(x)]^\circ$ (since $x \notin \cup\tilde{\mathbb{R}}_i$), so $\exists\ n$ such that (see Footnote 14) $tx \in [C_G(x_n)]^\circ \subseteq C^\circ$, as claimed. □

When $p = 2$, Lemma 7.2 applies to all compact subgroups $G \subseteq \mathcal{O}_2$ except $\mathcal{C}_2^1$ and $\mathcal{H}_2^1$, Lemma 7.3 applies to all compact subgroups of $\mathcal{O}_2$ except $\mathcal{C}_2^1$, $\mathcal{H}_2^1$, $\mathcal{C}_2^2$, and $\mathcal{H}_2^2$, while Lemma 7.2 applies to $\mathcal{H}_2^2$. Thus, from the equivalence of (a′) and (c) in Lemma 7.1 we obtain the following sharpened versions of Theorems 6.1 and 6.2:

THEOREM 7.1 *Suppose that* $X \sim EC_2(\Sigma)$. *If* $C \in \mathcal{C}_G$, *then* $\Sigma_1 \leq \Sigma_2$, $\Sigma_1 \neq \Sigma_2 \Rightarrow P_{\Sigma_1}(C^\circ) \geq P_{\Sigma_2}(\bar{C})$ *provided that* Σ_1 *is* G*-invariant and* G *acts effectively on* $\mathbb{R}^2$ *(i.e.,* $G \neq \mathcal{C}_2^1$ *or* $\mathcal{H}_2^1$*).*

THEOREM 7.2 *Suppose that* $X \sim EC_2(\Sigma)$. *If* $C \in \hat{\mathcal{M}}_G$, *then* $\Sigma_1 \leq \Sigma_2$, $\Sigma_1 \neq \Sigma_2 \Rightarrow P_{\Sigma_1}(C^\circ) \geq P_{\Sigma_2}(\bar{C})$ *provided that* Σ_1 *is* G*-invariant and* G *is irreducible or is the direct product of irreducible compact groups (i.e.,* $G \neq \mathcal{C}_2^1$, $\mathcal{H}_2^1$, *or* $\mathcal{C}_2^2$*), but also* $G \neq \mathcal{C}_2^3$.

Finally, Fefferman, Jodeit, and Perlman (1972, p. 118) presented several sufficient conditions for the *strict* inequality $\nu(C^\circ) > \nu(D\bar{C})$ to hold when $D \neq I$ and $C \in \mathcal{C}_1$. Their discussion remains valid when $C \in \mathcal{C}_G$ and G acts effectively, and when $C \in \hat{\mathcal{M}}_G$ and G is irreducible or is the direct product of irreducible compact groups. (When $C \in \hat{\mathcal{M}}_G$, their argument on p. 118 showing that $d\bar{C} \subseteq C^\circ$ must be replaced by our arguments in the proofs of Lemmas 7.3 and 7.4 showing that $t\bar{C} \subseteq C^\circ$ and $t\bar{C} \subseteq C^\circ \cup (\cup\tilde{\mathbb{R}}_i)$, respectively.)

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