

MOMENT ESTIMATION FOR STATIONARY POINT PROCESSES IN \mathbb{R}^d

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ABSTRACT

In this paper, we collect some results on the statistical analysis of moments of stationary point processes. We do not advance any new material, but try to bring together some important elements of the theory. The reader is referred to the original publications for complete proofs and explanations.

Key words: spatial point processes, Brillinger-mixing, moment, cumulant, asymptotic statistic.

1. Introduction

Point processes are natural models of the observed reality, when the phenomenon under study can be considered as the random dispersion of *small* objects. Even if we restrict our attention to applications of point processes in the plane to life sciences, many references can be found in the literature. Such processes are used to model the distribution of a parasitic fungus on wheat leaf area [17], of trees [19], of bird nesting areas [14], among many others.

When such a random dispersion of objects is examined, the existence of an underlying structure is a natural question. Indeed, the interactions between points of the same realization, using the language of particle physic, must be studied as a central problem to understand the generation and the pattern of the process.

One natural answer to that natural question is the estimation of moments and cumulants of the process, because they measure the co-occurrence of points, or the frequency of given configurations of a fixed number of points.

In that field, the precursor seems to be Bartlett [1], extending to point processes on the line the ideas of spectral analysis of stationary time series. Brillinger, following the same way, give a general rigorous theory of spectral cumulants estimation on the real line [2], and later put forwards a method of direct estimation of the density for some moment measures, with partial development of asymptotic theory [3]. Shortly after, Ripley [18] shows that the decisive tool for second order analysis of stationary point processes on a general space is the disintegration of invariant measures. That question was first introduced in the framework of moment measures for point processes by Krickeberg ([9], [10]).

The aim of that paper is to provide a general account of the asymptotic theory of moment estimation, for a general class of point processes in \mathbf{R}^d , in continuation of those pioneering works. There is no new material to be found in the sequel, all the results have been published elsewhere before. So we omit all the proofs. The basic notions necessary to work with such point processes are only briefly recorded. There are many basic references, among which the courses of Krickeberg [11] and Neveu [13] at Saint-Flour are to be recommended.

2. Notations and aims

Let P be a stationary point process on \mathbf{R}^d , that is a probability measure on the space $\{\mathcal{M}, \mathcal{B}(\mathcal{M})\}$ of the simple point measures on \mathbf{R}^d . μ is a typical realization of P , that is an element of \mathcal{M} chosen at random according to P . μ is then a point measure on \mathbf{R}^d , with mass 1 where there is a point of the realization, and 0 elsewhere.

Let g be any real bounded measurable function on $(\mathbf{R}^d)^k$, with compact range. The k th order moment measure $\nu_P^{(k)}(\cdot)$ of P is defined by the following integrals, provided they exist:

$$\nu_P^{(k)}(g) = \int_{\mathcal{M}} P(d\mu) \int_{(\mathbf{R}^d)^k} g(x_1, \dots, x_k) \mu(dx_1) \dots \mu(dx_k).$$

Let f_1, \dots, f_k be any k uple of real bounded measurable functions on \mathbf{R}^d . Then the k th

order cumulant $\gamma_P^{(k)}$ of P is defined by the following expected formula:

$$\gamma_P^{(k)}(f_1 \otimes \dots \otimes f_k) = \sum_{s=1}^k \sum_{\mathcal{Q}_s} (-1)^s (s-1)! \prod_{r=1}^s \nu_P^{\#q_r} \left(\bigotimes_{j \in q_r} f_j \right)$$

where \mathcal{Q}_s is the set of partitions of $1, 2, \dots, k$ in s parts q_1, \dots, q_s , and $\#q_r$ is the number of elements of the subset q_r .

