ON DIRECTED POLYMERS IN A RANDOM ENVIRONMENT

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We consider random walks whose laws are perturbed in an irregular way by a second random mechanism, the so-called random environment. The perturbation acts in such a way that visits to certain randomly chosen points are favoured or disfavoured. There have been proposed several models in the mathematical physics literature, sharing the common feature that only very few facts are known on a rigorous mathematical level. We present here some of these models and some problems connected with them.

We always look at a random walk on the d-dimensional lattice $\mathbf{Z}^d$ but we start by introducing the random environment, the "disorder". It is given by i.i.d. random variables $X(i)$, $i \in \mathbf{Z}^d$, satisfying

$$X(i) > 0 \text{ almost surely} \tag{1}$$

$$EX(i) = 1. \tag{2}$$

$X = 1$ is then just the case where no perturbation occurs. Sometimes, it is convenient to have a one-dimensional parameter $\beta \in \mathbb{R}$ regulating the amount of disorder. This can be done by considering

$$X_\beta(i) = e^{\beta Y(i)}/m(\beta)$$

where $Y(i)$ are i.i.d. real random variables such that $m(\beta) = E(e^{\beta Y}) < \infty$ for β in a neighborhood of 0.

The unperturbed random walk is an ordinary random walk $\xi_0 = 0, \xi_1, \ldots, \xi_T$ on $\mathbf{Z}^d$ whose jump distribution is given by $p(x)$, $x \in \mathbb{Z}^d$, $\Sigma_x p(x) = 1$, i.e. we have

$$P(\xi_1 = i_1, \ldots, \xi_T = i_T) = 1_0(\xi_0) \prod_{j=1}^{T} p(i_j - i_{j-1})$$

where $i_0 = 0$. We always assume that for some $\varepsilon > 0$, we have

$$\sum_{x \in \mathbb{Z}^d} e^{\varepsilon |x|} p(x) < \infty \quad \text{and} \quad \Sigma\, x p(x) = 0.$$

This random walk and the random environment are chosen to be independent.

Formally, we consider the probability space $\Omega = \Omega_{\text{path}} \times \Omega_{env}$, $\Omega_{\text{path}} = (\mathbf{Z}^d)^{\mathbf{N}}$, $\Omega_{env} = (\mathbf{R})^{\mathbf{Z}^d}$, with the appropriate product field and equipped with the probability measure $Q = P \otimes \mu^{\mathbf{Z}^d}$, where P is the law of the above defined random walk on $\mathbf{Z}^d$ and μ is the law of the X(i). The ξ and the X variables are given by the appropriate projections.

We denote by $< \; >$ the expectation with respect to ξ and by E(.) the expectation with respect to the environment variables. Strictly speaking, $< \; >$ is the conditional expectation given the X variables and E(.) is the conditional expectation given the random walk. The total expectation is then given by $E(< \; >)$.

One natural model for a perturbation would be to introduce the weight factor

$$\kappa_T(X, \xi) = \prod_{j=1}^{T} X(\xi_j)$$

and to transform the measure P on the path space Ω_{path} for each realization of the X variables, by considering $\hat{P}_{T,X}$ defined by

$$d\hat{P}_{T,X}/dP(\omega) = \kappa_T(X,\xi(\omega))/<\kappa_T(X,\xi)>, \qquad \omega \in \Omega_{\text{path}}$$

A typical quantity one is interested in is the mean square displacement

$$\int |\xi_T|^2 d\hat{P}_{T,X} = <|\xi_T^2| \kappa_T(X,\xi)>/<k_T(X,\xi)>$$

and its asymptotic behaviour for large T.

For the model just introduced, essentially nothing seems to be known. It is related to some problems for random Schrödinger operators. To see the relation, it is convenient to switch to a continuous time random walk which, for simplicity is assumed to be just a nearest neigbour symmetric random walk. Furthermore, we assume that X(i) is of the form exp(Y(i))/m, where the Y(i) are i.i.d. Bernoulli random variables taking values 1 or -1. Then

$$\kappa_T(X, \xi) = \exp\left(\int_0^T Y(\xi_s)\, ds\right)/m^T .$$

Assuming furthermore d=1, we consider the discrete Laplacian Δ on $\mathbf{Z}$: $\Delta f(i) = (f(i+1) + f(i-1) - 2f(i))/2$. It is known that for almost all realizations of the Y variables the operator $\Delta + Y$ acting on $l_2(\mathbf{Z})$ (Y just by multiplication) has a pure point spectrum which is dense in [-3,1] and where each eigenvalue has exponentially decaying eigenfunctions (see [2]). Therefore $<\kappa_T(X,\xi)1_i(\xi_T)>$

can be expanded in terms of these eigenfunctions ψ_j:

$$< \kappa_T(X,\xi)\, 1_i(\xi_T) > \; = \sum_j e^{\lambda_j T}\, \psi_j(0)\, \psi_j(i) \, .$$

In the limit $T \to \infty$, only those eigenvalues count which are near the upper edge of the spectrum. However, there seems to be no information available, where the eigenfunctions are situated. Up to my knowledge, even for d = 1, the mean square displacement in the above model is not known.

