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INEQUALITIES FOR THE PARAMETERS $\lambda(F)$,$\mu(F)$ WITH APPLICATIONS IN NONPARAMETRIC STATISTICS[1]

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The parameter $\lambda(F) = P(X_1 < X_2+X_3-X_4; X_1 < X_5+X_6-X_7)$, where $X_1,\ldots,X_7$ are independent and identically distributed (iid) according to a continous distribution F, was first considered by Lehmann (1964) in the context of certain nonparametric methods for the two-way layout. The parameter $\mu(F) = P(X_1 < X_2; X_1 < X_3 + X_4 - X_5)$ was first studied by Hollander (1966), also in the context of nonparametric techniques for the two-way layout. The best known bounds on these probabilities are

$$.28254 \approx 89/315 \leq \lambda(F) \leq 7/24 \approx .29167,$$

and

$$3/10 \leq \mu(F) \leq (\sqrt{2}+6)/24 \approx .30893.$$

The upper bound on $\lambda(F)$ is due to Lehmann (1964), the lower bound on $\lambda(F)$ to Spurrier (1991), the upper bound on $\mu(F)$ to Hollander (1967), and the lower bound on $\mu(F)$ to Spurrier (1991). We briefly review the development of these bounds and then present some new applications motivated by the recent bounds due to Spurrier. The applications include studying the extent to which the new bounds can improve large sample approximations to certain nonparametric test statistics and providing tighter upper and lower bounds on certain correlation coefficients involving these parameters.

1. Introduction

Consider the two-way layout with one observation per cell. Let

$$(1.1)\quad X_{ij} = \mu + b_i + \theta_j + e_{ij}, i = 1,\ldots,n, j = 1,\ldots,k \ (\Sigma b_i = \Sigma\theta_j = 0)$$

where the θ's are the parameters of interest, the b's are the nuisance parameters, and the e's are iid according to a common continuous distribution F. Let $Y_{uv}^{(i)} = |X_{iu} - X_{iv}|$ and $R_{uv}^{(i)}$ = rank of $Y_{uv}^{(i)}$ in the ranking from

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least to greatest of $\{Y_{uv}^{(i)}\}_{i=1}^{n}$. The Wilcoxon signed-rank statistic between treatments u and v is

$$T_{uv} = \sum_{i=1}^{n} R_{uv}^{(i)} \Psi_{uv}^{(i)}$$

where

$$\Psi_{uv}^{(i)} = \begin{cases} 1 & \text{if } X_{iu} < X_{iv} \\ 0 & \text{otherwise} \end{cases}$$

Hollander (1966) showed that under $H_0 : \theta_1 = \theta_2 = \cdots = \theta_k$, the null correlation coefficient $\rho_0^n(F)$ between T_{uv} and T_{uw} ($u \neq v$, $u \neq w$) is given by

$$\begin{aligned} \rho_0^n(F) &= [(24\lambda(F) - 6)n^2 + (48\mu(F) - 72\lambda(F) + 7)n \\ &\quad + (48\lambda(F) - 48\mu(F) + 1)][(n+1)(2n+1)]^{-1} \end{aligned} \tag{1.2}$$

where

$$\mu(F) = P(X_1 < X_2; X_1 < X_3 + X_4 - X_5) \tag{1.3}$$

and

$$\lambda(F) = P(X_1 < X_2 + X_3 - X_4; X_1 < X_5 + X_6 - X_7) \tag{1.4}$$

where $X_1, \ldots, X_7$ are iid according to F.

