Stochastic Orders and Decision under Risk
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MULTIVARIATE STOCHASTIC ORDERINGS AND GENERATING CONES OF FUNCTIONS

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Let C be a convex cone of functions $\phi : \mathbb{R}^n \to \mathbb{R}$; let X and Y be random vectors. Write $X \leq_C^{st} Y$ to mean $E\phi(X) \leq E\phi(Y)$ for all functions ϕ in C such that the expectations exist. Familiar examples of stochastic orderings that take this form are obtained from the convex cones of (1) increasing functions, (2) convex functions, (3) Schur-convex functions, and (4) centrally symmetric quasi-concave functions. In the literature, various properties of the corresponding orderings have been given, mostly on a case by case basis. The purpose of this paper is to gain some understanding of how some of these properties arise directly as consequences of conditions satisfied by the underlying convex cone C. A collection of examples is given.

1. Introduction. For a class C of (measurable) functions $\phi : \mathbb{R}^n \to \mathbb{R}$, and for random vectors X and Y, this paper is concerned with the condition

$$E\phi(X) \leq E\phi(Y) \text{ for all functions } \phi \in C \text{ such that the expectations exist.} \tag{1.1}$$

Write $X \leq_C^{st} Y$ to mean that (1.1) holds, or equivalently, when X has distribution F and Y has distribution G, write $F \leq_C^{st} G$.

Orderings of the form $\leq_C^{st}$ constitute a large class of what are sometimes called *stochastic orderings*. Other possible approaches to stochastic orderings are reviewed by Mosler and Scarsini (1991), but in this paper, only the definition via (1.1) is considered.

If (1.1) holds, then it is immediate that C can be replaced by the smallest convex cone containing C, so in this paper it is usually assumed from the start that C is a convex cone.

A number of orderings of the form $\leq_C^{st}$ have been defined and studied in the literature, usually with C specified, but sometimes with considerable generality. Some of these orderings are based upon the cone of functions isotone

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with respect to some preorder of $\mathbb{R}^n$. Of course, the example of prime interest in many contexts is the cone of increasing functions, i.e., the cone of all functions isotonic with respect to the usual preorder of $\mathbb{R}^n$. The cone of continuous convex functions is also of compelling interest in statistics, economics, and potential theory, but this cone is not the class of functions isotone with respect to some preorder; for it, somewhat different results and generalizations are to be found. These and other examples are briefly discussed in Section 4.

The focus of this paper is on the class of orderings rather than on any specific example, but the properties examined are those that are natural and are known for the cone of increasing functions, where the notation $\leq^{st}$ is used in place of $\leq^{st}_C$. The ordering $\leq^{st}$ with $n = 1$ possibly appeared first in the context of statistics, but it has been widely applied; for example, it can be used to define many of the classes of distributions studied in reliability theory. The corresponding ordering for $n > 1$ was studied by Lehmann (1955) and a number of subsequent authors. The following properties are well known (see e.g., Marshall and Olkin, 1979, Chapter 17 for a summary and for references).

1.1. PROPERTY. If $P(X \in A) \leq P(Y \in A)$ for all sets A with increasing indicator functions, then $X \leq^{st} Y$.

1.2. PROPERTY. For random vectors X and Y, $X \leq^{st} Y$ if and only if, in the univariate sense, $\phi(X) \leq^{st} \phi(Y)$ for all increasing functions ϕ.

1.3. PROPERTY. If $X \leq^{st} Y$, then there exist random variables $\widetilde{X}$ and $\widetilde{Y}$ such that X and $\widetilde{X}$ have the same distribution, Y and $\widetilde{Y}$ have the same distribution, and $P(\widetilde{X} \leq \widetilde{Y}) = 1$.

1.4. PROPERTY. If X_k converges weakly to X, Y_k converges weakly to Y, and $X_k \leq^{st} Y_k$ for all k, then $X \leq^{st} Y$.

1.5. PROPERTY. If $X \leq^{st} Y$ and $U \leq^{st} V$ where X and U, Y and V are independent, then $X + U \leq^{st} Y + V$.

1.6. PROPERTY. IF $X \leq^{st} Y$ and $Y \leq^{st} X$ then X and Y have the same distribution.

Orderings of the kind $\leq^{st}_C$ described above have to a large extent been introduced and studied individually; often, but not always, properties analogous to Properties 1.1–1.6 have been found. The purpose of this paper is to obtain some such results directly from conditions on the underlying convex cone. This approach offers an opportunity for a unified study of the various orderings, and it provides some insight into why some of the properties do not hold for all convex cones C.

Some required results concerning preorderings and/or partial orderings are obtained initially (Section 2). Even though the primary focus of this paper is on the space $\mathbb{R}^n$, it is worth working within the content of a real topological vector space L because the additional generality involves little additional complexity. Results concerning stochstic orderings in the general sense are given in Section 3, and Section 4 is devoted to examples.

Most of the results of this paper are stated without proof because the proofs are easily supplied. Thus, the intended contribution of this paper is its viewpoint more than its propositions.

2. Preorderings. Let L be a real topological vector space, let M be a subset of L, and let C be a convex cone of functions $\phi : M \to \mathbb{R}$.

2.1. DEFINITION. For x, y in M, write $x \leq_c y$ to mean that $\phi(x) \leq \phi(y)$ for all ϕ in C. In this case, the ordering $\leq_c$ is said to be *generated* by the cone C.