If P is assumed stationary with respect to the translations in \mathbf{R}^d , then the moments and the cumulants can be disintegrated: if $\nu^{(k)}$ is defined, then there exists a measure $\nu'^{(k)}$ called the reduced k th order moment measure, defined by

$$\nu^{(k)}(g) = \int_{(\mathbf{R}^d)^k} g(u_1, \dots, u_{k-1}, x_k) \nu'^{(k)}(du_1, \dots, du_{k-1}) dx_k \quad (1)$$

with $u_i = x_i - x_k$, for $i = 1, \dots, k-1$. The first order moment (or cumulant) of a point process is called the intensity: if the process is stationary, it is proportional to the Lebesgue measure on \mathbf{R}^d and the coefficient of proportionality is called the density. The second order cumulant is also called the covariance measure of the process.

Furthermore, if P is stationary, there exists a unique probability P_0 on $\{\mathcal{M}, \mathcal{B}(\mathcal{M})\}$, called the Palm probability associated to P , such that

$$\int_{\mathcal{M} \times \mathbf{R}^d} f(T_x \mu, x) P(d\mu) \mu(dx) = z \int_{\mathcal{M} \times \mathbf{R}^d} f(x, \mu) P_0(d\mu) dx \quad (2)$$

for any positive measurable function on $\mathcal{M} \times \mathbf{R}^d$, with T_x the translation of vector x on \mathbf{R}^d and z the density of the process.

We observe that equation (2) changes a stochastic integral with respect to the point measure μ into an integral with respect to the intensity of the process, which is deterministic. On the other hand, let

$$f(\mu, x_i) = \mathbf{1}_{\{G\}}(x_i) \int_{(\mathbf{R}^d)^{k-1}} h(x_1, \dots, x_{k-1}) \mu(dx_1) \dots \mu(dx_{k-1})$$

for G a bounded Borelian subset of \mathbf{R}^d . Taking $g(\cdot) = \mathbf{1}_{\{G\}}(x_k) h(x_1, \dots, x_{k-1})$ in equation (1), we observe that $\nu'^{(k)}$ and $z\nu_0^{(k-1)}$ coincide, where $\nu_0^{(k-1)}$ is the $(k-1)$ th moment of the Palm probability P_0 . Moreover, it is known that the Palm probability is concentrated on the part of \mathcal{M} consisting of point measures with a point at the origin. All these remarks are leading to take as a natural estimator of $\nu'^{(k)}(h)$ a sort of *mean*: if G is (a part of) the window of observation of the process, we will consider for each point y of the realization of μ , the random variable $\eta(y, \mu) = \int_{(\mathbf{R}^d)^{k-1}} h(x_1 - y, \dots, x_{k-1} - y) \mu(dx_1) \dots \mu(dx_{k-1})$. The random variable $N_G(h, \mu)$ defined by

$$N_G(h, \mu) = \frac{1}{\lambda(G)} \sum_{x_k \in G \cap \text{supp} \cdot \mu} \eta(x_k, \mu) \quad (3)$$

is an estimator of $\nu'^{(k)}(h)$. λ is the Lebesgue measure on \mathbf{R}^d and then $\lambda(G)$ is the volume of G . Note that $N_G(h, \mu)$ can be also written

$$N_G(h, \mu) = \frac{1}{\lambda(G)} \int_{(\mathbf{R}^d)^k} \mathbf{1}_{\{G\}}(x_k) h(x_1 - x_k, \dots, x_{k-1} - x_k) \mu(dx_1) \dots \mu(dx_k).$$

Combining equations (2) and (1) leads to the

Proposition 1. (Krickeberg, [11]) *If $\eta(0, \mu)$ is in $L^1(P_0)$, then $N_G(h, \mu)$ is an unbiased estimator of $\nu^{(k)}(h)$.*

The next section of the paper will be devoted to the asymptotic properties of $N_G(h, \nu)$, for h a given function. Existence of densities for the moments will not be assumed. In that paper, asymptotic is always understood as growth of the window G to the whole space \mathbf{R}^d . In section 4, we do assume the existence of densities for the moments with respect to the Lebesgue measure on the convenient space: they are estimated taking for h a kernel, hence a function varying with G . These two types of problems are nonparametric. In the last part of the paper we give some ideas of how to develop a parametric theory of estimation for the covariance density.