To our knowledge this was the first appearance of the parameter $\mu(F)$. Lehmann (1964) had introduced $\lambda(F)$ earlier in a related context. Lehmann considered

$$Y_{st} = \text{med}[\frac{1}{2}(X_{is} - X_{it} + X_{js} - X_{jt})]$$

where the median is over all $i \leq j$, as an estimator of $\theta_s - \theta_t$. Let G denote the common distribution of the difference between two e's. Then if G has a density g satisfying the regularity conditions of Lemma 3(a) of Hodges and Lehmann (1961), Lehmann showed that the joint limiting distribution of the Y_{st}'s is the $\binom{k}{2}$-variate normal distribution with zero mean and covariance matrix $\Sigma^* = (\sigma^*_{st,uv})$ where the variances are given by

$$\sigma^*_{st,st} = 1/12(\int g^2(x)dx)^2 \quad \text{all } s, t \tag{1.5}$$

and the covariances by

$$\begin{aligned} \sigma^*_{st,uv} &= 0 && \text{if } s,t,u,v \text{ are distinct} \\ &= \{\lambda(F) - 1/4\}/\{\int g^2(x)dx\}^2 && \text{if } s = u \text{ or } t = v \\ &= \{1/4 - \lambda(F)\}/\{\int g^2(x)dx\}^2 && \text{if } s = v \text{ or } t = u \,. \end{aligned} \tag{1.6}$$

The parameters $\mu(F)$, $\lambda(F)$ try very hard to be distribution–free but just don't make it. To get a feel for these parameters consider first

$$\alpha(F) = P(X_1 < X_2; X_1 < X_3)$$

where X_1, X_2, X_3 are iid according to the continuous distribution F. Of course $\alpha(F)$ is distribution–free being equal to 1/3, the chance that X_1 is the smallest of X_1, X_2, X_3.

Consider

$$\mu(F) = P(X_1 < X_2; X_1 < X_3 + X_4 - X_5)$$

and note that $X_3 + X_4 - X_5$ is a little like X_3 itself (because $X_4 - X_5$ is symmetric about 0) but the variance of $X_3 + X_4 - X_5$ is three times that of X_3. Thus it is slightly harder for X_1 to be simultaneously smaller than both X_2 and $X_3+X_4-X_5$ than it is to be simultaneously smaller than X_2 and X_3. Thus we should expect the value of $\mu(F)$ to be pulled slightly below the 1/3 value of $\alpha(F)$. A similar argument indicates the values of $\lambda(F)$ should be pulled down even further. That this is the case can be observed in Table 1.

F	Uniform	Normal	Logistic	Exponential	Cauchy
$\mu(F)$	0.3083	0.3075	0.3064	0.3056	0.3043
$\lambda(F)$	0.2909	0.2902	0.2898	0.2894	0.2879

Table 1. Values of $\mu(F)$ and $\lambda(F)$ for Various Distributions

In Section 2 we review the best known bounds for $\mu(F)$ and $\lambda(F)$, and indicate how they were derived. Section 3 is devoted to some applications. The bounds, motivated by the improved bounds due to Spurrier (1991), enable us to

(i) present improved upper and lower bounds for the correlation coefficient $\rho_0^n(F)$ defined by (1.2),

(ii) present improved upper and lower bounds for the correlation coefficient $r_0^n(F)$ defined by (2.3) of Section 2,

(iii) improve Hollander's (1967) estimator of the asymptotic null variance of his Y–test for ordered alternatives (see Section 3.3) and compare observed levels of this modified test with their asymptotic nominal values,

(iv) fine–tune Hsu's (1982) selection procedure for the best treatment (see Section 3.4) and compare observed coverage probabilities of this fine–tuned version of Hsu's procedure with their asymptotic nominal values.

2. Bounds for $\mu(F)$, $\lambda(F)$

The best bounds to date on $\mu(F)$,$\lambda(F)$ are

$$3/10 \le \mu(F) \le (\sqrt{2}+6)/24 \approx .30893, \tag{2.1}$$

and

$$.28254 \approx 89/315 \le \lambda(F) \le 7/24 \approx .29167. \tag{2.2}$$

The upper bound on $\mu(F)$ is due to Hollander (1967), the lower bound on $\mu(F)$ to Spurrier (1991), the upper bound on $\lambda(F)$ to Lehmann (1964) and the lower bound on λ(F) to Spurrier (1991). It remains an open question whether any of these bounds in (2.1) and (2.2) are best possible. We briefly sketch their derivations, referring the reader to the original articles for more details.

THEOREM 1 (Hollander (1967)) $\mu(F) \le (\sqrt{2}+6)/24$.

