Stochastic Orders and Decision under Risk
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OPTIMAL STOPPING OF LIFE-TESTING: USE OF STOCHASTIC ORDERINGS IN THE CASE OF CONDITIONALLY EXPONENTIAL LIFETIMES

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Some items with conditionally independent and exponential lifetimes are tested simultaneously. One wants to determine the optimal time to stop the experiment and the optimal decision between two, where optimality is relative to a given cost structure. We show how to formulate this problem as an optimal stopping problem for a suitable continuous time Markov process, which we prove to be stochastically monotone. Next we discuss how this monotonicity property is crucial to obtain an explicit solution of the system of variational inequalities arising from the optimal stopping problem.

Let $T_1, T_2, \cdots, T_n$ be the lifetimes of n items for which the statistical model $\{f^{(n)}(\cdot \mid \lambda)\}$ is specified ($T_1, \cdots, T_n$ are conditionally independent given Λ and $f^{(n)}(\cdot \mid \lambda)$ denotes the conditional density of $T_1, \cdots, T_n$ given $\Lambda = \lambda$.)

We begin to test the items simultaneously and at every instant $t > 0$ we can decide whether to stop or to continue the experiment. During the experiment we observe events of the form

$$\begin{aligned}\{T_{(1)} > t\}, &\{T_{(k+1)} > t, T_{(1)} \le t_1, \cdots, T_{(k)} \le t_k\},\\ &\qquad 0 \le t_1 \le \cdots \le t_k \le t,\ k = 1, \cdots, n-1, \\ &\qquad \text{or } \{T_{(1)} \le t_1, \cdots, T_{(n)} \le t_n\}, 0 \le t_1 < \cdots < t_n \le t,\end{aligned} \tag{1}$$

where $T_{(1)}, \cdots, T_{(n)}$ are the order statistics of $T_1, T_2, \cdots, T_n$. For every $t > 0$, let $\mathcal{F}_t$ be the σ-algebra generated by the events of the form (1). The flow $\{\mathcal{F}_t\}_{t \ge 0}$ (where $\mathcal{F}_0$ denotes the trivial σ-algebra) is the *observed history*.

When we stop the experiment we must choose between the two actions a_1 and a_2. The choice of a_i gives rise to a cost per item l_i which is a function of Λ. In addition, there may be a cost for running the experiment. We face

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the decision problem of determining the optimal time to end the test. On the other hand, in a Bayesian context, we also want to be able to decide whether to end the test or not, based on the information collected up to the current time, so we will restrict the choice of the terminal time σ to the class of $\{\overline{\mathcal{F}}_+\}$-stopping times, where, for each t, $\overline{\mathcal{F}}_{t+}$ is a σ-algebra which differs from $\mathcal{F}_{t+}$ $\left(\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_t\right)$ only by zero probability sets, that must be introduced for technical reasons (essentially in order to make entrance times into all Borel sets stopping times, cf. Shiryaev (1973), page 17). If the life-testing procedure is arrested at a stopping time σ, the Bayes terminal decision will give rise to the risk

$$\gamma_\sigma = \min\{\mathbb{E}[l_1(\Theta) \mid \overline{\mathcal{F}}_{\sigma+}], \ \mathbb{E}[l_2(\Theta) \mid \overline{\mathcal{F}}_{\sigma+}]\}. \tag{2}$$

Of course the simplest situation to analyze is obtained by assuming that $T_1, \cdots, T_n$ are conditionally independent exponentially distributed given Λ, where Λ is a nonnegative random variable:

$$f^{(n)}(t_1, \cdots, t_n \mid \lambda) = \lambda^n \exp\left\{-\lambda \sum_i t_i\right\} \qquad \lambda \geq 0. \tag{3}$$

The most relevant property of the statistical model (3) is that $\{f^{(1)}(t \mid \lambda)\}$ is an *exponential family* and, as such, it has, in particular, a *monotone likelihood ratio*: for $t_1 > t_2$ and $\lambda_1 > \lambda_2$

$$f^{(1)}(t_1 \mid \lambda_1) f^{(1)}(t_2 \mid \lambda_2) - f^{(1)}(t_1 \mid \lambda_2) f^{(1)}(t_2 \mid \lambda_1) < 0. \tag{4}$$

Moreover there exists a one-dimensional sufficient statistic.