The relation $\leq_c$ is a *preordering* of M, i.e., for x, y, and z in M,

$$x \leq_c x, \tag{2.1}$$

$$x \leq_c y; \text{ and } y \leq_c z \text{ implies } x \leq_c z. \tag{2.2}$$

Although (2.1) always holds, it can happen that $x \leq_c y$ only when $x = y$; this is the case, e.g., when $L = \mathbb{R}^n$, $M = [0,1]^n$ and C is the cone of continuous convex functions defined on M. At the other extreme, $x \leq_c y$ for all x, y in M when C is the cone of constant functions.

2.2. REMARK. It is of interest to note that *all* preorderings of L arise from a convex cone of real functions in the manner of Definition 2.1. To see this, let $\leq$ be a preordering of L, and for each x in L, let I_x denote the indicator function of $H_x = \{y : x \leq y\}$. If C is the convex cone generated by these indicator functions, then $\leq$ is just the ordering of $\leq_c$.

2.3. DEFINITION. Let C^* denote the set of all functions $\phi : M \to \mathbb{R}$ that preserve the ordering $\leq_c$; i.e., C^* consists of all functions ϕ such that $x \leq_c y$ implies $\phi(x) \leq \phi(y)$. The set C^* is called the *completion* of C. If $C = C^*$, then C is said to be *complete*.

Of course, C is a subset of C^*, and it can be a proper subset. For example, when $L = \mathbb{R}$ and C is the cone of convex functions mentioned in Remark 2.2, C^* is the cone of all functions mapping M to $\mathbb{R}$; indeed this is the case whenever $\phi(x) = x$ and $\phi(x) = -x$ both belong to C, because then $x \leq_c y$ only if $x = y$. Nevertheless, cones containing these two functions can be interesting in the context of stochastic orderings, as in Examples 3.18 and 4.2 below.

2.4. PROPOSITION. *If a convex cone C is complete, then it contains the constant functions, it is closed under the topology of pointwise convergence and it is closed under the formation of maxima and minima.*

2.5. PROPOSITION. *Let A be a set of functions $\phi : M \to \mathbb{R}$ and let C be the smallest convex cone containing A. Then $x \leq_c y$ in the sense of Definition 2.1 if and only if $\phi(x) \leq \phi(y)$ for all $\phi \in A$.*

Definition 2.1 uses a class of order preserving functions to define an ordering; it is perhaps more usual to start with the ordering and use it to define the order-preserving functions. In such cases, the ordering often comes from a convex cone as in the following definition.

2.6. DEFINITION. Let $H \subset L$ be a convex cone, and write $x \leq y$ to mean that $y - x \in H$. The ordering $\leq$ of L is said to be a *cone ordering*.

In addition to satisfying (2.1) and (2.2), cone orderings also satisfy

$$x \leq y \text{ implies } x + z \leq y + z \text{ for all } z \in L, \tag{2.3}$$

$$x \leq y \text{ implies } \lambda x \leq \lambda y \text{ for all } \lambda > 0. \tag{2.4}$$

If H is pointed, i.e., $x \in H$ and $-x \in H$ together imply $x = 0$, then

$$x \leq y \text{ and } y \leq x \text{ implies } x = y, \tag{2.5}$$

so that in this case, the ordering is a true partial ordering.

2.7. EXAMPLE. Let $L = \mathbb{R}^n$, and let $\leq$ be the ordering of majorization. This ordering fails to satisfy (2.5), and it is not a cone ordering. But on the set $D = \{x : x_1 \geq x_2 \geq \cdots \geq x_n\}$, majorization coincides with the cone ordering determined by the convex cone

$$\left\{ z : \sum_1^i z_j \geq 0, i = 1, 2, \cdots, n-1, \sum_1^n z_j = 0 \right\}.$$

If C_1 is the convex cone of permutation symmetric convex functions, then C_1 determines the ordering of majorization via Definition 2.1, but C_1 is not complete; its completion C_1^* is the cone of Shur convex functions. The Schur convex functions are the completion of other convex cones besides C_1; examples include the cone of symmetric convex functions ϕ such that $\phi(0) = 0$, and the cone of functions having the form $\phi(x_1, \cdots, x_n) = \Sigma\psi(x_i)$, where $\psi : \mathbb{R} \to \mathbb{R}$ is convex.

2.8. PROPOSITION. *Let $\leq$ denote the preordering of L determined by the convex one $H \subset L$. If $\phi(x) \leq \phi(y)$ for all functions $\phi \in C^*$ that preserve the ordering $\leq$ of L, then $x \leq y$.*

PROOF. Let $\phi_x(z) = 1$ if $z \in x + H$, and $\phi_x(z) = 0$, otherwise. To see that $\phi_x \in C^*$, suppose that $\phi_x(u) = 1$ and $u \leq v$. Then, $u \in x + H$, i.e., $u - x \in H$. But also, $v - u \in H$, so $(u - x) + (v - u) = v - x \in H$, i.e. $v \in x + H$, and thus $\phi_x(v) = 1$.

But clearly, $1 = \phi_x(x) \leq \phi_x(y)$; this implies $y \in x + H$, i.e., $x \leq y$. ▮

Because of Proposition 2.8, all cone orders can be defined in terms of the corresponding order-preserving functions as in Definition 2.1.

Of course both C and C^* determine the same ordering via the procedure of Definition 2.1. The following theorem gives conditions under which this ordering is a cone ordering.

2.9. THEOREM. *For each $\phi \in C^*$, each real vector u and each positive number λ, let ϕ_u and $\phi_{(\lambda)}$ be defined by*

$$\phi_u(x) = \phi(x + u), \quad \phi_{(\lambda)}(x) = \phi(\lambda x),$$

If $\phi \in C^*$ implies that for all real vectors u, $\phi_u \in C^*$, then $\leq$ satisfies (2.3). If $\phi \in C^*$ implies that for all positive numbers λ, $\phi_{(\lambda)} \in C^*$, then $\leq$ satisifes (2.4). If both conditions (2.3) and (2.4) are satisfied, then $\leq$ is a cone ordering.