PROOF Let $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n$ be iid according to the continuous distribution F. Let U_1 be the Mann–Whitney–Wilcoxon statistic, $U_1 = \sum_{i=1}^n \sum_{j=1}^n \phi(X_i, Y_j)$ where $\phi(a,b) = 1$ if $a < b$, 0 otherwise and let U_2 denote Wilcoxon's signed rank statistic applied to a random pairing of the X's with the Y's. Using a representation due to Tukey, U_2 can be represented as $U_2 = \sum_{i<j}^n \phi(X_i + X_j, Y_i + Y_j) + \sum_{i=1}^n \phi(X_i, Y_i)$. The correlation coefficient between U_1 and U_2 is directly obtained to be

$$r_0^n(F) = \frac{[n^2(24\mu(F)-6) + n(23-72\mu(F)) + (48\mu(F)-14)]}{(2n+1)[n(n+1)/2]^{1/2}} \tag{2.3}$$

and its limiting value is

$$r^*(F) \stackrel{\text{def}}{=} \lim_n r_0^n(F) = (24\mu(F)-6)/\sqrt{2}.$$

The result follows since $r^*(F) \le 1$.

THEOREM 2 (Spurrier (1991)) $\mu(F) \ge 3/10$.

PROOF Spurrier's method, motivated by Mann and Pirie (1982), is to exhibit an unbiased estimator $\hat{\mu}(F)$ of $\mu(F)$ that assumes just two values. These values are 3/10 and 19/60 and since $E(\hat{\mu}(F)) = \mu(F)$ it follows that $\mu(F) \ge 3/10$. Spurrier's estimator is as follows. Let

$$\begin{aligned} \hat{\mu}(F) &= \sum I(X_{i_1}, \ldots, X_{i_5})/120 \\ &= \sum I(X_{(i_1)}, \ldots, X_{(i_5)})/120, \end{aligned}$$

where $I(a,b,c,d,e) = 1$ if $a < e$ and $a < b+c-d$; 0 otherwise, $X_{(1)} < \cdots < X_{(5)}$ are the order statistics of $X_1, \ldots, X_5$ and the Σ is over all permutations $(i_1, \ldots, i_5)$ of (1,2,3,4,5). The function I is invariant under interchanges of its second and third arguments, thus $\hat{\mu}(F)$ can be rewritten as

$$\hat{\mu}(F) = \sum I(X_{(i_1)}, \ldots, X_{(i_5)})/60, \tag{2.4}$$

where the summation is over all permutations $(i_1, \ldots, i_5)$ of (1,2,3,4,5) such that $i_2 < i_3$. Spurrier partitions the 60 summands of (2.4) into four groups. Group 1 consists of 13 summands which identically equal 1, group 2 consists of 37 summands which identically equal 0, and group 3 consists of 8 summands which can be organized into 4 pairs of summands, with the total of the two summands in each pair equal to 1. Thus the first three groups have 58 I functions summing to 17 w. p. 1. The last group consists of two summands and form the random part of $\hat{\mu}(F)$. At least 1 of these 2 must equal 1. Thus either 18 or 19 of the 60 summands equal 1 and $\hat{\mu}(F) = 3/10$ or $19/60$.

THEOREM 3 (Lehmann (1964)) $\lambda(F) \leq 7/24$.

PROOF Consider the covariance matrix defined by (1.5) and (1.6) for the random variables Y_{12}, Y_{13}, Y_{23}. The determinant of the matrix is proportional to $(1+\gamma)^2(1-2\gamma)$ where $\gamma = 3(4\lambda(F) - 1)$. The determinant can be nonnegative only if either $\gamma = -1$ in which case $\lambda = 1/6$ or if $\gamma \leq 1/2$ in which case $\lambda \leq 7/24$. The case $\lambda = 1/6$ is ruled out directly as Lehmann used Schwarz' inequality to show $\lambda \geq 1/4$.

THEOREM 4 (Spurrier (1991)) $\lambda(F) \geq 89/315$.

