Stochastic Orders and Decision under Risk
IMS Lecture Notes - Monograph Series (1991)

REMARKS ON A RANDOM SURFACE

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A simple discrete random surface is defined. Its stochastic ordering/ inequality properties are discussed and some open problems are presented.

In this paper we discuss a simple discrete random surface introduced within a statistical mechanics context in [AN1, AN2]. Our purpose here is to survey some stochastic ordering/inequality properties and some easily stated open problems. For the sake of simplicity, we will mainly deal with a limiting case (corresponding to infinite temperature) of the model treated in [AN1, AN2]. We begin by discussing some of the physical motivation behind such random surface models. For more physical background and for other random surface models, see the papers in [DD] and the references in [AN1, AN2].

Consider a flat horizontal smooth solid substrate, in thermal equilibrium at temperature T, with two immiscible fluids lying above it – one a liquid labelled A (e.g., a lubricant) and the other a gas labelled B (e.g., air). It can happen that above some temperature T_w, there is a macroscopic slab of A between the substrate and B, while below T_w, A is squeezed out (or is of microscopic thickness). Above T_w, one says that A wets the substrate perfectly, and the transition at T_w, is known as a wetting phase transition. To model this phenomenon, one may regard the interface between fluids A and B as a two-dimensional surface and postulate an energy function E on some space of allowed configurations of the surface. At temperature T, the surface is random with a probability density proportional to $\exp(-E/T)$. When $T = \infty$, all allowed configurations are equally likely (see property (v) below).

* Research supported in part by National Science Foundation Grants DMS-8514834 and DMS-8902156, and by AFOSR Contract F49620-86-C0130 to the Arizona Center for Mathematical Sciences under the U.R.I. Program.

AMS 1980 Subject Classification: Primary 60K35; Secondary 82A25, 60D05

Key words and phrases: Random surface, association, FKG inequalities, wetting, roughening.

In [AN1, AN2], certain discrete versions of such random surfaces were proposed. These are parametrized by both T and a finite set Λ describing the horizontal extent of the substrate. The wetting transition is manifested by the divergence (resp., boundedness) of the surface height as the substrate becomes infinite, when $T > T_w$ (resp., $T < T_w$). It is not known whether the surface height divergence is accompanied by "roughening" – i.e., by the divergence of height *differences* above widely separated points on the substrate (see (2) below). Henceforth, we restrict ourselves to the random surface when $T = \infty$. For this case, we will prove that the surface height does indeed diverge as Λ becomes infinite and we will discuss the open problem of roughening.

The surface is described by a function which gives an integer valued height above each point i in the discrete plane $\mathbf{Z}^2$. We will "pin down" the surface in the complement Λ^c of some finite subset Λ (e.g., a rectangle) of $\mathbf{Z}^2$; the height function will then be denoted $H^\Lambda \equiv \{H_i^\Lambda : i \in \mathbf{Z}^2\}$. For any finite $\Lambda \subset \mathbf{Z}^2$, the random surface (pinned down outside Λ) is defined by the following properties of the height random field H^Λ:

(i) $H_i^\Lambda = 0$ for each $i \in \Lambda^c$.

(ii) H_i^Λ is a non-negative integer for each $i \in \Lambda$.

(iii) $|H_i^\Lambda - H_j^\Lambda| \leq 1$ when i and j are nearest-neighbors in $\mathbf{Z}^2$ (i.e., when $\|i - j\| = 1$, where $\|\cdot\|$ denotes Euclidean length).

(iv) For every $i \in \Lambda$, there is some nearest-neighbor path from i to Λ^c along which the height is decreasing (i.e., non-increasing). [This restriction allows the surface to have multiple peaks and saddle points but no "hidden valleys".]

(v) All height functions satisfying (i)–(iv) are equally likely.

Proposition 1, from [AN1, AN2], gives some of the basic properties of the H^Λ's. Its proof is based on a representation of the random surface, given in Proposition 2 below, in terms of i.i.d. ± 1 valued variables.