Most of the existing literature deals with *discrete-time sequential testing* (the lifetimes are observed in sequence and after each observation one must decide whether to go on with a further observation or to stop and choose one of the terminal decisions). The optimal sequential Bayes procedure turns out to be a *generalized sequential ratio test* (see Sobel (1953), Brown, Cohen, and Strawderman (1979)) and thus to have a certain monotonicity property. However, most real life-testing problems cannot be modeled as discrete time sequential problems, since one wants to allow to stop the observation at a generic time instant (which may give rise to "survival data"). Moreover the actual necessity of saving time usually suggests to test the items simultaneously. For these reasons one is led to consider the problem of optimal stopping in continuous time.

We shall introduce below a continuous time two-dimensional stochastic process $\{Z(t)\}$ corresponding to the statistical model (3). Existence of a fixed dimension sufficient statistic for the statistical model (3) translates into the

Markov property for $\{Z(t)\}$, while (4) yields that $\{Z(t)\}$ is stochastically non-decreasing. The life-testing problem introduced above can then be reduced to an *optimal stopping problem* for $\{Z(t)\}$. In control theory, the stopping region for a Markov process can be characterized by means of a system of equations and inequalities on the *value function*, involving the infinitesimal generator (see Grigelionis and Shiryaev (1966), Shiryaev (1973)). With the terminology of the theory of partial differential equations, the system thus obtained gives rise to a *free-boundary problem*. It is not, in general, possible to solve such a problem.

The main purpose of this paper is to illustrate how one can overcome this difficulty by bringing together the point of view of optimal control theory and that of discrete-time sequential analysis. In fact one can exploit monotonicity properties of the stochastic process $\{Z(t)\}$ – obtained by stochastic ordering arguments – to specialize and ultimately solve the above mentioned free-boundary problem.

For each $t \geq 0$, let $H(t)$ denote the number of failures already observed at t, $H(t) = \sum_i I_{(T_i \leq t)}$, and $Y(t)$ denote the "total time on test" for the random lifetimes $T_1, \cdots, T_n$:

$$Y(t) = \sum_{i=1}^{n} \min(t, t_i) = t_1 + \cdots + t_{H(t)} + (n - H(t))t$$

and let $Z(t) \equiv (K(t), Y(t))$, where we set $K(t) = n - H(t)$. Now $K(t)$ denotes the number of items still alive at time t and we can write

$$Y(t) = \int_0^t K(s)ds.$$

The stochastic process $\{Z(t)\}$ describes completely our life-testing experiment. The qualitative behavior of $\{Z(t)\}$ is the following: $K(0) = n$, $Y(0) = 0$. At random times $T_{(1)}, \cdots, T_{(n)}$, $\{K(t)\}$ decreases by 1, until it reaches 0, where the process is absorbed. Until the absorption, $\{Y(t)\}$ is always increasing, but with decreasing rate $\{K(t)\}$.

As is well known, under the statistical model (3), for every $t > 0$, $(K(t), Y(t))$ is sufficient with respect to Λ (see e.g. Barlow and Proschan (1988)), i.e. letting $\Pi_{k,y}$ be the probability distribution on $[0, \infty)$ defined by

$$d\Pi_{k,y}(\lambda) = \frac{\lambda^{n-k} \exp\{-\lambda y\} d\Pi(\lambda)}{\int_0^\infty \lambda^{n-k} \exp\{-\lambda y\} d\Pi(\lambda)}, \quad k = 0, 1, \cdots, n, \quad y \in [0, \infty)$$

(where Π denotes the "prior" distribution on the parameter Λ and is assumed to have at least n finite moments), $\Pi_{K(t),Y(t)}$ is the conditional distribution of

Λ, given $\overline{\mathcal{F}}_{t^+}$. In particular, for any measurable function $f : [0, \infty) \to \mathcal{R}$ such that $\int_0^\infty |v(\lambda)| d\Pi(\lambda) < \infty$, it holds

$$\mathbb{E}[v(\Lambda) \mid \overline{\mathcal{F}}_{t^+}] = \int_0^\infty v(\lambda) d\Pi_{K(t),Y(t)}(\lambda). \tag{5}$$