2.10. DEFINITION. Let $\leq$ be a preorder of L. If $\{(x, y) : x \leq y\}$ is a closed subset of L^2, then $\leq$ is said to be a *closed preorder*.

2.11. PROPOSITION. *Suppose that $\leq$ is the ordering determined by a convex cone H as in Definition 2.6. Then $\leq$ is closed if and only if H is closed.*

Because of this proposition, it is often assumed that the convex cone H determining an ordering is closed.

For orderings determined by Definition 2.1 the counterpart of Proposition 2.11 fails, and a counterexample can easily be constructed with $L = \mathbb{R}$ using the cone of functions that are multiples of the Heaviside function $\phi(x) = 0$ if $x < 0$, $\phi(x) = 1$ if $x \geq 0$. On the other hand the result can be true under some circumstances.

2.12. PROPOSITION. *Suppose that there exists a subset A of C^* such that (i) functions in A are continuous and (ii) C^* is the smallest complete convex cone containing A. Then $\leq_c$ is closed.*

3. Stochastic Orderings. Let us turn our attention from orderings of points in a linear topological space L to orderings of random variables taking values in L. The sigma-field of subsets of L that is required for this purpose is assumed without further mention to consist of the Borel subsets of L.

3.1. DEFINITION. Let C be a convex cone of real functions ϕ defined on a measurable subset M of a real topological vector space L. For random variables X and Y taking values in M, write $X \le_C^{st} Y$ to mean that

$$E\phi(X) \le E\phi(Y) \text{ for all } \phi \in C \text{ such that the expectations exist.} \qquad (3.1)$$

When $X \le_C^{st} Y$, X is said to be *stochastically less than* Y (with respect to C). In case $M = \mathbb{R}^n$ and C is the cone of increasing functions, write $X \le^{st} Y$ in place of $X \le_C^{st} Y$.

Some comments about this definition follow.

(i) A more usual procedure is to start with a preorder of M, and to take C to be the cone of order-preserving functions (see, e.g., Kamae, Krengel, and O'Brien, 1977, or Marshall and Olkin, 1979). In contrast to Definition 3.1, this procedure requires that C be complete. Elimination of this requirement accommodates some interesting examples and raises some interesting questions.

(ii) Although represented here as an ordering of random variables, it is important to realize that the ordering $\le_C^{st}$ is more properly considered an ordering of distribution functions or of probability measures, because the ordering depends only upon the marginal distributions of X and Y and not upon the random variables themselves.

(iii) If C and D are convex cones of functions $\phi : M \to \mathbb{R}$ and $C \subset D$, then $X \le_D^{st} Y$ implies $X \le_C^{st} Y$.

3.2. DEFINITION. Let M be a measurable subset of the topological vector space L and let C be a convex cone of functions $\phi : M \to \mathbb{R}$. The *stochastic completion* of C is the convex cone C^+ of all functions $\phi : M \to \mathbb{R}$ for which $X \le_C^{st} Y$ implies $E\phi(X) \le E\phi(Y)$.

Definition 3.2 can be understood in terms of iii) above; C^+ is the largest convex cone for which $C \subset C^+$ but still $X \le_C^{st} Y$ implies $X \le_{C^+}^{st} Y$.

3.3. PROPOSITION. *For all convex cones C, $C \subset C^+ \subset C^*$, and C^+ contains the constant functions. Moreover, C^+ is closed in the sense of uniform convergence, and any monotone (pointwise) limit of functions in C is in C^+.*

PROOF. Let $x, y \in M$ and suppose that $x \le y$ in the sense that $\phi(x) \le \phi(y)$ for al ϕ in C. If X and Y are degenerate at x and y respectively, then $\phi(x) = E\phi(X) \le E\phi(Y) = \phi(y)$ for all ϕ in C^+. Thus, $C^+ \subset C^*$. The remaining parts of the proof are even easier and are omitted. ∎

Unlike the completion of C, the stochastic completion of C need not be closed under the formation of maxima and minima.

Regarded as an ordering of probability measures, $\leq_C^{\mathrm{st}}$ is always a cone ordering. To see this, some definitions and notations are convenient.

Let S be the convex cone of finite signed measures σ defined on the Borel subsets of M such that $\int d\sigma = 0$. Let $K \subset S$ be a convex cone and let C be the convex cone of all measurable functions $\phi : M \to \mathbb{R}$ such that $\int \phi \, d\sigma \geq 0$ for all $\sigma \in K$. The K is said to be *complete* if $\int \phi \, d\sigma \geq 0$ for all $\phi \in C$ implies $\sigma \in K$.

3.4. PROPOSITION. *(i) Let C be a convex cone of measurable functions $\phi : M \to \mathbb{R}$, and let $K \subset S$ be the (complete) convex cone of signed measures σ for which $\int \phi \, d\sigma \geq 0$ whenever ϕ is in C and the integral exists. Alternatively, (ii) let $K \subset S$ be a convex cone and let C be the (stochastically complete) convex cone of all measurable functions $\phi : M \to \mathbb{R}$ such that $\int \phi \, d\sigma \geq 0$ for all $\sigma \in K$ such that the integral exists. Then $X \leq_C^{\mathrm{st}} Y$ if and only if $F \leq^{(K)} G$, where F and G are the respective probability measures induced by X and Y, and $\leq^{(K)}$ is the cone ordering of S determined by K.*

Proposition 3.4 shows that there is a relationship between complete cones K in S and cones C of real measurable functions defined on M; indeed, K is often called the *polar* of C. Proposition 3.4 also shows that the ordering $\leq_C^{\mathrm{st}}$ is a cone ordering when properly regarded as an ordering of distribution functions or probability measures. The following proposition is a trivial consequence, but a direct proof is given.

