The Genealogy of Patterns of ESS's

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Summary

In a finite conflict a given payoff matrix **A** may possess **ESS** 's with supports $T_1, T_2, ..., T_k$. Such a set of supports is called a pattern and is attainable since there exists a payoff matrix **A** with **ESS** 's with those supports. This paper presents a summary of previous work aimed at specifying the set of attainable patterns, the most recent results for conflicts with 5 pure strategies and defines the notion of a genealogy of patterns. This genealogy, in which a subpattern is regarded as an 'offspring' is displayed for n = 4 and n = 5 where the genealogy has been produced so as to minimize the number of line-intersections using simulated annealing.

Introduction. A finite conflict is defined by a pair $\{\mathbf{U}, \mathbf{A}\}$ where $\mathbf{U} = \{1, 2, 3 ...n\}$ is the strategy space i. e. there are n pure strategies labeled 1 through n, and **A** is an $n \times n$ matrix whose elements a_{ij} are the payoffs; a_{ij} is the payoff to an individual who plays strategy i and whose opponent plays strategy j. We consider, as in classical game theory (Von Neumann and Morgenstern, 1953), the mixed extension so that the strategies are $\mathbf{p} = (p_1, p_2 ... p_n)$ where $p \in \Delta, \Delta = \{p; p \geq 0, \Sigma p_i = 1\}$. The payoff for strategy **p** against **q** has the expected value $\mathbf{p}^T \mathbf{A}\mathbf{q}$ and this is also the payoff of strategy **p** in a population which is playing **q** on average.

An **ESS**, evolutionarily stable strategy, is a **p** such that for every $\mathbf{q} \neq \mathbf{p}$

(1) $\mathbf{E}(\mathbf{p}, \mathbf{p}) \geq \mathbf{E}(\mathbf{q}, \mathbf{p})$ and

(2) if equality in (1) then $\mathbf{E}(\mathbf{p}, \mathbf{q}) > \mathbf{E}(\mathbf{q}, \mathbf{q})$.

For the generic case of a finite conflict, a strategy **p** is an **ESS** iff (if and only if) $\mathbf{E}(i, \mathbf{p}) = \mathbf{E}(\mathbf{p}, \mathbf{p})$ all $i \in \mathbf{R}(\mathbf{p})$, where $\mathbf{R}(\mathbf{p}) = \{i: p_i > 0\}$ the support of **p**, $\mathbf{E}(j, \mathbf{p}) < \mathbf{E}(\mathbf{p}, \mathbf{p})$, all $j \notin \mathbf{R}(\mathbf{p})$, and if $\mathbf{B} = (a_{ij})$, $i, j \in \mathbf{R}(\mathbf{p})$ then $\mathbf{z}^t\mathbf{B}\mathbf{z}$ is negative definite where z such that $\Sigma z_i = 0$, Haigh(1975).

If **P** is the power set of **U** and $\mathbf{T} = \{T_1, T_2 ... T_k\} \subset P$ then **T** is said to be a pattern, and in the context of conflict theory **T** is said to be an attainable pattern if there exists an **A** (a real $n \times n$ matrix) such that there are precisely k **ESS**'s for **A** and the supports are T_i, $i = 1,2...k$ Vickers and Cannings(1988a). A pattern $\mathbf{V} = \{V_1, V_2 ... V_l\}$ is said to be a subpattern of $\mathbf{T} = \{T_1, T_2 ... T_k\}$ if $k > l$ and for some permutation of $\{1,2, ... l\}$, $\{i_1, i_2 ... i_l\}$ $V_j = T_{i_j}$ $j = 1, ... l$. In this case we write $\mathbf{V} \subset \mathbf{T}$. A fundamental conjecture of Vickers and Cannings(1988a) is that if **T** is attainable and $\mathbf{V} \subset \mathbf{T}$ then **V** is attainable(for $n \geq 2$). In a series of papers Cannings and Vickers(1988, 1989), Vickers and Cannings (1988a, 1988b) the authors have begun an attempt to specify the details of the set of attainable patterns. Results have been essentially of two types; exclusion results which specify certain features which an attainable pattern may not possess, and existence results which demonstrate how the payoff matrix can be constructed to achieve specific classes of patterns. These two constitute the main building blocks of the theory. However the theory is far from complete; there are many patterns which are neither excluded nor demonstrated by these results. This mortar has to be filled in by ad hoc methods and there are still many gaps in the edifice. After briefly reviewing the results previously published, and adding some new ones, we shall consider the problem of representing the whole genealogy of attainable patterns for $n = 4,5$. For given n the genealogy of attainable patterns is the digraph $\mathbf{G} = \{\mathbf{V}, \mathbf{E}\}$ where the vertex-set **V** consists of all attainable patterns(if there is uncertainty then all patterns which have not been shown unattainable are included) and the edge-set $\mathbf{E} = \{(\mathbf{U}, \mathbf{W}) : \mathbf{U} \in \mathbf{V}, \mathbf{W} \in \mathbf{V}, \mathbf{U} \subset \mathbf{W}\}$. Essentially we regard **W** as a 'parent' of **U** and **U** as an 'offspring' of **W**. In contrast to genealogies of biological organisms, at least those known to man, an individual here may have many parents. **G** is of course just a digraph but since (**V**, descendant of) is a poset it seems natural to refer to it as a genealogy and exploit the 'generations' which, see below, exist.