3. Nonparametric estimation of $\nu^{(k)}(h)$

Let us make precise the notion of growth of G to \mathbf{R}^d . In fact we wish G to behave more or less like a ball. The following definition gives a rather general answer.

Definition 1. Let \mathcal{C} be the family of nonvoid compact convex subsets of \mathbf{R}^d and $B(0, r)$ the open ball with center at the origin and radius r . A family $\{G_r\}_{r \in \mathbf{R}_+}$ of subsets of \mathbf{R}^d is said a regular family if $G_r \in \mathcal{C}$, $\delta_r = \sup\{|x| : x \in G_r\}$ tends to infinity with r and if there exists a real positive number a such that $\lambda(G_r) \geq a\lambda(B(0, \delta_r))$ for each r .

An essential property of a regular family is that, for r sufficiently great, side effects can be neglected.

3.1. Consistency

The following ergodic theorem for random measures proves the consistency of $N_G(h, \mu)$.

Theorem 1. (Nguyen, Zessin, [15]) *If P is an ergodic stationary point process and if the random field $h(x, \mu)$ is such that, for any bounded Borelian B in \mathbf{R}^d*

$$\int_{\mathcal{M}} \left(\int_B |h(x, \mu)| \mu(dx) \right)^p P(d\mu) < \infty, \tag{4}$$

if $\{G_r\}_{r \in \mathbf{R}_+}$ is a regular family, then $\lim_{r \rightarrow \infty} N_{G_r}(h, \mu)$ exists P -almost surely and in $L^p(P)$ and is equal to $\mathbf{E}(N_U(h, \mu))$, U , any bounded Borel subset of \mathbf{R}^d . Moreover, if $\eta(0, \mu)$ is in $L^1(P_0)$ then equation 4 is true for $p = 1$, and for any B ,

$$\int_{\mathcal{M}} N_B(h, \mu) P(d\mu) = z\nu_0^{(k-1)}(h).$$

Consequently, $N_{G_r}(h, \mu)$ is a consistent unbiased estimator of $\nu^{(k)}(h)$.

The proof of the theorem rests mainly on the fact that $N_G(h, \mu)$ can be bounded above and below by quantities to which a classical ergodic theorem can be applied.

3.2. Convergence in distribution

A rather general theorem is proved under the hypothesis that the process P is mixing in a sense defined by Brillinger.

Definition 2. A stationary point process P is said Brillinger-mixing if all its reduced cumulants exist and are σ -finite.

Because the reduced cumulants can be seen as measure of the frequency of co-occurrence of points in a certain configuration, that definition of mixing implies that the *interaction* between points of a realization vanish rather quickly when points are far from each other.

That mixing condition, well adapted to the point processes, is sufficient to prove a central limit theorem for $N_G(h, \mu)$.

Theorem 2. (Jolivet, [5]) *Let h be a positive function with compact support on $(\mathbb{R}^d)^{k-1}$ and $\{G_r\}_{r \in \mathbb{R}_+}$ be a regular family.*

If P is a stationary Brillinger-mixing process, and if $\eta(0, \mu)$ is in $L^1(P_0)$, then, when r tends to infinity, the random variable $X_{G_r}(h, \mu) = \sqrt{\lambda(G_r)}[N_{G_r}(h, \mu) - \nu^{(k)}(h)]$ converges in distribution to a Gaussian random variable, with expectation 0 and variance depending on the $2p$ first moments of P and on h .

This result is proved by showing that the k th order moment of X_{G_r} , vanish for $k \geq 3$ when r tends to infinity.

However, the hypothesis of Brillinger-mixing is a rather strong one, because it assumes existence of moments of any order for P . A recent result by Heinrich [4] on a particular class of point processes shows that it is not minimal. Let's consider the class of stationary cluster point processes with a Poisson process as primary process. Heinrich call them Poisson cluster processes. Let Γ be the number of points of one realization of the secondary process. If $E(\Gamma^n)$ exists for any n , then the process is Brillinger-mixing. But the following weaker result can be shown.