PROOF The proof is similar to Spurrier's proof of Theorem 2 but is more tedious and involves a computerized evaluation of numerous cases. Let $X_{(1)} < \cdots < X_{(7)}$ denote the order statistics of $X_1, \ldots, X_7$ and let

$$\hat{\lambda}(F) = \sum J(X_{(i_1)}, \ldots, X_{(i_7)})/5040,$$

where $J(a,b,c,d,e,f,g) = 1$ if $a < b+c-d$ and $a < e+f-g$; 0 otherwise, and the Σ is over all permutations $(i_1, \ldots, i_7)$ of (1,2,...,7). Using invariance and symmetry $\hat{\lambda}(F)$ can be rewritten as

$$\hat{\lambda}(F) = \sum J(X_{(i_1)}, \ldots, X_{(i_7)})/630, \tag{2.5}$$

where the summation is over all permutations $(i_1, \ldots, i_7)$ of (1,...7) such that $i_2 < i_3$, $i_4 < i_7$ and $i_5 < i_6$. Spurrier partitions the 630 summands in (2.5) into four groups. The first group contains 89 summands identically equal to 1, the second group contains 331 summands identically equal to 0, and the third group contains 132 summands which can be partitioned into pairs such that one pair is 0 and the other is 1. The 552 summands in the first three groups sum to 155. The remaining 78 summands comprise the fourth group and they are the random component of $\hat{\lambda}(F)$. Using a computer, Spurrier shows the minimum sum of those 78 is 23. Thus the minimum possible value of $\hat{\lambda}(F)$ is $(155+23)/630 = 89/315$. Since $\hat{\lambda}(F)$ is an unbiased estimator of $\lambda(F)$, it follows that $\lambda(F) \geq 89/315$.

3. Applications

3.1. *Upper and Lower Bounds on the Null Correlation Coefficient Between Overlapping Signed Rank Statistics*

The null correlation coefficient between two overlapping signed rank statistics $\rho_0^n(F)$, given by (1.2), depends on F except for $n = 1$ when its value is 1/3. Upper bounds ρ_U^n can be obtained by substituting the upper bounds for $\mu(F)$ and $\lambda(F)$ given respectively by the right–hand inequalities of (2.1) and (2.2), into (1.2). Lower bounds ρ_L^n can be obtained by substituting the lower bounds for $\mu(F)$ and $\lambda(F)$, given respectively by the left–hand–inequalities of (2.1) and (2.2), into (1.2). These bounds are displayed in Table 2.

n	1	2	3	4	5	6	7	8	9	10
ρ_U^n	.3333	.3886	.4163	.4330	.4441	.4521	.4581	.4627	.4665	.4695
ρ_L^n	.3333	.3600	.3701	.3752	.3784	.3804	.3819	.3830	.3839	.3845
n	11	12	13	14	15	20	25	40	50	∞
ρ_U^n	.4720	.4741	.4760	.4776	.4790	.4840	.4871	.4918	.4934	.5000
ρ_L^n	.3851	.3856	.3859	.3863	.3866	.3876	.3881	.3890	.3893	.3905

Table 2. Upper bounds ρ_U^n and lower bounds ρ_L^n for $\rho_0^n(F)$

3.2. *Upper and Lower Bounds on the Null Correlation Coefficient Between the Mann–Whitney–Wilcoxon Statistic and the Randomly Paired Signed Rank Statistic*

The null correlation between U_1 and U_2 (defined in the proof of Theorem 1) is given by (2.3). The null correlation $r_0^n(F)$ depends on F except for $n = 1$ and $n = 2$ where its values are 1 and .9238, respectively. Upper bounds r_U^n can be obtained by substituting the upper bound for $\mu(F)$ given by the right–hand inequality of (2.1) into (2.3). Lower bounds r_L^n can be obtained by substituting the lower bound for $\mu(F)$ given by the left–hand inequality of (2.1) into (2.3). These bounds are displayed in Table 3.