2. Exclusion Results. (1) Bishop and Cannings(1976) proved that if **p** is an ESS with support **R(p)** and **q** has $\mathbf{R}(\mathbf{q}) \subset \mathbf{S}(\mathbf{p})$, where $\mathbf{S}(\mathbf{p}) = \{i: i \in \mathbf{U}, \mathbf{E}(i, \mathbf{p}) = \mathbf{E}(\mathbf{p}, \mathbf{p})\}$ then **q** is not an ESS. In the generic case **R(p)** = **S(p)** so we have that if **p** and **q** are ESS's then neither $\mathbf{R}(\mathbf{p}) \subset \mathbf{R}(\mathbf{q})$ nor $\mathbf{R}(\mathbf{q}) \subset \mathbf{R}(\mathbf{p})$. Therefore an attainable pattern is an antichain (see Anderson(1988)).

The four following results are given in Vickers and Cannings(1988a). Here $\mathbf{Q} \subset \mathbf{U}$ and $(1, 2) \cup \mathbf{Q}$ is written as $(1, 2, \mathbf{Q})$ etc.

(2)If $\mathbf{Q} \subset \mathbf{U} \backslash (1,2,3)$, $T_1 = (1,2,\mathbf{Q})$, $T_2 = (1,3,\mathbf{Q})$, $T_3 = (2,3,\mathbf{Q})$, then no pattern $\{T_1, T_2, T_3\}$ is attainable.

Special cases of interest

(*a*) Q $=\emptyset$ so for any triplet $(i, j, k) \subset$ U the pairs (i,j), (i,k) and (j,k) cannot all occur as supports for **ESS**'s.

(*b*) Q = U\(1,2,3) then clearly there can be at most two supports with n - 1 elements.

(3) If Q $\subset$ U\(1,2,3...l), $T_i = (i$,Q$)$, $i = 1,...l$ and $T_{l+1} = (1,2,3...l)$then no pattern T $= (T_1,T_2...T_l,T_{l+1})$ is attainable.

(4) If Q $\subset$ U\(1,2,3...l), $T_i = (i, l,$ Q$)$, $i = 1,...l$ - 1 and $T_l = (1,2,3...l - 1,$ Q$)$ then no pattern T $= (T_1,T_2...T_l)$ is attainable.

(5) If Q $\subset$ U\(1,2,3), $T_1 = (1,$Q$)$, $T_2 = (2,3,$Q$)$, $T_3 = (1,2)$ and $T_4 = (1,3)$ then no pattern T $= (T_1,T_2,T_3,T_4)$ is attainable. This last result is given in Cannings and Vickers(1989).

3. Maximal Patterns for n=2, 3, 4. An attainable pattern is said to be maximal if it is not a proper subpattern of any attainable pattern. Thus if the conjecture above holds we need only specify the maximal patterns. The application of the results (1) to (5) above leave 2, 4 and 9 possibilities respectively for n=2, 3, and 4. These have all been shown to be attainable and the patterns are as follows:

$n = 2$ {(1,2)}, {(1),(2)}

$n = 3$ {(1,2,3)},{(1,2),(1,3)},{(1,2),(3)},{(1),(2),(3)}

$n = 4$ {(1,2,3,4)},{(1,2,3),(1,2,4)},{(1,2,3),(2,4),(3,4)}

{(1,2,3),(4)},{(1,2),(1,3),(1,4)},{(1,2),(2,3),(3,4),(1,4)}

{(1,2),(1,3),(4)},{(1,2),(3),(4)},{(1),(2),(3),(4)}.