Theorem 3. (Heinrich, [4]) *Let P be a stationary Poisson cluster process such that $E(\Gamma^{2k})$ exists, and $\{G_r\}_{r \in \mathbb{R}_+}$ be a regular family.¹ Let h obeys the following conditions:*

- h has a bounded support,
- $|h|$ is bounded,
- $\int_{(\mathbb{R}^d)^{(k-1)}} h^2(x_1, \dots, x_{k-1}) \nu^{(k)}(dx_1, \dots, dx_{k-1}) > 0$.

Then $X_{G_r}(h, \mu)$ converges in distribution to a Gaussian variable.

¹ In fact, the system of subsets considered by Heinrich is slightly more general.

The proof of this result is done by approximating P with truncated Poisson cluster processes, that is considering the restrictions of the secondary processes on spheres centered at the masses of the primary Poisson process, and then showing that $N_{G_r}(h, \mu)$ for such a process is a sum of m -dependent random variables.

These two last theorems have a multivariate version in the following sense: if we consider now h as a function on $C_u \times (\mathbf{R}^d)^{k-1}$, C_u being the unit cube of \mathbf{R}^d , and if the conditions of theorem 2 (resp. 3) are fulfilled for $h(t; \cdot)$ for any t in C_u , then for any n -uple t_1, \dots, t_n of fixed points of C_u , there is a multidimensional version of the central limit theorems for the vector $X_{G_r}(t_1), \dots, X_{G_r}(t_n)$.

3.3. Functional limit theorems

In applications, we are often interested in simultaneous estimation of $\nu^{(k)}(h)$ for various h . Let's take the following example. Let B be any bounded borelian subset of \mathbf{R}^d , and $Z(B, \mu) = \mu(B)$ the number of points of the realization in B . The functional $\mathcal{A}(B) = \mathbf{E}(Z^2(B, \mu))\mathbf{E}^{-1}(Z(B, \mu))$ can be used as an indicator of the aggregative pattern of a stationary point process, inspired by the methods of quadrat counts (see [16], for instance). Let $h(x) = \lambda(B \cap T_x B)\lambda^{-1}(B)$:

$$N_G(h, \mu) = \frac{1}{\lambda(G)} \int_{(\mathbf{R}^d)^2} \mathbf{1}_{\{G\}}(x_2) \frac{\lambda(B \cap T_{x_1-x_2} B)}{\lambda(B)} \mu(dx_1)\mu(dx_2)$$

is a consistent unbiased estimator of $\nu^{(2)}(h) = \mathcal{A}(B)$.

But $\mathcal{A}(B)$ is an indicator of clustering of the process *at the scale of B* . And it is obvious that it is not an indicator of the aggregative pattern of the process as a whole. But we can consider the family $\mathcal{A}(C_t)$ for the set $\{C_t\}_{t \in [0,1]^d}$ of all the parallelepipeds in the unit cube of \mathbf{R}^d . And then we are interested with the process $\{\widehat{\mathcal{A}}_G(C_t)\}_{t \in [0,1]^d}$ with $\widehat{\mathcal{A}}_G(C_t) = N_G(h(t; \cdot), \mu)$ and $h(t; x) = \lambda(C_t \cap T_x C_t)\lambda^{-1}(C_t)$.

Going back to the general case, assuming the hypothesis of theorem 2 are fulfilled by $h(t; \cdot)$ for each t in C_u , and some strong continuity property for h with respect to t , we prove the

Theorem 4. (Jolivet, [5]) *If P is a stationary, Brillinger-mixing process, if $\{h(t; \cdot)\}_{t \in [0,1]^d}$ fulfils the above mentioned conditions, and if $\eta_t(0, \mu)$ is in $L^1(P_0)$ for each $t \in [0, 1]^d$, then the process*

$$t \rightarrow X_{G_r}(h(t; \cdot), \mu)$$

has almost surely continuous sample paths and, as r tends to infinity, converges in distribution into the Skohorod space $D([0, 1]^d)$ to a Gaussian process with mean 0 and covariance function given by

$$K(s, t) = \lim_{r \rightarrow \infty} \int_{\mathcal{M}} X_{G_r}(h(s; \cdot), \mu) X_{G_r}(h(t; \cdot), \mu) P(d\mu).$$

The proof requires two steps:

- almost sure continuity of the sample paths of X_{G_r} ,
- convergence of its modulus of continuity to 0.