3.5. PROPOSITION. *Suppose that $X_\theta \leq_C^{\mathrm{st}} Y_\theta$ for all θ in the set B and suppose that the distributions F_θ and G_θ are measurable in $\theta \in B$. If X and Y have respective distributions F and G where*

$$F(x) = \int F_\theta(x) dH(\theta), \qquad G(x) = \int G_\theta(x) dH(\theta),$$

then $X \leq_C^{\mathrm{st}} Y$.

PROOF. For any $\phi \in C$, $E\phi(X) = E\{E[\phi(X) \mid \Theta]\} \leq E\{E[\phi(Y) \mid \Theta]\} = E\phi(Y)$ where Θ has the distribution H. ∎

The following proposition is related to Proposition 3.4 since it provides a stochastic version of Theorem 2.9.

3.6. PROPOSITION. *Suppose that $\phi \in C$ implies that $\phi_u \in C^+$ where ϕ_u is defined in Theorem 2.9. If $X \leq_C^{\mathrm{st}} Y$ and $U \leq_C^{\mathrm{st}} V$ where X and U, Y and V are independent, then $X + U \leq_C^{\mathrm{st}} Y + V$. If $\phi \in C$ implies $\phi_{(\lambda)} \in C$ where $\phi_{(\lambda)}$ is defined in Theorem 2.9, then $X \leq_C^{\mathrm{st}} Y$ implies $aX \leq_C^{\mathrm{st}} aY$ for all $a > 0$.*

3.7. PROPOSITION. *Let C_i be a convex cone of functions defined on a measurable subset M_i of a topological vector space L_i, $i = 1, 2$, and let C be* *the convex cone of all functions ϕ defined on $M_1 \times M_2$ with the property that for each fixed $y \in M_2$, $\phi(\cdot, y) \in C_1$ and for each fixed $x \in M_1$, $\phi(x, \cdot) \in C_2$. If $X \leq_{C_1}^{\text{st}} Y$, and $U \leq_{C_2}^{\text{st}} V$, where X and U, Y and V are independent, then $(X, U) \leq_C^{\text{st}} (Y, V)$.*

PROOF. If $\phi \in C$, then $E\phi(X, u) \leq E\phi(Y, u)$ for all u in M_2 and all $X \leq_{C_1}^{\text{st}} Y$. Consequently, $E\phi(X, U) \leq E\phi(Y, U)$. Similarly, $E\phi(Y, U) \leq E\phi(Y, V)$ for all $U \leq_{C_2}^{\text{st}} V$. ∎

The following is a companion to Proposition 3.7.

3.8. PROPOSITION. *Under the set-up of Proposition 3.7, suppose that $\psi : M_1 \times M_2 \to M_1$ has the property that $(x, u) \leq_C (y, v)$ implies $\psi(x, u) \leq_{C_1} \psi(y, v)$. If $(X, U) \leq_C^{\text{st}} (Y, V)$, then $\psi(X, U) \leq_{C_1}^{\text{st}} \psi(Y, V)$.*

PROOF. Because of the condition on ψ, it is immediate that for any $\phi \in C_1$, the composition $\phi \circ \psi$ is in C. ∎

Propositions 3.7 and 3.8 can be used to obtain the first part of Proposition 3.6.

Much of the rest of this section is focused on the question of how Properties 1.1–1.6 extend to more general cones.

3.9. PROPOSITION. *Let A be a set of functions $\phi : M \to \mathbb{R}$, and let C be the smallest convex cone containing A. Then $X \leq_C^{\text{st}} Y$ if and only if $E\phi(X) \leq E\phi(Y)$ for all ϕ in A.*

Of course the condition $X \leq_C^{\text{st}} Y$ can be replaced by $X \leq_{C+}^{\text{st}} Y$, or C can be replaced by the smallest convex cone containing C that is closed under monotone limits. In that form, Proposition 3.7 yields Property 1.1 as a special case.

Now, consider Property 1.2 and its extension to the general setting, where it becomes much more interesting. For this purpose some notation is convenient

3.10. NOTATION. Under the set-up of Definition 3.1, write

$$X \leq_C^P Y \text{ if } \phi(X) \leq^{\text{st}} \phi(Y) \text{ for all } \phi \in C. \tag{3.2}$$

Further, for any convex cone C of functions $\phi : M \to \mathbb{R}$, let

$$\widetilde{C} = \{f : \text{for some } \phi \in C \text{ and increasing function } \psi : \mathbb{R} \to \mathbb{R},\ f = \psi \circ \phi\}.$$

3.11. THEOREM. *The orderings $\leq_C^P$ and $\leq_{\widetilde{C}}^{\text{st}}$ are equivalent. Consequently,*

$$X \leq_C^P Y \text{ implies } X \leq_C^{\text{st}} Y. \tag{3.3}$$

The orderings $\leq_C^P$ *and* $\leq_C^{st}$ *are equivalent if and only if* $\widetilde{C} \subset C^+$.