We shall term an attainable pattern T $= (T_1,T_2,T_3...T_k)$ degenerate if for any V $\subset$ U, $\bigcup_{i \in V} T_i$ is disjoint from $\bigcup_{i \notin V} T_i$. Thus the maximal nondegenerate patterns are

$n = 2$ {(1,2)}

$n = 3$ {(1,2,3)},{(1,2),(1,3)}

$n = 4$ {(1,2,3,4)},{(1,2,3),(1,2,4)},{(1,2,3),(2,4),(3,4)}

{(1,2),(1,3),(1,4)},{(1,2),(2,3),(3,4),(1,4)}.

Any degenerate pattern is clearly attainable if the patterns in **V** and not -**V** are attainable, and not otherwise.

4. Existence Results Suppose that **C** is the class of matrices such that for $A \in C$ we have $a_{ij} = a_{ji}$, $a_{ii} = 0$, $a_{ij} = +1$ or -1. Then Cannings and Vickers(1988) prove that the supports of the **ESS**'s of any $A \in C$ are precisely the cliques (maximal complete subgraphs) of the graph **G** = (**U**,**E**) where $(i,j) \in E$ iff $a_{ij} = +1$. This class generates all the above examples with the exception of {(1,2,3),(2,4),(3,4)}. In addition we can easily see that any pattern with $|T_i| = 2$ all i is attainable by a matrix of **C** with **G** having an edge matching each T_i iff that G is triangle-free. Since (2) of the exclusion results asserts that there can be no triplets with all three pairs we see that that is the only restriction in this set of patterns. We note in passing that the extensive literature on graph theory can be exploited in various ways to yield information on patterns; for example random graphs can represent possible evolutions of conflicts, Cannings and Vickers(1988).

5. New Results. We present here some new results with respect to **ESS** patterns for the case n=5. These add to the previously most complete set of information which was given in Cannings and Vickers(1989).

5.1 410A *Lemma 1.* If $bf + ad < df$ or $de + bc < be$ or $fc + ae < ac$ or $a + c < 0$ or $b + e < 0$ or $d + f < 0$ then

$$\begin{vmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{vmatrix}$$

has no internal **ESS** .

Proof. The conditions are a subset of those of Theorem 3 of Vickers and Cannings(1988).

Lemma 2. The above matrix is equivalent to a symmetric matrix iff

$$a+d+e=b+c+f.$$

Lemma 3. If a,b,c,d,e,f are all positive, then the above matrix cannot have 2 **ESS**'s.

Proof. The only possibility to be excluded is that it has two ESS's of size 2, with support {1,2} and {2,3} say. However if strategy 1 cannot invade the ESS with support {2,3} then $f > a$ and this implies that strategy 3 can invade the ESS with support {1,2}.

Lemma 4. If a and c have opposite sign and if d and f have opposite sign, then the above matrix does not have an internal ESS if it is equivalent to a symmetric matrix.

Proof. We argue by contradiction. With the assumptions regarding the signs of a,c,d,f the matrix has one of the following sign arrangements:

$$\begin{array}{ccc} 0 & + & \\ - & 0 & + \\ & - & 0 \end{array} \qquad \begin{array}{ccc} 0 & + & \\ - & 0 & - \\ & + & 0 \end{array} \qquad \begin{array}{ccc} 0 & - & \\ + & 0 & - \\ & + & 0 \end{array} \qquad \begin{array}{ccc} 0 & - & \\ + & 0 & + \\ & - & 0 \end{array}$$

The last of these can be eliminated because it has a pure strategy 2 as an ESS and hence no internal ESS. The signs of b and e can now be inferred for the remaining three cases by applying

(a) no pure ESS is allowed so each column must contain at least one +,

(b) $a + d + e = b + c + f$ (from Lemma 2) The three possibilities are

$$\begin{array}{ccc} 0 & + & + \\ - & 0 & + \\ + & - & 0 \end{array} \qquad \begin{array}{ccc} 0 & + & + \\ - & 0 & - \\ + & + & 0 \end{array} \qquad \begin{array}{ccc} 0 & - & + \\ + & 0 & - \\ + & + & 0 \end{array}$$

For the first of these we have that $de + bc > be$ and hence $d > b$ thus $a + d + e = b + c + f$ gives $0 < a + e < c + f < 0$. The third matrix is dealt with in a similar fashion. For the second we see that $de + bc > be$ implies an immediate contradiction.