n	1	2	3	4	5	6	7	8	9	10
r_U^n	1.0000	.9238	.9231	.9306	.9382	.9448	.9503	.9549	.9587	.9620
r_L^n	1.0000	.9238	.8981	.8854	.8779	.8729	.8693	.8667	.8646	.8630
n	11	12	13	14	15	20	25	40	50	∞
r_U^n	.9648	.9673	.9694	.9712	.9729	.9790	.9828	.9890	.9911	1.0000
r_L^n	.8616	.8605	.8596	.8588	.8581	.8557	.8542	.8521	.8514	.8485

Table 3. Upper bounds r_U^n and lower bounds r_L^n for $r_0^n(F)$

3.3. *Observed Levels of Hollander's Test*

For testing, in the two–way layout, $H_0 : \theta_1 = \theta_2 = \cdots = \theta_k$ versus the ordered alternatives $H_a : \theta_1 \leq \theta_2 \leq \cdots \leq \theta_k$ (with at least one inequality strict), Hollander (1967) proposed tests based on

$$Y = \sum_{u<v}^{k} T_{uv} \tag{3.1}$$

where the T_{uv} are the Wilcoxon signed–rank statistics defined in Section 1. The statistic Y, suitably standardized, is asymptotically normal but it is not distribution–free under H_0 as the finite–dimensional joint distributions of the $\{ T_{uv} \}$ depend on F; in particular the null variance $\sigma_0^2(Y)$ depends on F as

$$\sigma_0^2(Y) = n(n+1)(2n+1)k(k-1)(3+2(k-2)\rho_0^n(F))/144 \tag{3.2}$$

where $\rho_0^n(F)$ is given by (1.2). Y is not asymptotically distribution–free as the asymptotic null variance of Y depends on F through $\lambda(F)$. Through a Monte Carlo study, we determined the levels of a modification of Hollander's (1967) test which rejects H_0 at the approximate α–level if $Y \geq k(k-1)n(n+1)/8 + z_\alpha\hat{\sigma}_0(Y)$, and accepts otherwise. Here z_α is the upper α percentile point of a $N(0,1)$ distribution, $\hat{\sigma}_0^2(Y)$ is obtained by replacing $\rho_0^n(F)$ by $\hat{\rho}_n = 12\hat{\lambda}(F) - 3$ in (3.2), and $\hat{\lambda}(F)$ is Lehmann's estimator given in (3.3) below provided $89/315 \leq \hat{\lambda}(F) \leq 7/24$, otherwise we take $\hat{\lambda}(F)$ to be 89/315 if (3.3) goes below the lower bound or 7/24 if (3.3) goes above the upper bound. This is a modification of Hollander's test because we are using Spurrier's new lower bound for $\lambda(F)$ to fine–tune $\hat{\lambda}(F)$. Results for the simulation are given in Table 4. For the range of distributions considered, the observed levels are reasonably close to the nominal α's and indicate the asymptotic test can be trusted in applications. (In Table 4, w is the number of times $\hat{\lambda}$ was within bounds.)

Lehmann's original estimator of $\lambda(F)$ is $\hat{\lambda}(F)$ given by

(3.3) $$n(n-1)(n-2)k(k-1)(k-2)\hat{\lambda}(F) = \sum_{(i,j,l)\in C_n} \sum_{(u,v,w)\in C_k} \eta(X_{iv} - X_{iu} + X_{ju} - X_{jv})\eta(X_{iw} - X_{iu} + X_{lu} - X_{lw})$$

where $\eta(t) = 1$ as $t \geq 0$ and is otherwise 0. The sets $C_n(C_k)$ are defined as the collection of all permutations of three integers chosen from the first $n(k)$ integers.