From now on we shall use the notation

$$\hat{v}(k, y) \equiv \int_0^\infty v(\lambda) d\Pi_{k,y}(\lambda) \tag{6}$$

At each time t, conditionally on $K(t)$ and Λ, the residual lifetimes of the $K(t)$ items still alive are independent exponentials, independent of $\overline{\mathcal{F}}_{s^+}$, for every $s < t$. On the other hand, due to the sufficiency of $(K(t), Y(t))$, the "posterior" distribution of Λ given $\overline{\mathcal{F}}_{t^+}$ is $\Pi_{K(t),Y(t)}$. Therefore $\{Z(t)\}$ is an $\{\overline{\mathcal{F}}_{t^+}\}$-Markov process. It can be easily seen that $\{Z(t)\}$ is time-homogeneous and that its infinitesimal generator is

$$\begin{aligned} Af(k, y) &= \lim_{\Delta t \to 0} \frac{\mathbb{E}[f(Z(t + \Delta t)) \mid Z(t) = (k, y)] - f(k, y)}{\Delta t} \\ &= k \left\{ \frac{df}{dy} + \hat{\lambda}(k, y)[f(k-1, y) - f(k, y)] \right\}. \end{aligned}$$

For a more complete analysis of the process $\{Z(t)\}$ see Costantini and Spizzichino (1990).

Now we wish to emphasize the dependence of the law of $\{Z(t)\}$ on the initial number of components and the prior distribution of the parameter Λ by setting, for every probability distribution Π on $[0, +\infty)$, for every $n \in \mathbb{N}$, and for every bounded measurable function u defined on $\mathbf{Z}_+ \times [0, \infty) \to \mathcal{R}$

$$\mathbb{E}^{(n,\Pi)}[u(Z(t))] \equiv \mathbb{E}[u(Z(t))],$$

$$\mathbb{E}_{k,y}^{(n,\Pi)}[u(Z(t))] \equiv \mathbb{E}[u(Z(t+s)) \mid K(s) = k, Y(s) = y],$$
$$\text{for } s > 0, t \geq 0, y \geq 0, k = 0, 1, \cdots, n.$$

For our purposes the following properties of $\{Z(t)\}$ are particularly relevant:

(a) For any nondecreasing function $v : [0, \infty] \to \mathcal{R}$ such that $\int_0^\infty |v(\lambda)| d\Pi(\lambda) < \infty$, the function $\hat{v}(k, y)$ defined by (6) is nonincreasing both in k and y.

(b) For any nondcreasing function $u : \mathbf{Z}_+ \times [0, \infty) \to \mathcal{R}$, $\mathbb{E}^{(n,\Pi)}[u(Z(t))]$ is nondecreasing in n and the following implication holds:

$$\Pi' \leq^{st} \Pi \Rightarrow \mathbb{E}^{(n,\Pi')}[u(Z(t))] \geq \mathbb{E}^{(n,\Pi)}[u(Z(t))]. \tag{7}$$

(c) For any nondecreasing function $u : \mathbf{Z}_+ \times [0, \infty) \to \mathcal{R}$ it holds

$$\mathbb{E}_{k,y}^{(n,\Pi)}[u(Z(t))] \leq \mathbb{E}_{k',y'}^{(n,\Pi)}[u(Z(t))] \; \forall \; k \leq k', y \leq y', \forall t \geq 0 \tag{8}$$

i.e.. $\{Z(t)\}$ is stochastically nondecreasing.

Property (a) can be obtained first by observing that for the family of the conditional densities (with respect to Π) of Λ given $(K(t) = k,\; Y(t) = y)$ we have the monotonicity in λ of the likelihood ratio, both with respect to k and y, so that we can apply Lemma 2, page74 of Lehmann (1959). As far as property (b) is concerned, the first statement can be easily verified, while (7) follows by observing that $\mathbb{E}^{(n,\Pi)}[u(Z(t)) \mid \Lambda = \lambda]$ is nonincreasing in λ (see again Lemma 2, page 74 of Lehmann (1959)). Finally, by the conditional independence of $T_1, T_2, \cdots, T_n$ and the lack of memory of the exponential distribution, we have

$$\mathbb{E}_{k,y}^{(n,\Pi)}[u(Z(t))] = \mathbb{E}^{(k,\Pi_{k,y})}[u(Z(t))]. \tag{9}$$

Inequality (8) can then be obtained by combining (a) and (b).