Theorem. The pattern **410A** (see Vickers and Cannings, 1988 for the key) is not attainable by any symmetric matrix.

Proof. Suppose the contrary. Then the matrix

$$\left|\begin{array}{ccccc} 0 & \alpha & \beta & & \\ \gamma & 0 & a & b & \\ \delta & c & 0 & d & \theta \\ & e & f & 0 & \phi \\ & & \chi & \omega & \end{array}\right|$$

can have ESS's with supports {1,2} , {1,3} , {2,3,4} , {3,5} , {4,5} . By the first lemma, a and c are not both negative and by the third they are not both positive. Hence a and c have opposite signs and likewise so do d and f. Lemma 4 thus implies the given contradiction.

The pattern **410A** can be obtained e.g. by the following matrix

0	3.1	10	α	α
3.1	0	2	-1	α
4	-1	0	2	3.1
α	2	−1	0	4
α	α	3.1	10	0

provided that α is sufficiently large and negative when the ESS's are specified by the columns of

1/2	5/7	0	0	0
1/2	0	1/3	0	0
0	2/7	1/3	1/2	0
0	0	1/3	0	2/7
0	0	0	1/2	5/7 .

This example and the theorem correct an error in Vickers and Cannings(1988a). In that paper the matrix which was given as corresponding to the pattern **410A** has in fact the pattern **410C** . The theorem demonstrates that symmetric matrices (which are relevant to the single-locus multi-allelic problem) will not show the full range of patterns available to general matrices.

5.2 111 The pattern **111** can be produced by the symmetric matrix

20	28	18	18	20.5
28	-44	50	50	-180
18	50	0	16	50
18	50	16	0	50
20.5	-180	50	50	8

and the ESS's are specified by the columns of

1/4	0	25/26
1/4	0	0
1/4	1/4	0
1/4	1/4	0
0	1/2	1/26

5.3 021A The pattern **021A** is produced by the symmetric matrix

-146.82	77.27	66.54	60.78	-61.70
77.27	-9.92	-5.56	-3.44	3.29
66.54	-5.56	-3.34	-0.25	2.41
60.78	-3.44	-0.25	0.32	1.46
-61.70	3.29	2.41	1.46	0.49

and the **ESS**'s are specified by the columns of

0.272654	0.0	0.0
0.594505	0.117229	0.0
0.093022	0.085717	0.1763
0.039819	0.0	0.156412
0.0	0.797055	0.667287

6 Genealogy for n=4. For n=4 we saw above that there were only five maximal non-degenerate attainable patterns. However there are eleven non-degenerate patterns and these are shown in Figure 1 in their genealogy. This genealogy has been drawn in generations , each generation corresponding to the number of elements in the pattern. The form shown also has the desirable property, from the point of view of drawing, that none of the edges cross. Note that the founders are the maximal patterns and that non-founders have one, two or three parents.

7 Genealogy for n=5. For n=5 the application of all our methods plus some special cases leaves us with potentially 61 patterns of which 55 are known to be attainable, there being 20 maximal ones of which 16 are known attainable. There are 6 generations and the geneaology is shown in Figure 2. Simulated annealing (see next section) allowed the production of a genealogy which had 165 intersections of the edges. In an attempt to produce a representation which is easier to read the genealogy was split into two pieces, the top four generations and the bottom three, so that the same individuals appear at the bottom of the first and the top of the second segments. This resulted in some reduction of the total number of intersections there being 102 and 35 intersections in the two segments. The reduction is less than might have been anticipated the top segment being fairly similar to the top of the whole genealogy and so is not shown here.