PROPOSITION 1. *(a) H^Λ is stochastically increasing in Λ; i.e., $\Lambda' \supset \Lambda$ implies $E(f(H^{\Lambda'})) \geq E(f(H^\Lambda))$ for any increasing function f (of finitely many height variables).*

(b) H^Λ is associated in the sense of [EPW]; i.e.,

$$\mathrm{Cov}(f(H^\Lambda), g(H^\Lambda)) \geq 0$$

for any increasing functions f and g.

(c) $H^\Lambda \to \infty$ as $\Lambda \to \mathbf{Z}^2$; i.e., for any $i \in \mathbf{Z}^2$ and any finite h,

$$P(H_i^\Lambda \leq h) \to 0 \;\; as \;\; \Lambda \to \mathbf{Z}^2. \tag{1}$$

Part (c) of Proposition 1 shows that as Λ becomes large, the surface height diverges; it is not known whether the surface also becomes "rough":

Open Problem A. Property (iii) implies that for fixed j and k, $H_j^\Lambda - H_k^\Lambda$ stays bounded as $\Lambda \to \mathbf{Z}^2$; does the limiting height difference diverge as $\|j - k\| \to \infty$? I.e., is it true that for any h,

$$\lim_{\|j-k\|\to\infty} \limsup_{\Lambda\to\mathbf{Z}^2} P(|H_j^\Lambda - H_k^\Lambda| \leq h) = 0 \text{ ?} \tag{2}$$

The next proposition, from [AN1, AN2], will be used to simplify this open problem (see Open Problem A′ below) and to prove Proposition 1.

PROPOSITION 2. *Let $X \equiv \{X_i : i \in \mathbf{Z}^2\}$ be an i.i.d. symmetric ± 1 valued random field. For each finite Λ, denote by $\Gamma(\Lambda, i)$ the set of nearest-neighbor paths from i to Λ^c. Then the random field H^Λ given by*

$$H_i^\Lambda = \min_{\gamma\in\Gamma(\Lambda,i)} (\textit{no. of sign changes on } X^\Lambda \textit{ along } \gamma) \tag{3}$$

satisfies Properties (i)–(v). Here X^Λ is defined to agree with X in Λ and to be identically $+1$ in Λ^c.

PROOF. Properties (i)–(iii) are fairly obvious. Property (iv) can be seen by choosing the minimizing path in definition (3). Property (v) holds because the possible values of $\{X_i^\Lambda : i \in \Lambda\}$ are equally likely and because (3) defines a one-to-one mapping between these possible values and those of $\{H_i^\Lambda : i \in \Lambda\}$. This latter fact can be seen by noting that

$$X_i^\Lambda = (-1)^{H_i^\Lambda}. \tag{4}$$

PROOF OF PROPOSITION 1. The fields H^Λ defined by (3) are easily seen to be pointwise increasing in Λ (i.e., $\Lambda' \supset \Lambda$ implies $H_i^{\Lambda'}(\omega) \geq H_i^\Lambda(\omega)$ for each i and each ω in the probability space of X) and hence stochastically increasing.

The proof of Part (b) is somewhat complicated because the H^Λ given by (3) is not a monotonic function of X^Λ. However, we define μ^Λ by $\mu_i^\Lambda = 1$ if $H_i^\Lambda = 0$ (otherwise $\mu_i^\Lambda = 0$) and note that μ^Λ *is* an increasing function of X^Λ. We also define the random set

$$L_\Lambda = \{j \in \Lambda : H_j^\Lambda > 0 \text{ and } j \text{ is not a nearest-neighbor of some } i \text{ with } H_i^\Lambda = 0\}; \tag{5}$$