PROOF. Suppose that $X \leq_C^P Y$. Then for all increasing functions $\psi : R \to R$ and $\phi \in C$, $E\, \psi \circ \phi(X) \leq E\, \psi \circ \phi(Y)$, i.e., $X \leq_{\widetilde{C}}^{st} Y$. Suppose that $X \leq_{\widetilde{C}}^{st} Y$. Then for all increasing $\psi : \mathbb{R} \to \mathbb{R}$ and $\phi \in C$, $E\, \psi \circ \phi(X) \leq E\, \psi \circ \phi(Y)$ i.e., $\phi(X) \leq^{st} \phi(Y)$. This proves the first part of the theorem, from which (3.3) follows because $C \subset \widetilde{C}$. To prove the last part of the theorem, suppose first that $\widetilde{C} \subset C^+$. It then follows that $\leq_C^P$ and $\leq_C^{st}$ are equivalent if it is shown that $X \leq_C^{st} Y$ implies $X \leq_C^P Y$. So suppose that $X \leq_C^{st} Y$. Then $E\phi(X) \leq E\phi(Y)$ for all $\phi \in C^+$, and in particular, this inequality holds for all $\phi \in \widetilde{C}$. Thus, $E\, \psi \circ \phi(X) \leq E\, \psi \circ \phi(Y)$ for all increasing ψ and all $\phi \in C$. Consequently, $\phi(X) \leq^{st} \phi(Y)$, i.e., $X \leq_C^P Y$. Now, suppose that $\leq_C^P$ and $\leq_C^{st}$ are equivalent, and suppose that $\psi \circ \phi \in \widetilde{C}$. Then $X \leq_C^{st} Y$ implies $X \leq_C^P Y$ which in turn implies $E\, \psi \circ \phi(X) \leq E\, \psi \circ \phi(Y)$. Thus, $\psi \circ \phi \in C^+$. ∎

Property 1.3 is especially interesting and useful, so its extension is important. Such an extension is given by Strassen (1965); see also Kamae, Krengel and O'Brien (1977) or Marshall and Olkin (1979, p. 483). All of these references make the assumption that the cone C is complete.

3.12. THEOREM. *Suppose that* L *is a complete separable metric space, and suppose that the preorder of* L *generated by the convex cone* C *via Definition 2.1 is closed. Then the conditions*

(i) $X \leq_C^{st} Y$ *and*

(ii) There exists a pair $\widetilde{X}, \widetilde{Y}$ *of random variables such that*

- *(a)* X *and* $\widetilde{X}$ *are identically distributed,* Y *and* $\widetilde{Y}$ *are identically distributed,*
- *(b)* $P\{\widetilde{X} \leq_C \widetilde{Y}\} = 1$

are equivalent if and only if $C^+ = C^*$, *i.e., the stochastic completion* C^+ *of* C *is complete.*

PROOF. Suppose first that C^+ is complete. The fact that (i) and (ii) then hold is given by Strassen (1965) as an application of his Theorem 11; see also Marshall and Olkin (1979, p. 483) where the reference to Strassen's equation (10) should read (30). On the other hand, if (i) and (ii) hold then for all $\phi \in C^*$, $P\{\phi(\widetilde{X}) \leq \phi(\widetilde{Y})\} = 1$, and hence $E\phi(X) = E\phi(\widetilde{X}) \leq E\phi(\widetilde{Y}) = E\phi(Y)$ for all $\phi \in C^*$. Thus, $C^+ = C^*$. ∎

Extensions of Property 1.4 can be obtained in several ways. Perhaps the following is the most obvious because it is an immediate consequence of the definition of weak convergence.

3.13. PROPOSITION. *Suppose that C is the smallest convex cone containing A and that $\phi \in A$ implies that ϕ is bounded and continuous. If X_n converges weakly to X, Y_n converges weakly to Y, and $X_n \leq_C^{\text{st}} Y_n$, $n = 1, 2, \cdots$, then $X \leq_C^{\text{st}} Y$.*

The proof that Kamae, Krengel and O'Brien (1977) give of their Proposition 3 is easily modified to yield the following variation of Proposition 3.13.

3.14. PROPOSITION. *Suppose that the conditions of Theorem 3.12 hold and that $C^+ = C^*$. If X_n converges weakly to X, Y_n converges weakly to Y, and $X_n \leq_C^{\text{st}} Y_n$, $n = 1, 2, \cdots$, then $X \leq_C^{\text{st}} Y$.*

Property 1.5 has been extended by various authors in various ways; see Proposition 3.6. The following extension of Proposition 3.6 and Property 1.5 essentially follows Scarsini and Shaked (1987), who treat the special case discussed in Example 4.6 below. Their proof requires little modification here.

3.15. THEOREM. *Suppose that $\phi \in C$ implies $\phi_u^{(i)} \in C^+$, $i = 1, 2$ where for some function $\psi : M \times M \to M$,*

$$\phi_u^{(1)}(x) = \phi \circ \psi(x, u) \in C^+, \quad \phi_u^{(2)}(x) = \phi \circ \psi(u, x) \in C^+.$$

If $X \leq_C^{\text{st}} Y$ and $U \leq_C^{\text{st}} V$ where X and U, Y and V are independent, then $\psi(X, U) \leq_C^{\text{st}} \psi(Y, V)$.

PROOF. Since $\phi_u^{(1)} \in C^+$, it follows that $E\phi_u^{(1)}(X, u) \leq E\phi_u^{(1)}(Y, u)$, and hence, $E\phi_u^{(1)}(X, U) \leq E\phi_u^{(1)}(Y, U)$. Similarly, $E\phi_u^{(2)}(U, Y) \leq E\phi_u^{(2)}(V, Y)$. These two inequalities together yield the desired result. ∎

Property 1.6 extends to the general setting when the cone C is sufficiently rich.