Besides the drawback of Brillinger-mixing condition, which is rather sharp, the continuity condition just referred to ensure almost sure continuity of sample paths can be regarded as very strong. That assumption is relaxed in Heinrich’s version of functional limit theorem for Poisson cluster processes. Nevertheless, the conditions he assumes for the increments of h as a function on $[0, 1]^d$ are not easy to summarize, and we refer the reader to his original paper.

4. Density estimation

In that section, we restrict ourselves to the estimation of the covariance density, assuming its existence. We assume h to be a bounded continuous function with integral 1, and we define the family of kernel functions

$$h_r(\cdot) = \beta_r^{-d} h(\beta_r^{-1} \cdot)$$

where $r \rightarrow \beta_r$ is a nonnegative application on \mathbf{R}_+ such that β_r tends to 0 when r tends to infinity. Let q be the density of the covariance of the process:

$$\gamma^{(2)}(g) = \int_{(\mathbf{R}^d)^2} g(x, y) q(y - x) dx dy.$$

An estimator of $q(u)$, $u \in \mathbf{R}^d$, is given by

$$\hat{q}_r(u) = \lambda^{-1}(G_r) \int_{(\mathbf{R}^d)^2} \chi(x_1, x_2) \mathbf{1}_{\{G_r\}}(x) h_r(x - y - u) \mu^c(dx) \mu^c(dy)$$

where $\mu^c(\cdot) = \mu(\cdot) - z\lambda(\cdot)$, and χ is the indicator function of the complement of the diagonal set $\Delta = \{x \in \mathbf{R}^d, y \in \mathbf{R}^d; x = y\}$. It was proposed by Krickeberg [11].

As a direct consequence of the cumulant study developed for the proof of theorem 2, we have the following proposition:

Proposition 2. (Jolivet, [7]) *If h and β_r fulfil the above mentioned conditions, if q is continuous at u , and if $\{G_r\}$ is a regular family, then, as r tends to infinity, the k th order cumulant of $\hat{q}_r(u)$ is equivalent to*

$$(\lambda(G_r)\beta_r^d)^{1-k} q(u) \int_{\mathbf{R}^d} h^k(x) dx.$$

Two direct consequences are asymptotic unbiasedness of these estimators and convergence in distribution of $\lambda^{1/2}(G_r)(\hat{q}_r(u) - q(u))$ to a Gaussian variable. Various extensions of that result are possible: we examine two of them in the following paragraphs.

4.1. Convergence in distribution

We assume the existence of densities for the moments of order 2, 3 and 4, being some integrability conditions (for details, see the cited reference). Define

$$\Delta_r(u) = (\lambda(G_r)\beta_r^d)^{1/2} (\hat{q}_r(u) - \mathbf{E}(\hat{q}_r(u))).$$

Theorem 5. (Heinrich, [4]) *Let P be a stationary Poisson cluster process with moments verifying the above mentioned conditions and $\mathbf{E}(\Gamma^4) < \infty$; let $\{G_r\}$ be a regular family. Let u_1, \dots, u_n be n fixed points of continuity of q such that $u_i \neq \pm u_j$, for $i \neq j$. Then the vector $\Delta_r(u_1), \dots, \Delta_r(u_n)$ converges in distribution to a vector of n centered independent Gaussian variables. Furthermore, the quadratic form*

$$\left(\int_{\mathbf{R}^d} h^2(x) dx \right)^{-1} \sum_{i=1}^n \frac{\Delta_r(u_i)}{q(u_i)}$$

converges in distribution to a χ^2 with n degrees of freedom when r tends to infinity.

Obviously, that result can be used to test if a Poisson cluster process has a given covariance density q . Another question is how fast \hat{q}_r approaches q . The next paragraph answers that question.