Note that, just as the sufficiency of the statistic $(K(t), Y(t))$ for Λ, for every $t > 0$,, translates into the Markov property for the process $\{Z(t)\}$, the monotone likelihood ratio property (4) translates into (c).

Now we turn to the discussion of the problem of determining the optimal time to end our life testing experiment.

Let a_1 and a_2 be the two terminal actions between which we must choose at the end of the test and let $l_1(\lambda)$ and $l_2(\lambda)$ be the corresponding loss functions, which we assume to be continuous. Typically we can think of a_1 and a_2 as a "conservative" and an "optimistic" action respectively, and we can assume that $l_1(\lambda) - l_2(\lambda)$ is a nonincreasing function of λ; we will also suppose

$$\lim_{\lambda \to \lambda_0^+} [l_1(\lambda) - l_2(\lambda)] > 0, \;\text{ where }\; \lambda_0 = \inf\{\lambda \geq 0 \mid \Pi([0, \lambda]) > 0\}.$$

Using the notation introduced in (6), the risk γ_σ in (2) gives rise to the terminal cost

$$g(k, y) = k \min\{\hat{l}_1(k, y), \hat{l}_2(k, y)\} + c(n - k) \tag{10}$$

whre c is the cost of the failure of a component during the test.

Suppose now that the cost of running the test up to a time σ is of the form $\int_0^\sigma \psi(K(s), Y(s))ds$. In such a case, the problem of finding the optimal Bayes stopping strategy for the life-testing experiment can be formulated as an optimal stopping problem for the Markov process $\{Z(t)\}$: find a Markov time σ^* such that $\mathbb{E}\{J_{\sigma^*}\} \leq \mathbb{E}\{J_\sigma\} \; \forall \; \sigma \in \mathcal{M}$, where

$$J_\sigma = g(K(\sigma), Y(\sigma)) + \int_0^\sigma \psi(K(s), Y(s))ds$$

and $\mathcal{M}$ is the class of all $\{\overline{\mathcal{F}}_{t+}\}$-stopping times. Note that in the case when ψ has the form $\psi(k,y) = pk$ $(p > 0)$, J_σ becomes

$$J_\sigma = g(K(\sigma), Y(\sigma)) + pY(\sigma),$$

so that the running cost becomes itself a terminal cost.

We shall illustrate only the above case. Let us define the "value function" s

$$s(k,y) = \inf_{\sigma\in\mathcal{M}} \mathbb{E}_{k,y}[g(K(\sigma), Y(\sigma)) + pY)\sigma)]. \tag{11}$$

By Theorem 4, page 104 in Shiryaev (1973), an optimal stopping time σ^* is given by $\sigma^* = \inf\{t \geq 0 : Z(t) \in \Gamma\}$, where Γ is the region defined by $\Gamma \equiv \{(k,y) : s(k,y) = g(k,y) + py\}$. The value function s satisfies the following system of conditions, for $k = 0, \cdots, n$

$$g(k,y) + py - s(k,y) \geq 0$$

$$As(k,y) \geq 0 \tag{12}$$

$$[g(k,y) + py - s(k,y)]As(k,y) = 0.$$

As already mentioned, it is not in general possible to find the stopping region Γ by means of analytical tools; in particular it may happen that system (12) has more than one solution so that it does not determine the value function s. However one can exploit the regularity and the monotonicity properties of $\{Z(t)\}$ to get additional information on s and Γ.

A first hint on the shape of Γ comes from a fundamental consequence of (4), namely the *diminishing variation of sign property* (see Karlin and Rubin (1956), that allows us to conclude that there exist numbers y_k $(k = 1, \cdots, n)$ such that $\min\{\hat{l}_1(k,y), \hat{l}_2(k,y)\} = \hat{l}_1(k,y)$ if and only if $y \leq y_k$ (for a more detailed discussion of loss functions with intrinsic meaning in life-testing see Clarotti and Spizzichino (1989) and Spizzichino (1990)). A more inspiring observation is the following: the analog, in continuous time, of the fact that the optimal Bayes strategy is a generalized likelihood ratio test is that $\Gamma \cap \{(k,y) : k \geq 1\}$ is the disjoint union of a nonincreasing closed set and a nondecreasing one. Recall that a subset of a partially ordered set is *nondecreasing* (*nondecreasing*) if its indicator function is nondecreasing (nonincreasing) with respect to the given partial order. In the state space of $\{Z(t)\}$, a nondecreasing (nonincreasing) closed set is simply given by