The model of directed polymers in a random environment seems to have grown out of an even more difficult problem, where one replaces the random walk ξ by a polymer, i.e. a self avoiding random walk. There seems to exist absolutely no results for this on a mathematical level, but there is a considerable interest in this in the physics literature. Form background information, see the introduction of [3] and the references there.

A directed polymer is a caricature of a true polymer, where one makes the random walk self avoiding just by stepping in one of the dimensions deterministically one step in each time point (we take time discrete again). Of course, we can identify this special dimension with the discrete time axis. The model is therefore the same as that introduced in the beginning, with the only difference that the environment variables now change also independently in time, i.e. we consider random variables X(i,t), $i \in \mathbb{Z}^d$, $t \in \mathbb{N}$, which satisfy (1) and (2), and we put now

$$\kappa_T(X,\xi) \;=\; \prod_{s=1}^{T} X(\xi_s, s) \quad .$$

There is some disagreement what the "dimension" of such a model is. If the time axis is considered as one of the space dimensions, the dimension is d + 1.

It appears clear that this model is much easier than a true polymer in a random environment, but despite its simplicity, not much is known on a rigorous level. There seems to be the general belief that for d = 1 the mean square displacement of ξ_T is for almost all X realizations of order $T^{4/3}$ (see e.g. [5], [7]). The following result is a generalization of the results in [1] and [6]:

Theorem. *Let* d $\geq$ 3. *There exist* $\delta > 0$, *depending only on* d *and the transitions probabilities* p(x), *such that if* var(X) $\leq \delta$, *one has almost surely*

a) $$\lim_{T\to\infty} T^{-1} \sum_{i \in \mathbb{Z}^d} (\lambda, i)^2\, \mu_{T,X}(i) \;=\; (\Sigma\lambda,\lambda)$$

for all $\lambda \in \mathbf{R}^d$. Here, $(.,.)$ denotes the inner product and Σ is the covariance matrix for p.

b) $\mu_{T,X}(\sqrt{T}\,.) \to N(0,\Sigma)$ *in distribution*

This has been proved in [1] for the special case where $X = 1 + \varepsilon Z$, $Z = \pm 1$ with probability 1/2 and for the nearest neighbor random walk.

We prove here a) in our slightly more general situation. b) can be proved by showing the convergence of all moments.

To prove a), one first remarks that for any realization of the random walk ξ $\kappa_T(X,\xi)$ is an F_T- martingale, where $F_T = \sigma(X(i,s) : i \in \mathbf{Z}^d, s \leq T)$, which satisfies $E(\kappa_T(X,\xi)) = 1$. It converges to 0 almost surely, except in the trivial case $X = 1$. Then also $<\kappa_T(X,\xi)>$ is an F_T-martingale, with $E(<\kappa_T(X,\xi)>) = 1$. The transience of the d-dimensional random walk for $d \geq 3$ is now used to prove that this martingale converges to a strictly positive limit if var(X) is small enough. To see this, we write

$$(E(< \kappa_T(X,\xi) > 2) = E(< \kappa_T(X,\xi) \quad) > < \kappa_T(X,\xi'))>)$$

where ξ' is an independent copy of ξ. Interchanging E() with < >, we get

$$E(<\kappa_T(X,\xi)>^2) = \ < \prod_{s=1}^{T} E(X(\xi_s, s)X(\xi'_s, s)) > \ = < (1+\mathrm{var}(X))^{n_T(\xi,\xi')} >$$

where

$$n_T(\xi,\xi') = \sum_{S=1}^{T} 1_{\xi_s = \xi'_s} = \sum_{S=1}^{T} 1_{\xi_s - \xi'_s = 0} \leq \sum_{S=1}^{\infty} 1_{\xi_s - \xi'_s = 0} = n_\infty(\xi,\xi')$$

n_∞ obviously has an exponential moment, so, if var(X) is small enough, we have $\sup_T E(<\kappa_T(X,\xi)>^2) < \infty$ and, by the martingale convergence theorem, $<\kappa_T(X,\xi)>$ converges to a limit, say θ with $E(\theta) = 1$. The event

$$\{\lim_{T\to\infty} < \kappa_T(X,\xi) > \ = 0\ \}$$

is clearly measurable with respect to the field $\sigma(X(i,s) : i \in \mathbf{Z}^d, s \geq t)$ for any t, so, by the Kolmogorov 0-1-law, we get $P(\theta > 0) = 1$, showing that

$$P(\lim_{T\to\infty} \ <\kappa_T(X,\xi)> \text{ exists and is} > 0) = 1. \tag{3}$$