4.2. Speed of convergence

Here we assume that h fulfils the assumptions of Proposition 2 and is strongly symmetric, that is, for any $\alpha \in \{-1, 1\}^d$, $h(x) = h(\alpha x)$. Let $D^s q(u)$ be the s th differential of q at u and $\|D^s q(u)\|$ its norm as a s -linear form on \mathbf{R}^d . Let

$$\|D^s q\|_2^2 = \int_{\mathbf{R}^d} \|D^s q(u)\|^2 du.$$

Then we have the

Theorem 6. (Jolivet, [7]) *If P is a stationary Brillinger-mixing point process, and if it admits reduced cumulants of order 2, 3 and 4 with bounded continuous densities, if q is s times differentiable, and if $\|D_s q\|_2^2$ exists, then*

$$\limsup_{r \rightarrow \infty} \lambda(G_r)^{\frac{2s}{2s+d}} \mathbf{E} \int_K (\hat{q}_r(u) - q(u))^2 du \leq C \left[\|q\|_1 Q^{-\frac{d}{2s+d}} + \|D^s q\|_2^2 Q^{\frac{2s}{2s+d}} \right]$$

where K is a compact subset of \mathbf{R}^d , C a constant depending only on s and g , and the relation between the size of G_r and β_r being given by $Q = \lambda(G_r)\beta_r^{d+2s}$, for Q a constant.

We then have an overestimation of the speed of convergence of \hat{q}_r to q as measured by an integrated square error on any compact K . That result is optimal in that it is not possible to achieve a better speed, as is claimed by the following theorem. Here \mathbf{Q}_M is the whole set of stationary point processes on \mathbf{R}^d with density z and s times differentiable covariance density such that $\|D_s q\|_2^2$ is bounded above by M .

Theorem 7. (Jolivet, [6]) *If P is a point process in \mathbf{Q}_M , and if \tilde{q} is any estimator of its covariance density, then*

$$\liminf_{r \rightarrow \infty} \inf_{\tilde{q}} \sup_{P \in \mathbf{Q}_M} \lambda(G_r)^{\frac{2s}{2s+d}} \int_{\mathbf{R}^d} (\hat{q}_r(u) - q(u))^2 du \geq C$$

where C is a constant.

All the results until Theorem 6 are purely nonparametric and rest on a careful study of the cumulants of the statistics involved. The (Brillinger or m) mixing conditions play a key role in the theory. To prove the Theorem 7, the methods change radically: we need to study the likelihood of a family of point processes very close to the Poisson process, here a family of Gauss-Poisson processes. Studying the likelihood open the way to estimation in parametric models of the moments: it is the aim of the following section.

5. Likelihood - Parametric models

The general form of the likelihood of a point process which cumulants admit densities with respect to the Lebesgue measure was first given by Kuznetsov and Stratonovich [12]. Let $p^{(k)}$ be the density of $\gamma^{(k)}$. If we observe $\mu = \varepsilon_{x_1} + \dots + \varepsilon_{x_k}$ on G , ε_x being the measure with mass 1 concentrated at x , then the density of the likelihood is given by

$$\mathcal{L}(\mu, G) = \frac{1}{k!} \exp \left(\sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \int_{G^j} p^{(j)}(y_1, \dots, y_j) dy_1 \dots dy_j \right) \sum_{l=1}^k \sum_{\mathcal{Q}_l} \prod_{r=1}^l \left(\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int_{G^j} p^{(\#q_r)}(x_{\alpha_1^r}, \dots, x_{\alpha_{\#q_r}^r}, y_1, \dots, y_j) dy_1 \dots dy_j \right) \tag{5}$$

where \mathcal{Q}_l is the set of partitions of $\{1, \dots, k\}$ into l parts q_1, \dots, q_l , the elements of q_r being $\alpha_1^r, \dots, \alpha_{\#q_r}^r$.

That very unattractive expression is simplified in the case of a Gauss-Poisson process. Let's recall that a stationary Gauss-Poisson process is, roughly speaking, a Poisson cluster process whose random cluster size is 1 or 2. Its cluster members are independently distributed around the cluster centers according to a distribution on \mathbf{R}^d which is the reduced cumulant conveniently normalized. Then a stationary Gauss-Poisson process is completely determined by the densities of its two first cumulants.