3.16. PROPOSITION. *The condition*

(i) *$X \leq_C^{\text{st}} Y$ and $Y \leq_C^{\text{st}} X$ implies that X and Y have the same distribution*

is equivalent to the condition

(ii) *$E\phi(X) = E\phi(Y)$ for all $\phi \in C$ implies that X and Y have the same distribution.*

It is not necessarily easy to determine if condition (ii) holds, but clearly it holds if the associated cone K of signed measures (defined just before Proposition 3.4) is pointed. Condition (ii) has been proposed by Kimeldorf and Sampson (1987) as one condition for the ordering $\leq_C^{\text{st}}$ to be a "positive dependence ordering."

A collection of examples follows that are too simplistic to be of much interest except for the purpose of illustrating the above ideas and conclusions.

3.17. Example. Let $\phi : M \to \mathbb{R}$ be fixed and let C consist of all functions of the form $a\phi + b$, where $a \geq 0$. If $E\phi(X)$ and $E\phi(Y)$ exist, then $X \leq_C^{\text{st}} Y$ if and only if $E\phi(X) \leq E\phi(Y)$. The special case $M = \mathbb{R}$ and $\phi(x) = x$ gives the ordering $X \leq_C^{\text{st}} Y$ if and only if $E\phi(X) \leq E\phi(Y)$; then, $x \leq_C y$ means $x \leq y$ in the usual sense, so C^* is the cone of all increasing functions. On the other hand, C is stochastically complete, so C^+ and C^* differ markedly. For this example, it is clear that $\widetilde{C} = C^*$.

3.18. Example. Let $M = \mathbb{R}$, and let C consist of all functions of the form $\phi(x) = ax^2 + bx + c$, $a \geq 0$. Then $X \leq_C Y$ means $EX = EY$ and Var $X \leq$ Var Y. This ordering is not without interest but its properties are limited. Note that $x \leq_C y$ means $x = y$, so C^* consists of all functions $\phi : \mathbb{R} \to \mathbb{R}$. In this example, the inclusions $C \subset \widetilde{C}$ and $C^+ \subset C^*$ are both proper, but $C = C^+$.

3.19. Example. The cone C_1 of permutation symmetric convex functions discussed in Example 2.6 is not complete and does not contain $\widetilde{C}_1$, but it is stochastically complete. This means that Theorems 3.10 and 3.11 do not apply. To see that C_1 is stochastically complete, suppose first that there is a function ϕ in C^+ that is symmetric but not convex. Then for some x, y in M, α in $(0,1)$, and some ϕ in C^+,

$$\alpha\phi(x) + (1-\alpha)\phi(y) > \phi(\alpha x + (1-\alpha)y).$$

Let X be a random variable such that $X = x$ with probability $\beta\alpha$, $X = y$ with probability $\beta(1-\alpha)$, and $X = \alpha x + (1-\alpha)y$ with probability $1-\beta$. Let Y be a random variable such that $Y = x$ with probability α, $Y = y$ with probability $1-\alpha$. Then the condition $E\phi(X) \leq E\phi(Y)$ is equivalent to

$$a\phi(x) + (1-\alpha)\phi(y) \leq \phi(\alpha x(1-\alpha)y),$$

a contradiction.

Next, let ψ be a function that is not symmetric. Then there is a point x in M and a permutation π such that $\psi(x) > \psi(\pi x)$. If $X = x$ with probability 1 and $Y = \pi x$ with probability 1, then $E\phi(X) \leq E\phi(Y)$ for all functions ϕ in C because equality holds, but $E\psi(X) > E\psi(Y)$. Thus, ψ is not in C^+.

It is not yet clear how different C and C^+ can be, and this question is the point of the following problem.

3.20A. Open Problem. Determine necessary and sufficient conditions under which $C = C^+$. These cones are not necessarily equal, and the most obvious examples of inequality are those for which C does not contain all of the constant functions. Other examples of inequality arise when C is not closed. A partial result related to this problem is given in the following theorem.

3.20b. THEOREM. *Suppose that M is compact. If C is closed (in the topology of uniform convergence), contains the constant functions, and functions in C are continuous, then all continuous functions in C^+ are already in C.*

PROOF. First note that the space B of bounded measurable functions on M is locally convex in the uniform topology (Taylor, 1961, p. 146), and moreover B is locally compact. Next suppose that the theorem is false, and instead that there exists a *continuous* function ψ in C^+ that is not in C. Because C is closed and convex, there exists a continuous linear functional f on L such that $f(\psi) < \inf\{f(\phi) : \phi \in C\}$ (see, e.g. Day, 1962, p. 22). The functional f can be written in the form $f = f_+ - f_-$ where f_+ and f_- are positive linear functionals defined on the space of continuous functions which vanish off the compact set M (see, e.g. Halmos, 1950, p. 249). The positive linear functionals f_+ and f_- can be represented as integrals with respect to measures, say μ_+ and μ_- (see, e.g. Halmos, 1950, p. 247). This means that f itself can be represented in the form

$$f(\phi) = \int_M \phi \, d\sigma$$

where $\sigma - \mu_+ - \mu_-$ is a signed measure.

Since C contains the constant functions, it follows that constant functions integrate to 0 with respect to σ, and this means that by a renormalization, it is possible to take μ_+ and μ_- to be probability measures. Because $f(\psi) < 0$, it follows that $\psi \notin C^+$, a contradiction. ∎

4. Some Specific Cases. The following examples include the best known special cases; other examples have been studied by Mosler and Scarsini (1991) and by Bergmann (1991).