To prove a) in our theorem, we use the well known fact that for any $\lambda \in \mathbf{R}^d$ $(\lambda,\xi_T)^2 - T(\lambda,\Sigma\lambda)$ is a martingale for the filtration of the random walk. Using that $\kappa_T(X,\xi)$ is an F_T-martingale for a fixed path, we conclude that

$$M_T = <((\lambda,\xi_T)^2 - T(\lambda,\Sigma\lambda))\kappa_T(X,\xi)>$$

is an F_T-martingale, and so

$$U_T = \sum_{s=1}^{T} \frac{1}{s}(M_s - M_{s-1}) \qquad (M_o = 0),$$

too. Remark that

$$M_s - M_{s-1} = <((\lambda,\xi_s)^2 - s(\lambda,\Sigma\lambda))\kappa_{s-1}(X,\xi)(X(\xi_s,s) - 1)>. \tag{4}$$

By a calculation which is slightly more complicated than that above for $E<\kappa_T(X,\xi)>^2$, using (4), one sees that if var(X) is small enough, one has

$$\sup_T E(U_T^2) < \infty,$$

so U_T converges almost surely, and by the Kronecker lemma, we have

$$\lim_{T\to\infty} \frac{1}{T} M_T = 0.$$

From this and (3), part a) in the theorem follows.

It is generally believed that the properties in the theorem no longer hold true if var(X) is too large, although this is essentially based only on Monte Carlo simulations (see [4] and the references cited there). This is related to a large deviation problem. We will give a short discussion of this point. To discuss this, it is convenient to have the one dimensional parameter β as introduced in the beginning, i.e. we consider

$$X_\beta(i,t) = e^{\beta Y(i,t)}/m(\beta)$$

where $m(\beta) = E(e^{\beta Y}) < \infty$ for $|\beta|$ small enough. Our theorem states that for small enough $|\beta|$, one has the properties a) and b). It is expected that even if Y is bounded (so $m(\beta)$ exists for all β) this central limit behavior breaks down for large enough $|\beta|$. This is not at all understood. The question is connected with large deviation problem in the following way: We have

$$\kappa_T(X,\xi) = \exp(\beta S_T)/m(\beta)^T,$$

where $S_T = \sum_{s=1}^{T} Y(\xi_s, s)$. From the above proved fact that for small enough $|\beta|$ one has

$$\lim_{T\to\infty} \langle \kappa_T(X,\xi) \rangle > 0$$

one concludes that

$$\lim_{T\to\infty} \frac{1}{T} \log \langle e^{\beta S_T} \rangle = \log m(\beta), \text{ almost surely.} \tag{5}$$

So, one has the somewhat strange fact that for almost all realizations of the environment variables Y, S_T has as a random variable depending on the path space the same large deviation behaviour as do sums of i.i.d. Y variables, at least in a neighborhood of EY. A first step towards an understanding of the large $|\beta|$ region would be a discussion of the question if (5) breaks down for large $|\beta|$. As far as I know, such a breakdown has not been proved. A strong indication that this happens is however given in a discussion of even simplified (so called mean-field) model by Derrida & Spohn [3]. They discussed a random walk on the so called Caley-tree which can be interpreted as a walk on an infinite dimensional lattice. The path space Ω_{path} here is the set of sequences

$$\xi_0 = 0, \xi_1, \xi_2, \ldots.$$

with $\xi_t \in \{-1,1\}^t$, such that if $\xi_t = (a_1, \ldots, a_t)$ then $\xi_{t-1} = (a_1, \ldots, a_{t-1})$. The path measure is defined by the property that ξ_t adds to ξ_{t-1} as the t'th component +1 or -1 with probability 1/2.

The environment variables are given by

$$X_\beta(i) = \exp(\beta Y(i)) \text{ for } i \in A = \bigcup_{t=1}^{\infty} \{-1,1\}^t.$$

Then, one investigates

$$\kappa_T(X,\xi) = \prod_{s=1}^{T} X_\beta(\xi_s).$$

(It is unnecessary to take X(i,t) because the position in A already fixes the time point). Our analysis of $\langle\kappa_T(X,\xi)\rangle$ works as well here, so (5) is true for small enough $|\beta|$. Derrida and Spohn discussed the special case where Y is standard normally distributed, in which case $\log m(\beta) = \beta^2/2$. Here, in fact, there is a critical point for β and $\lim_{T\to\infty}(1/T) \log\langle e^{\beta S_T}\rangle$ agrees with $\beta^2/2$ only in some region $|\beta| \le \beta_{crit}$, and outside, this limit depends linearly on β.

If $|\xi_t|$ denotes the sum of the components in ξ_t then our analysis in the theorem works as well for this, proving e.g. that for small enough $|\beta|$

$$\lim_{T\to\infty} \frac{1}{T} \langle|\xi_T|^2 \kappa_T(X,\xi)\rangle / \langle\kappa_T(X,\xi)\rangle = 1$$

for almost all realization of the environment variables. It would be interesting to know what happens with this quantity for $|\beta| > \beta_{crit}$.

References

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