Let, as in the preceding section, q be the density of the reduced covariance. Then, by application of formula 5, we have

$$\mathcal{L}(\mu, G) = \frac{1}{k!} \exp \left(-z\lambda(G) + \frac{1}{2} \int_{G^2} q(x - y) dx dy \right) \sum_{j=0}^{[k/2]} \sum_{\sigma(j;k)} \prod_{i=1}^j q(x_{\sigma_{2i-1}} - x_{\sigma_{2i}}) \prod_{l=2j+1}^k \left(z - \int_G q(x_{\sigma_l} - y) dy \right) \tag{6}$$

when the observation of the realization μ on G is $\varepsilon_{x_1} + \dots + \varepsilon_{x_k}$. Here $\sigma(j; k)$ is the set of all the partitions of $\{1, \dots, k\}$ into j pairs and $k - 2j$ singletons, σ_{2i-1} and σ_{2i} being the elements of pair number j and σ_l being the element of singleton number l . A careful study of formula 6 (see [6] and [8]) leads to the following approximation

$$L_G(\mu, q) = \frac{1}{\lambda(G)} \left(\int_{G^2} (z^2 q(x - y) + \frac{1}{2} q^2(x - y)) - \left(\int_{G^2} \chi(x, y) q(x - y) \mu(dx) \mu(dy) \right) \right). \tag{7}$$

From now on, we forget this approximation is coming from a Gauss-Poisson model. If the covariance density q is assumed to vary into a parametric family $\{g(\cdot, \theta)\}_{\theta \in \Theta}$,

Θ being a compact subset of \mathbf{R}^m , then it is tempting to use $C_G(\mu, \theta) = L_G(\mu, q)$ for $q(\cdot) = g(\cdot, \theta)$, as a contrast, and to choose as estimator for θ

$$\hat{\theta}_r = \arg \min_{\alpha \in \Theta} C_{G_r}(\mu, \alpha).$$

The results given by the two next theorems are obtained by using the standard methods of asymptotic statistics.

Theorem 8. (Jolivet, [8]) *If the stationary process P_θ , with known density z and covariance density $g(\cdot, \theta)$ is ergodic and if*

- $g(\cdot, \theta)$ is non-negative, bounded, continuous and integrable for each θ in Θ ,
- $g(x, \cdot)$ is continuous for each x in \mathbf{R}^d ,

then $\hat{\theta}_r$ is a weakly consistent estimator of θ .

Theorem 9. (Jolivet, [8]) *If*

- P_θ is Brillinger-mixing,
- for any x , g is twice differentiable with respect to θ and follows the hypotheses of theorem 8, as well as any of its two first order derivatives,
- $\int_{\mathbf{R}^d} \left(\frac{\partial g(x, \theta)}{\partial \theta} \right)^T \frac{\partial g(x, \theta)}{\partial \theta} dx$ is a non-singular positive matrix

then $\lambda(G_r)^{1/2}(\hat{\theta}_r - \theta)$ converges in distribution to a centered Gaussian variable, as r tends to infinity.

6. Conclusion

The results collected in that paper show that the statistical analysis results of moments of stationary processes are very similar to the ones of classical statistical analysis of i.i.d. samples, although the methods are relatively different. The only requirement is a reasonable mixing condition.

The nonparametric approach seems to be rather complete. Nevertheless, minimal conditions on the process, in terms of mixing conditions, could be investigated. On the other hand, the functional central limits theorem quoted here are restricted to the class of parallelepipeds of an euclidean space: investigation on more general classes would be of interest from both theoretical and applied points of view.

The parametric theory is much less developed, because of the complicated structure of the likelihood of such processes, even for simple models like Gauss-Poisson. However, even the very restricted results obtained until now could be extended along the lines of actual works in asymptotic statistics: better knowledge of asymptotic behavior, introduction of resampling techniques could be some promising perspectives.

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