a) *Normal:*

n	5	10	11	12	13	15
α \k	3	4	8	5	7	6
.010	.010	.006	.010	.002	.002	.010
.025	.018	.020	.018	.024	.016	.030
.050	.052	.038	.038	.040	.042	.052
.100	.100	.078	.096	.082	.082	.092
w	161	377	499	463	497	497

b) *Cauchy:*

n	5	10	11	12	13	15
α \k	3	4	8	5	7	6
.010	.018	.016	.008	.008	.004	.004
.025	.042	.032	.024	.024	.016	.016
.050	.064	.054	.052	.052	.056	.034
.100	.122	.118	.110	.084	.090	.078
w	177	483	500	500	500	500

c) *Exponential:*

n	5	10	11	12	13	15
α \k	3	4	8	5	7	6
.010	.008	.004	.006	.012	.006	.006
.025	.028	.022	.020	.022	.022	.016
.050	.060	.044	.052	.048	.050	.048
.100	.114	.094	.094	.090	.118	.108
w	164	438	500	494	500	500

d) *Uniform:*

n	5	10	11	12	13	15
α \k	3	4	8	5	7	6
.010	.018	.010	.010	.014	.004	.010
.025	.038	.030	.024	.034	.020	.020
.050	.058	.052	.058	.044	.048	.038
.100	.120	.128	.104	.078	.090	.078
w	172	322	463	396	454	462

e) *Logistic:*

n	5	10	11	12	13	15
α \k	3	4	8	5	7	6
.010	.020	.014	.008	.008	.004	.006
.025	.040	.030	.024	.028	.016	.018
.050	.070	.054	.048	.048	.046	.034
.100	.118	.116	.108	.082	.096	.072
w	185	415	500	488	499	499

Table 4. Observed levels of Hollander's test

For the simulations of Sections 3.3 and 3.4, we generated independent samples from the following populations : (a) normal, (b) Cauchy, (c) exponential, (d) uniform, (e) logistic. Samples from the normal and exponential populations were obtained using the Zigurrat method as discussed in Marsaglia and Tsang (1984), samples from the uniform were obtained using a random number generator that combines, with subtraction mod 1, element c in arithmetic sequence generated by $c = c - cdmod(16777213./16777216.)$, period $2^{24} - 3$. All these are available from the Statistics Laboratory of the Florida State University. Samples from the remaining populations were obtained by transforming the numbers generated from the uniform distribution. All codes were written in FORTRAN and ran using an f77 compiler on the Sun Network system. All results were based on 500 iterations. Finally, Lehmann's estimator $\hat{\lambda}(F)$ was computed using the algorithm proposed by Mann and Pirie (1982).

3.4. *Observed Coverage Probabilities of Hsu's Procedure*

Hsu's (1982) procedure involves simultaneous inference with respect to a so called "best" treatment. Consider again model (1.1) where θ_j is the effect of treatment π_j. The treatment corresponding to the largest θ_j is said to be the "best" treatment. If there is more than one such treatment, then exactly one is arbitrarily designated the "best" treatment.

Let $\theta_{[k]} = \max_{1 \leq i \leq k} \theta_i$ and denote by (k) the unknown index of the "best" treatment, i.e., $\pi_{(k)}$ is the unique "best" treatment. Hsu's procedure gives a confidence set C for $\pi_{(k)}$, and simultaneously a set of simultaneous upper bounds $D = (D_1, \ldots, D_k)$ for $\phi = (\theta_{[k]} - \theta_1, \theta_{[k]} - \theta_2, \ldots, \theta_{[k]} - \theta_k)$.

Let $m = n(n+1)/2$ and let

$$A_{uv}^{[1]} \leq A_{uv}^{[2]} \leq \cdots \leq A_{uv}^{[m]}$$

denote the ordered averages $(X_{iu} - X_{iv} + X_{ju} - X_{jv})/2, 1 \leq i \leq j \leq n$. Then Hsu's procedure is outlined as follows:

For all $u \neq v$*, let*

$$\begin{aligned} T_{uv} &= A_{uv}^{[(m+1)/2]} && \textit{if } m \textit{ is odd} \\ &= (A_{uv}^{[m/2]} + A_{uv}^{[m/2+1]})/2 && \textit{if } m \textit{ is even} \end{aligned}$$