$$\bigcup_{k=0}^{n}\{k\}\times[y_k,+\infty),\ y_k\geq y_{k+1},\ k=0,\cdots,n-1$$

$$\left(\bigcup_{k=0}^{n}\{k\}\times[0,y_k),\ y_k\geq y_{k+1},\ k=0,\cdots,n-1\right).$$

Therefore, if $\Gamma\cap\{(k,y):k\geq 1\}$ is the disjoint union of a nonincreasing closed set and a nondecreasing one, then it is determined by pairs $(y_k^{(1)},y_k^{(2)})$, $k=1,2,\cdots,n-1$, where $0\leq y_k^{(1)}<y_k^{(2)}$ or $0=y_k^{(1)}=y_k^{(2)}$ ($(k,y_k^{(i)})$ are the boundary points of Γ), and, under some mild regularity assumptions, (12) yields the following system of equations, for $k=1,\cdots,n$:

$$\begin{aligned}
\frac{ds}{dy}(k,y)&=\hat{\lambda}(k,y)[s(k,y)-s(k-1,y)]\ \text{for}\ y_k^{(1)}<Y<y_k^{(2)}\ \text{and}\ y_k^{(2)}>0\\
s(k,y_k^{(1)})&=k\hat{l}_1(k,y_k^{(1)})+c(n-k)+py_k^{(1)}\ \text{for}\ y_k^{(1)}>0\\
s(k,y_k^{(2)})&=k\hat{l}_2(k,y_k^{(2)})+c(n-k)+py_k^{(2)}\\
s(0,y)&=nc+py\ \ \forall y.
\end{aligned}\tag{13}$$

By adding regularity conditions on the derivative of s at $y_k^{(1)}$ and $y_k^{(2)}$, namely

$$\frac{ds}{dy}(k,y_k^{(i)})=k\frac{d\hat{l}_i}{dy}(k,y_k^{(i)})+p\tag{14}$$

one obtains a system for which it is much more feasible to prove existence and uniqueness of solution than for (12).

This formulation brings together the point of view and the techniques of discrete-time sequential analysis and those of optimal control theory. Our program is to study the system (13)–(14) for specific choices of l_1 and l_2 and to show that the solution coincides with the value function.

In Costantini and Spizzichino (1990), this program has been carried out completely for

$$l_1(\lambda)\equiv c,\ \ l_2(\lambda)=\int_0^\infty C(r)\lambda\exp\{-\lambda r\}dr,\quad p=0,$$

where C is a decreasing function such that $\lim_{r\to 0^+}C(r)>c>\lim_{r\to\infty}C(r)$. This is the *burn-in* decision problem with $p=0$ as introduced in Clarotti and Spizzichino (1990) (as a general reference on non-Bayes burn-in procedures see Jensen and Petersen (1982)). In this case a_1 is the action of discarding the item while a_2 is the action of putting it into operation. Note that, since

$l_1(\lambda) \equiv c$ and $p = 0$, if we start the test at all and stop it before $\{K(t)\}$ reaches 0, we necessarily take the decision a_2 for all still alive items. For this problem, under some very mild regularity hypotheses and a monotonicity assumption on the function

$$Ag(k,y) = I_{\{(k,y):k\geq 1\}} k\hat{\lambda}(k,y)[c - \hat{l}_2(k-1,y)]$$

we have been able to show that $\Gamma \cap \{(k,y) : k \geq 1\}$ is a closed nondecreasing region ($y_k^{(1)} = 0$, $k = 1, \cdots, n$) and $(s(k,\cdot),\ y_k^{(2)})$, $k = 1, \cdots, n$ is the unique solution of (13)–(14). As a particular application we have considered prior distributions II in the *gamma* family and cost functions C linear or piecewise constant. for this class of applications we have verified that all required assumptions are satisfied and we have derived an algorithm that computes the solution explicitly.

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