8 Simulated Annealing. Simulated annealing is a probabalistic optimisation technique (e.g. Lundy,1985). Thomas(1989) has applied it to the production of diagrams of genealogies. His technique involves imposing a requirement that nodes be seperated by some minimal distance and then minimising the sum of squared distances between all pairs of nodes. The criterion adopted here is rather different. Each node is assigned to its generation and within a generation nodes are numbered (i. e. assigned an ordering), the objective function is taken as the number of intersections between edges. Thus two edges (i, j) and (k, l) , where i is a parent of j and k a parent of l, intersect only if i and k are in the same generation (j and l are then necessarily in the same generation), and either $m(i) < m(k)$ and $m(j) > m(l)$, or $m(i) > m(k)$ and $m(j) < m(l)$, where $m(i)$ is the position of i in its generation list.

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Figure 1

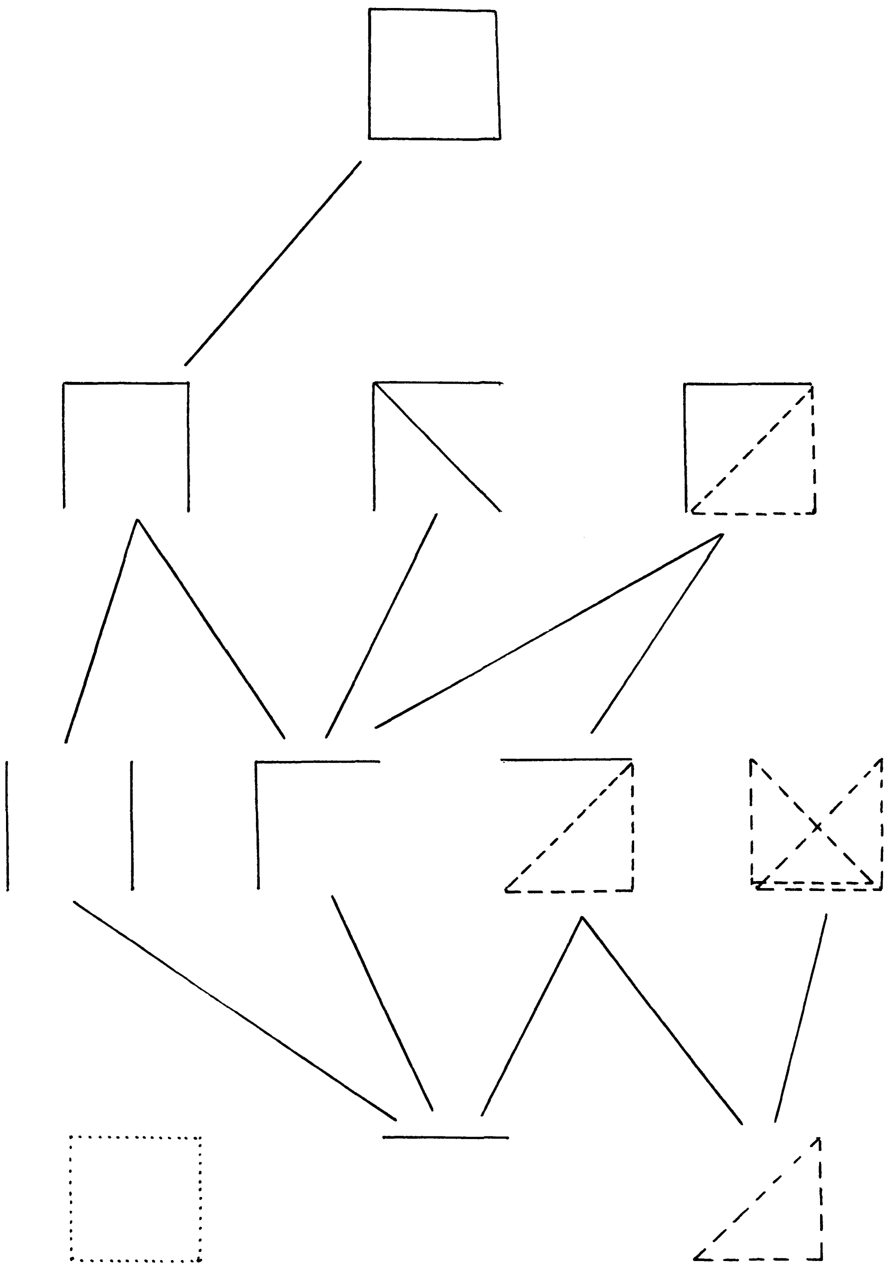

Figure 2, Part 1

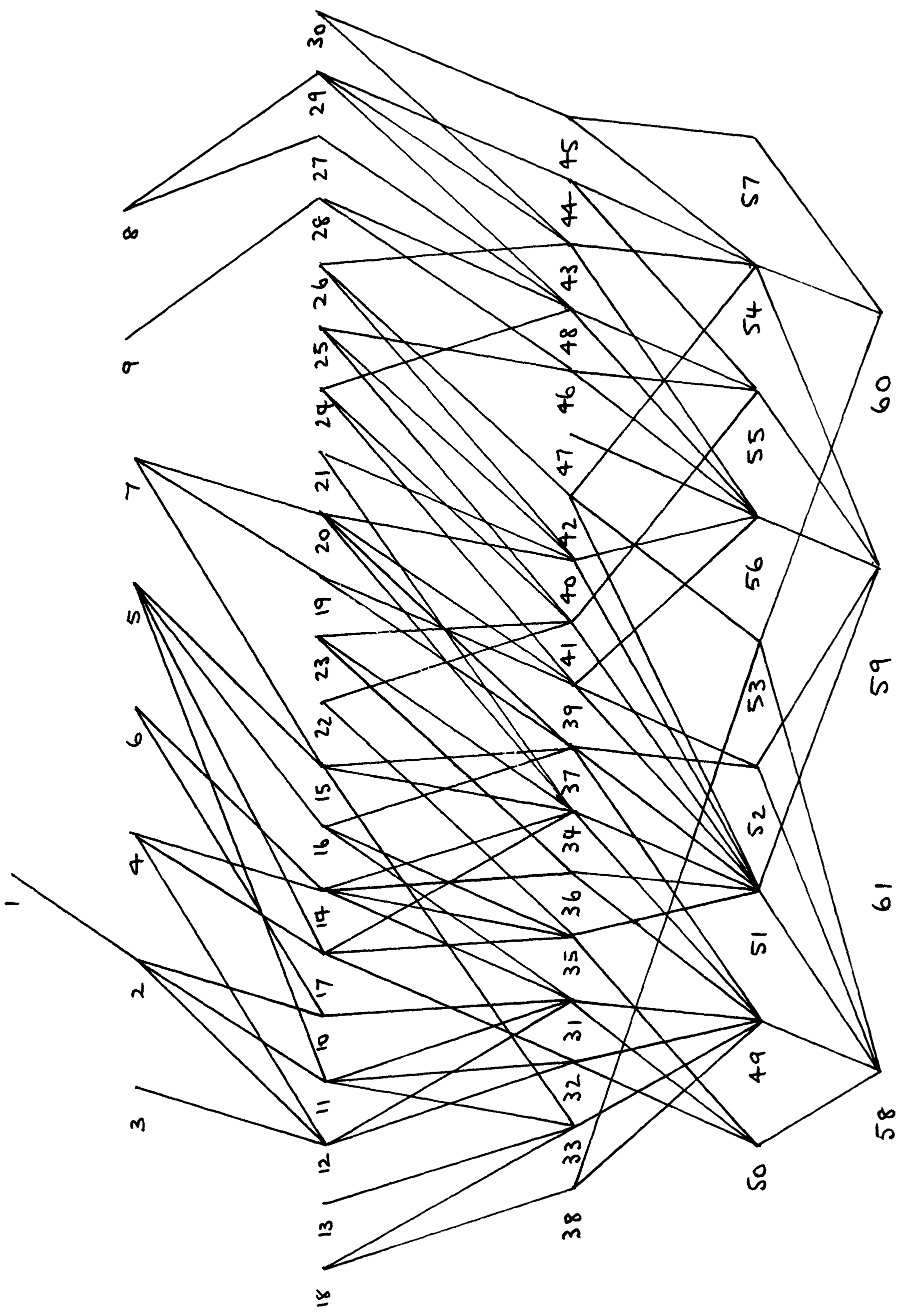

Figure 2, Part 2