For $u = 1, \ldots, k$ *, calculate*

$$T_{u.} = \sum_{v \neq u} T_{uv}/k$$

Choose c_α *such that*

$$P\{\ c_\alpha \leq W \leq m - c_\alpha\ \} = 1 - \alpha$$

where W *is the Wilcoxon signed-rank statistic on n observations and* $0 < \alpha < 1$*. For all* $u \neq v$*, calculate*

$$B_{uv} = (3n)^{1/2}(A_{uv}^{[m+1-c_\alpha]} - A_{uv}^{[c_\alpha]})/z_{\alpha/2}$$

where $z_{\alpha/2}$ *is the upper* $\alpha/2$ *percentile point of a N(0,1) distribution. Let*

$$B = \binom{k}{2}^{-1} \sum_{u<v} B_{uv}$$

and

$$\hat{\tau}^2 = [1/12 + (k-2)(\hat{\lambda}(F) - 1/4)]B^2/k$$

where $\hat{\lambda}(F)$ *is Lehmann's estimator of* $\lambda(F)$ *modified so that* $\hat{\lambda}(F)$ *is replaced by* $89/315$ *if it is less than this lower bound and by* $7/24$ *if it exceeds this upper bound. Take*

$$\begin{aligned} C &= \{\pi_i : \min_{j \neq i} T_{i.} - T_{j.} \geq -(2/n)^{1/2} d(k-1, 1/2, P^*)\hat{\tau}\} \\ D_i &= \max\{\max_{j \neq i} T_{j.} - T_{i.} + (2/n)^{1/2} d(k-1, 1/2, P^*)\hat{\tau}, 0\} \\ &\quad i = 1, \ldots, k \end{aligned}$$

where $0 < P^* < 1$ *and* $d(h, \rho, P^*)$ *denotes the number such that*

$$P(Z_i \geq -d(h, \rho, P^*) \textit{ for } i = 1, \ldots, h) = P^*$$

where $Z_1, \ldots, Z_h$ *are equally correlated N(0,1) random variables with correlation* ρ*. Values for* $d(k-1, 1/2, P^*)$ *were obtained from Gupta, Nagel, and Panchapakesan* (1973).

Now define the coverage probability of a procedure R with confidence set C for $\pi_{(k)}$ and bounds $D = (D_1, \ldots, D_k)$ for $\phi = (\theta_{[k]} - \theta_1, \ldots, \theta_{[k]} - \theta_k)$ by

$$P\{\ \texttt{coverage|R}\ \} \equiv P\{\pi_{(k)} \in C \text{and } \theta_{[k]} - \theta_i \leq D_i \text{ for } i = 1, \ldots, k\}$$

Hsu showed that

$$\lim_{n\to\infty} \inf_{\theta} P_\theta\{ \texttt{ coverage}|\texttt{R } \} = P^* \tag{3.4}$$

where R is the procedure we described above. For our Monte Carlo study we determined the observed coverage probability for various choices of P^* and model structures for θ_j's. Results of the simulation are given in Table 5. In most cases the observed probability is higher, as is to be expected, than the nominal P^* and in only a few cases are the observed levels less than P^*, a direction change that can be attributed to the Monte Carlo sampling rather than the theoretical requirement (3.4) not being met.

a) *Normal:*

Model 1: $\theta_1 = \theta_2 = \cdots = \theta_{k-1} = 0$,$\theta_k = 0.1$

n	5	5	6
P^* \k	4	5	4
.990	.990	.996	.998
.975	.982	.994	.992
.950	.968	.980	.982
.900	.942	.938	.958
.750	.806	.848	.826

Model 2: $\theta_1 = 0$, $\theta_k = \theta_{k-1} + 0.1$

n	5	5	6
P^* \k	4	5	4
.990	.992	.996	1.000
.975	.988	.994	.992
.950	.974	.986	.990
.900	.948	.958	.968
.750	.826	.862	.844

Model 3: $\theta_1 = 1$, $\theta_k = 2\theta_{k-1}$

n	5	5	6
P^* \k	4	5	4
.990	.988	.996	.996
.975	.976	.986	.990
.950	.960	.964	.972
.900	.928	.920	.940
.750	.758	.810	.784

Table 5(a). Observed coverage probabilities of Hsu's procedure (with $c_\alpha = 2$)

b) *Cauchy:*

Model 1: $\theta_1 = \theta_2 = \cdots = \theta_{k-1} = 0, \theta_k = 0.1$

n P^* \k	5 4	5 5	6 4
.990	.996	.996	.996
.975	.992	.996	.992
.950	.984	.990	.982
.900	.964	.978	.974
.750	.886	.922	.906