4.1. EXAMPLE (USUAL STOCHASTIC ORDERING). Let $L = M = \mathbb{R}$, and let C consist of all increasing functions. Then C is complete, and $\leq_C$ is the usual notion of stochastic order. This order is also generated by certain other convex cones of functions; for example, the cone of non-negative increasing functions, or the cone of increasing functions ϕ such that $\phi(0) = 0$. These alternative cones are not complete or even stochastically complete because neither cone contains all constant functions.

The set of non-negative increasing functions is the smallest convex cone containing the set A of increasing step functions that is closed under monotone convergence. Thus, according to Propositions 3.3 and 3.4 one can check that $X \leq^{\text{st}} Y$ by checking that $E\phi(X) \leq E\phi(Y)$ for all increasing step functions

ϕ. Of course this just says that the respective survival functions $\bar{F}$ and $\bar{G}$ of X and Y satisfy

$$\bar{F}(x) \leq \bar{G}(x) \text{ for all } x. \tag{4.1}$$

Various multivariate versions of Example 4.1 have been proposed, but one is of prime importance.

4.1.a. Let $L = M = \mathbb{R}^n$ and let $C = C_1 :=$ {increasing functions $\phi : \mathbb{R}^n \to \mathbb{R}$}. The cone C_1 is the smallest closed convex cone containing the indicator functions of all upper sets and the constant functions. It is easy to see that C_1 is complete.

4.1.b. Let $L = M = \mathbb{R}^n$ and let $C = C_2$ consist of all distribution functions (not necessarily normed) on $\mathbb{R}^n$. Then C_2 is the smallest convex cone closed in the sense of weak convergence that contains indicator functions of upper quadrants, i.e., sets of the form $\{x : x \geq x_0\}$, and so it provides a natural multivariate version of (4.1). but when $n > 1$, the cone C_2 is not complete, and in fact, $C_2^* = C_1$.

4.1.c. Let $L = M = \mathbb{R}^n$ and let $C = C_3$ consist of all survival functions (not necessarily normed) on $\mathbb{R}^n$. Then C_3 is the smallest convex cone containing the indicator functions of lower quadrants. Of course the ordering obtained from this example is not the same as that obtained from the cone of 4.1.b, but the two orderings are equivalent for comparisons of bivariate distributions with equal marginals.

4.2. EXAMPLE (BALAYAGE ORDERING). Suppose that M is a compact convex subset of a locally convex space and suppose that C consists of all continuous convex (or alternatively, concave) functions defined on M. In its simplest form, this example was studied by Karamata (1932), and in more general setting the ordering arises in the comparison of statistical experiments (Blackwell, 1953). It also arises in probabilistic potential theory, where it is often called *balayage ordering*, and it provides the setting for Choquet's theorem (see, e.g., Phelps, 1966). In this context other cones of functions, such as superharmonic functions or excessive function, are also of interest, and consequently some results for the cone of convex functions have been generalized (see Meyer, 1966, Chapter XI, Section 3, or Alfsen, 1971, Chapter I, Section 5).

The ordering of Example 4.1 might be thought of as an ordering of location, and the ordering here can then be thought of as an ordering of spread about a common expectation. The properties of interest here are mostly different from the properties studied in this paper. As already mentioned, for this example, C^* consists of all functions mapping M to $\mathbb{R}$, so $x \leq_C y$ only if $x = y$; the stochastic completion C^+ of C consists of upper semicontinuous convex functions.

4.3. Example (Second Order Stochastic Dominance). When C consists of the cone of increasing concave functions, the resulting order is of importance in economics for comparing risks (Hadar and Russell, 1969; Hanoch and Levy, 1969; and Rothschild and Stiglitz, 1970). In this context, the ordering of Example 4.1 is called "first order stochastic dominance," and functions in C are utility functions.

4.4. Example (Stochastic Majorization). Let $L = M = \mathbb{R}^n$, and let C be the cone of Schur-convex functions. As noted in Example 2.6, this cone is complete. For further results and references, see Marshall and Olkin (1979, Chapter 11) and Rüschendorf (1981).

4.4.a. The cone C_1 of permutation symmetric convex functions is often encountered in applications but as mentioned in Examples 2.7 and 3.13, its properties are somewhat limited.

4.4.b. The cone of quasi-convex functions arises in the context of Theorem 3.11 because this cone is just $\widetilde{C}_1$. As such, it may deserve more attention than it has yet received. See Levhari, Paroush and Peleg (1975).

4.5. Example (Peakedness). Let $L = M = \mathbb{R}^n$, and let C consist of all centrally symmetric non-negative quasi-concave functions. If X and Y are random variables with centrally symmetric distributions and if $X \leq_C^{\text{st}} Y$, then X is said to be *less peaked* than Y. This definition, due to Birnbaum (1948) in the univariate case, has been studied by various authors (see Dharmadhikari and Joag-Dev, 1988, p. 160, and Bergmann, 1991).

The cone of this example is not complete, and in fact its completion consists of all reflection symmetric functions ϕ such that $\phi(\alpha x) \leq \phi(x)$ for all x and all α in $[0, 1]$. Of course the cone C does not even include all constant functions. On the other hand, it is clear that $C = \widetilde{C}$ where $\widetilde{C}$ is defined in Notation 3.10. This means that $X \leq_C^{\text{st}} Y$ if and only if $\phi(X) \leq^{\text{st}} \phi(Y)$ for all ϕ in C.

Because C is generated as in Proposition 3.5 from indicator functions of centrally symmetric convex sets, as well as the centrally symmetric log-concave functions, various equivalent conditions for peakedness comparisons can be given.