L_Λ is strictly contained in Λ and L_Λ is a decreasing (set-valued) function of μ^Λ. Given μ^Λ (or equivalently, given L_Λ), the random variables $\{X_i^\Lambda : i \in L_\Lambda\}$

are still i.i.d. symmetric while the X_j^Λ's for j a nearest-neighbor of L_Λ are all -1 (and these j's have $H_j^\Lambda = +1$). It follows that

$$\{H_i^\Lambda : i \in \mathbf{Z}^2\} \cong \{(1-\mu_i^\Lambda)(1+H_i^{L_\Lambda}) : i \in \mathbf{Z}^2\}, \tag{6}$$

where $\cong$ denotes equidistribution. $1-\mu^\Lambda$ is an increasing function of $-X^\Lambda$ and hence is associated [H]. By induction on the number of sites in Λ, we may assume that conditional on $1-\mu^\Lambda$, H^{L_Λ} is associated. Furthermore, by Part (a), the conditional distribution of H^{L_Λ} is stochastically increasing as a function of $1-\mu^\Lambda$ (in the language of [J], H^{L_Λ} is a "monotone mixture" with $1-\mu^\Lambda$); it follows [J] that the double family $\{(1-\mu_i^\Lambda), H_j^{L_\Lambda}\}$ is associated. Formula (6) then shows that H^Λ is (equidistributed with) an increasing function of this double family and hence associated.

Finally we prove Part (c) by using (3) and some percolation theory. Let us denote by $\Gamma(i)$ the set of all infinite nearest-neighbor (self-avoiding) paths in $\mathbf{Z}^2$ starting at i. Then, using (3), H_i^Λ converges (a.s.) as $\Lambda \to \mathbf{Z}^2$ to the minimum over $\gamma \in \Gamma(i)$ of the no. of sign changes of X along γ. This will be infinite unless there is an infinite "cluster" of plus sites or of minus sites somewhere in $\mathbf{Z}^2$. But the plus (respectively minus) sites correspond to the occupied sites of a standard independent nearest-neighbor site percolation model on $\mathbf{Z}^2$ with density $\frac{1}{2}$. Since the critical density for percolation (i.e., for having infinite clusters) strictly exceeds $\frac{1}{2}$ [T], it follows that there are no infinite plus (respectively minus) clusters a.s. This completes the proof of Proposition 1.

We conclude the paper with another open problem and an explanation of why its resolution would also resolve the open problem presented earlier.

Open Problem A'. For $\Lambda \subset \mathbf{Z}^2$, define N_Λ as the minimum, over nearest-neighbor paths γ from the origin to Λ^c, of the number of sign changes along γ of X, an i.i.d. symmetric ± 1 valued random field on $\mathbf{Z}^2$. $N_\Lambda \to \infty$ as $\Lambda \to \mathbf{Z}^2$ a.s. Does the distribution spread out as it diverges; i.e., is

$$\lim_{\Lambda \to \mathbf{Z}^2} [\sup_n P(N_\Lambda = n)] = 0 \ ? \tag{7}$$

We remark that N_Λ differs from H_0^Λ by at most ± 1 and hence (7) should be regarded as a "slight" strengthening of (1). To show that (7) would imply (2), we argue as follows. For a given Λ, and $m = 0, 1, 2, \cdots$, define the random regions, $\tilde{\Lambda}_m = \{i \in \Lambda : H_i^\Lambda \geq m\}$. Define $D_m(i)$ to be the Euclidean distance from i to $\tilde{\Lambda}_m^c$ and for given j and k, define M to be the smallest m such that both $D_m(j)$ and $D_m(k)$ are less than K, where K is a function of $||j-k||$ which will be chosen below. Assume (without loss of generality) that $D_M(j) \geq$

$D_M(k)$. Because of the exponential tail of the size distribution of the plus and minus clusters of X, it follows (compare the arguments of [AN2, Sec. V]) that for some $b > 0$ and $C < \infty$,

$$P(D_M(j) \leq K/2) \leq Ce^{-bK}. \tag{8}$$

If $D_M(k) = 0$, then $|H_j^\Lambda - H_k^\Lambda| \geq H_j^{\tilde\Lambda_M}$, which by Proposition 1 and (8) can be made arbitrarily large with probability arbitrarily close to 1 as (first $\Lambda \to \mathbf{Z}^2$ and then) $\|j-k\| \to \infty$ by choosing $K \to \infty$. If $D_M(k) \neq 0$, then

$$H_j^\Lambda - H_k^\Lambda = H_j^{\tilde\Lambda_M} - H_k^{\tilde\Lambda_M}. \tag{9}$$