Model 2: $\theta_1 = 0$, $\theta_k = \theta_{k-1} + 0.1$

n P^* \k	5 4	5 5	6 4
.990	.996	.998	.996
.975	.992	.996	.992
.950	.984	.992	.984
.900	.968	.980	.976
.750	.898	.926	.916

Model 3: $\theta_1 = 1, \theta_k = 2\theta_{k-1}$

n P^* \k	5 4	5 5	6 4
.990	.996	.996	.996
.975	.992	.996	.992
.950	.982	.990	.984
.900	.958	.976	.968
.750	.882	.916	.904

Table 5(b). Observed coverage probabilities of Hsu's procedure (with $c_\alpha = 2$)

c) *Exponential:*

Model 1: $\theta_1 = \theta_2 = \cdots = \theta_{k-1} = 0, \theta_k = 0.1$

n P^* \k	5 4	5 5	6 4
.990	.998	.990	.996
.975	.992	.982	.990
.950	.978	.968	.980
.900	.938	.940	.954
.750	.826	.832	.832

Model 2: $\theta_1 = 0,\ \theta_k = \theta_{k-1} + 0.1$

n P^* \k	5 4	5 5	6 4
.990	1.000	.990	.996
.975	.992	.988	.990
.950	.988	.972	.982
.900	.944	.952	.968
.750	.836	.852	.846

Model 3: $\theta_1 = 1,\ \theta_k = 2\theta_{k-1}$

n P^* \k	5 4	5 5	6 4
.990	.996	.990	.996
.975	.986	.976	.984
.950	.972	.962	.968
.900	.916	.922	.940
.750	.794	.792	.790

Table 5(c). Observed coverage probabilities of Hsu's procedure (with $c_\alpha = 2$)

d) *Uniform:*

Model 1: $\theta_1 = \theta_2 = \cdots = \theta_{k-1} = 0, \theta_k = 0.1$

n P^* \k	5 4	5 5	6 4
.990	.994	1.000	1.000
.975	.988	.996	1.000
.950	.982	.988	.994
.900	.962	.964	.976
.750	.854	.884	.878

Model 2: $\theta_1 = 0, \theta_k = \theta_{k-1} + 0.1$

n P^* \k	5 4	5 5	6 4
.990	.996	.998	1.000
.975	.990	.988	.998
.950	.978	.984	.994
.900	.956	.962	.978
.750	.816	.866	.856

Model 3: $\theta_1 = 1$, $\theta_k = 2\theta_{k-1}$

n P^* \k	5 4	5 5	6 4
.990	.986	.990	.996
.975	.972	.976	.986
.950	.958	.960	.978
.900	.914	.912	.944
.750	.786	.814	.818

Table 5(d). Observed coverage probabilities of Hsu's procedure (with $c_\alpha = 2$)

e) *Logistic:*

Model 1: $\theta_1 = \theta_2 = \cdots = \theta_{k-1} = 0, \theta_k = 0.1$

n	5	5	6
P^* \k	4	5	4
.990	.992	.994	1.000
.975	.986	.986	.992
.950	.972	.976	.976
.900	.936	.942	.944
.750	.818	.818	.850

Model 2: $\theta_1 = 0$, $\theta_k = \theta_{k-1} + 0.1$

n	5	5	6
P^* \k	4	5	4
.990	.994	.994	1.000
.975	.988	.992	.994
.950	.974	.976	.982
.900	.948	.952	.954
.750	.844	.860	.876

Model 3: $\theta_1 = 1, \theta_k = 2\theta_{k-1}$

n	5	5	6
P^* \k	4	5	4
.990	.988	.994	1.000
.975	.982	.986	.990
.950	.966	.964	.976
.900	.922	.936	.940
.750	.796	.800	.842

Table 5(e). Observed coverage probabilities of Hsu's procedure (with $c_\alpha = 2$)

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