4.6. Example (Concordance). If $L = \mathbb{R}^2$ and suppose that C consists of the L-superadditive functions, i.e., functions ϕ for which

$$\phi(\alpha_1+\delta_1, \alpha_2+\delta_2)+\phi(\alpha_1-\delta_1, \alpha_2-\delta_2) \geq \phi(\alpha_1+\delta_1, \alpha_2-\delta_2)+\phi(\alpha_1-\delta_1, \alpha_2+\delta_2)$$

whenever δ_1, $\delta_2 \geq 0$ Then $X \leq_C^{\text{st}} Y$ if and only if X is "less concordant" than Y in the sense of Cambanis, Simon and Stout (1976), Tchen (1976), or Tchen (1980). Because of its connection with the notion of "positive quadrant dependence" (Lehmann, 1966), this ordering was introduced and studied by Yanagimoto and Okamoto (1969). See also Marshall and Olkin (1979, p. 382) and Rüschendorf (1980). It is easily shown that in this case, X and Y necessarily have the same marginals.

The convex cone of L-superadditive functions is stochastically complete but not complete; in fact, its completion consists of all real functions defined on $\mathbb{R}^2$.

4.7. EXAMPLE (SCALED ORDER STATISTICS). Scarsini and Shaked (1987) define a preordering $\leq$ of non-negative random vectors by the condition that the kth order statistic of $a_1X_1, \cdots, a_nX_n$ be stochastically smaller than the kth order statistic of $a_1Y_1, \cdots, a_nY_n$ for all $a_i > 0$, $i = 1, 2, \cdots, n$. They identify a set of functions ϕ for which (3.1) implies $X \leq Y$ in their ordering. It would be of interest to characterize the convex cone generated by their set of functions.

REFERENCES

ALFSEN, E. M. (1971). *Compact Convex Sets and Boundary Integrals.* Springer-Verlag: Berlin, Heidelberg, New York.

BIRNBAUM, Z. W. (1948). On random variables with comparable peakedness. *Ann. Math. Statist.* **19**, 76–81.

BERGMANN, R. (1991). Stochastic orders and their application to a unified approach to various concepts of dependence and association. This volume.

BLACKWELL, D. (1951). Comparison of experiments. *Proc. Second Berkeley Symp. Math. Statist. Probab.*, 93–102.

CAMBANIS, S., SIMONS, G.and STOUT, W. (1976). Inequalities for $EK(X, Y)$ when the marginals are fixed. *Z. Wahrsch. Verw. Gebiete* **36**, 285–294.

DAY, M. M. (1962). *Normal Linear Spaces.* Academic Press: New York.

DHARMADHIKARI, S., and JOAG-DEV, K. (1988). *Unimodality, Convexity, and Applications.* Academic Press: New York.

HADAR, J. and RUSSELL, W. R. (1969). Rules for ordering uncertain prospects. *Amer. Econom. Rev.* **59**, 25–34.

HANOCH, G. and LEVY, H. (1969). The efficiency analysis of choices involving risk. *Rev. Econom. Stud.* **36**, 335–346.

HALMOS, P. R. (1950) *Measure Theory.* Van Nostrand: Princeton, NJ.

KAMAE, T., KRENGEL, U., and O'BRIEN, G. L. (1977). Stochastic inequalities on partially ordered spaces. *Ann. Probab.* **5**, 899–912.

KARAMATA, J. (1932). Sur une inégalité rélative aux fonctions convexes. *Publ. Math. Univ. Belgrade* **1**, 145–148.

KIMELDORF, G., and SAMPSON, A. (1987). Positive dependence orderings. *Ann. Inst. Statist. Math.* **39A**, 113–128.

LEHMANN, E. (1955). Ordered families of distributions. *Ann. Math. Statist.* **26**, 399–419.

LEHMANN, E. (1966). Some concepts of dependence. *Ann. Math. Statist.* **37**, 1137–1153.

LEVHARI, D., PAROUSH, J. and PELEG, B. (1975). Efficiency analysis for multivariate distributions. *Rev. Econom. Stud.* **42**, 87–91.

MARSHALL, A. W. and OLKIN, I. (1979). *Inequalities: Theory of Majorization and its Applications.* Academic Press: New York.

MEYER, P. A. (1966). *Probability and Potentials.* Blaisdell: Waltham, MA.

MOSLER, K. C. and SCARSINI, M. (1991). Some theory of stochastic dominance. This volume.

PHELPS, R. R. (1966). *Lectures on Choquet's Theorem.* Van Nostrand: Princeton, NJ.

ROTHSCHILD, M. and STIGLITZ, J. (1970). Increasing risk: I. A definition. *J. Econom. Theory* **2**, 225–243.

RÜSCHENDORF, L. (1980). Inequalities for the expectation of Δ-monotone functions. *Z. Wahrsch. Verw. Gebiete* **54**, 341–349.

RÜSCHENDORF, L. (1981). Orderings of distributions and rearrangement of functions. *Ann. Probab.* **9**, 276–283.

SCARSINI, M. and SHAKED, M. (1987). Ordering distributions by scaled order statistics. *Z. Oper. Res.* **31**, A1–A13.

STRASSEN, V. (1965). The existence of probability distributions with given marginals. *Ann. Math. Statist.* **36**, 423–439.

TCHEN, A. H.-T. (1976). Inequalities for distributions with given marginals. Tech. Report No. 19, Department of Statistics, Stanford University, Stanford, CA.

TCHEN, A. H.-T. (1980). Inequalities for distributions with given marginals. *Ann. Probab.* **8**, 814–827.

YANAGIMOTO, T. and OKAMOTO, M. (1969). Partial orderings of permutations and monotonicity of a rank correlation statistic. *Ann. Inst. Statist. Math.* **21**, 489–506.

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