Define $\tilde\Lambda = \tilde\Lambda_M \cap \{i : \|i-j\| < \|j-k\|/2\}$ and $\tilde\Lambda'$ similarly with j and k interchanged so that $\tilde\Lambda$ and $\tilde\Lambda'$ are disjoint. Since $D_M(j) < K$, it follows that $H_j^{\tilde\Lambda_M}$ is less than some multiple of K and then, again by an exponential cluster size tail argument, that

$$P(H_j^{\tilde\Lambda_M} \neq H_j^{\tilde\Lambda}) \leq C'\|j-k\|^2 e^{-b'\|j-k\|/K} \tag{10}$$

for some $b' > 0$ and $C' < \infty$ with a similar inequality valid when j and k are interchanged along with $\tilde\Lambda$ and $\tilde\Lambda'$. We choose K so that the RHS of (10) goes to zero as $\|j-k\| \to \infty$, (e.g., $K = \sqrt{\|j-k\|}$). Then to show that $|H_j^\Lambda - H_k^\Lambda|$ is large with probability close to 1 as $\|j-k\| \to \infty$, it suffices to show that $|H_j^{\tilde\Lambda} - H_k^{\tilde\Lambda'}|$ is large with probability close to 1. We now condition on $\tilde\Lambda$ and $\tilde\Lambda'$ and note that since they are disjoint, $H_j^{\tilde\Lambda}$ and $H_k^{\tilde\Lambda'}$ are (conditionally) independent. Thus, by further conditioning on $H_k^{\tilde\Lambda'}$,

$$\begin{aligned} P(|H_j^{\tilde\Lambda} - H_k^{\tilde\Lambda'}| \leq h) &\leq \sup_{h'} \sum_{n=h'-h}^{h'+h} P(H_j^{\tilde\Lambda} = n) \\ &\leq \sup_{h'} \sum_{n=h'-h-1}^{h'+h+1} P(N^{\tilde\Lambda - j} = n) \\ &\leq (2h+3) \sup_n P(N^{\tilde\Lambda-j} = n). \end{aligned} \tag{11}$$

Since $\tilde\Lambda - j \to \mathbf{Z}^2$ with probability approaching 1 by (8), the last expression in (11) tends to zero by (7).

REFERENCES

[AN1] Abraham, D. B. and Newman, C. M. (1988). Wetting in a three-dimensional system: an exact solution, *Phys. Rev. Lett.* **61**, 1969–1972.

[AN2] Abraham, D. B. and Newman, C. M. (1989). Surfaces and Peierls contours: 3-d wetting and 2-d Ising percolation, *Comm. Math. Phys.* **125**, 181–200.

[DD] De Coninck, J. and Dunlop, F. (Eds.) (1990). *Wetting Phenomena*, Lect. Notes in Physics, Vol. 354, Springer-Verlag, Berlin – Heidelberg – New York.

[EPW] Esary, J. D., Proschan, F. and Walkup, D. W. (1967). Association of random variables with applications, *Ann. Math. Statist.* **38**, 1466–1474.

[H] Harris, T. E. (1960). A lower bound for the critical probability in a certain percolation process, *Proc. Camb. Phil. Soc.* **56**, 13–20.

[J] Jogdeo, K. (1978). On a probability bound of Marshall and Olkin, *Ann. Stat.* **6**, 232–234.

[T] Tóth, B. (1985). A lower bound for the critical probability of the square lattice site percolation, *Z. Wahrsch. Verw. Gebiete* **69**, 19–22.

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