DEPENDENCE IN CONDITIONALLY SPECIFIED DISTRIBUTIONS

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Suppose that (X, Y) is a two-dimensional random variable whose joint density is conditionally specified. If for each x, the conditional distribution of Y given $X = x$ is a member of a particular exponential family and for each y, the conditional distribution of X given $Y = y$ is a member of a possibly different exponential family, it is possible to determine sufficient conditions for negative (or positive) dependence of (X, Y). Analogous results are obtainable for certain conditionally specified distributions which do not involve exponential families.

1. Introduction. Castillo and Galambos (1987a,b) considered a spectrum of joint distributions for which the conditional distributions belonged to specified families. Subsequently Arnold (1987) and Arnold and Strauss (1987, 1988) considered broad classes of such conditionally specified distributions. Specific attention was focused on situations where the conditional distributions were posited to be members of particular exponential families of distributions. The initial example in this genre involved bivariate distributions in which both sets of conditional densities were negative exponential. Such joint distributions were shown to have densities of the form:

$$f_{X,Y}(x,y) = \beta\gamma\theta(\delta)\exp\{-[\beta x + \gamma y + \beta\gamma\delta xy]\}, \quad x > 0, y > 0, \tag{1}$$

where

$$\theta(\delta) \triangleq \left[\int_0^\infty e^{-u}(1+\delta u)^{-1} du\right]^{-1}. \tag{2}$$

In connection with the development of method of moments estimates for such a joint distribution, it was necessary to compute the correlation between X and Y. The resulting expression

$$\rho(X,Y) = \frac{\delta + \theta(\delta) - \theta^2(\delta)}{\theta(\delta)(1 + \delta - \theta(\delta))} \tag{3}$$

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was evaluated numerically for various values of δ. It was consequently evident that $\rho(X,Y) \leq 0$ for every $\delta \leq 0$. Retrospectively this could have easily been obtained by observing that, since for the density (1) we have

$$P(X > x \mid Y = y) = \exp[-(1+\gamma\delta y)\beta x], \tag{4}$$

it follows that X is stochastically decreasing in Y. Consequently using Theorem 5.4.2 of Barlow and Proschan (1981), X and Y are negatively quadrant dependent so that $\rho(X,Y) \leq 0$.

It is reasonable to ask to what extent does this negative dependence carry over to the broader classes of conditionally specified distributions discussed in Castillo and Galambos (1987a) and Arnold and Strauss (1987, 1988). The issue is addressed in Section 2. Certain conditionally specified distributions not of the exponential form are discussed in Section 3. The final section outlines certain multivariate versions of the results in the earlier sections. Since it is relatively easy to get experimenters to model conditional distributions and it is relatively easy to get them to discuss the signs of correlations between variables, it is of interest to be able to identify certain modelling inconsistencies which might result. For example we cannot have $X \mid Y = y$ exponential with parameter dependent on y, $Y \mid X = x$ exponential with parameter dependent on x and (X,Y) positively correlated. If we replace the word "exponential" with "normal," such a model is of course possible.

2. Bivariate Distributions Whose Conditionals Are in Prescribed Exponential Families. As much as possible we will use the notation of Arnold and Strauss (1987). Let $\{f_1(x;\underline{\theta}) : \underline{\theta} \in \Theta\}$ denote a k parameter exponential family of densities of the form

$$f_1(x;\underline{\theta}) = r_1(x)\beta_1(\underline{\theta}) \exp\left\{\sum_{i=1}^{k} \theta_i q_i^{(1)}(x)\right\} \tag{5}$$

where Θ is the natural parameter space and the densities are defined with respect to some convenient dominating measure on $\mathbf{R}$ (usually Lebesgue measure or counting measure). Analogously let $\{f_2(y;\underline{\tau}) : \underline{\tau} \in T\}$ denote an ℓ-parameter exponential family of the form

$$f_2(y;\underline{\tau}) = r_2(y)\beta_2(\underline{\tau}) \exp\left\{\sum_{j=1}^{\ell} \tau_j q_j^{(2)}(y)\right\} \tag{6}$$

where T is the natural parameter space. Assume that the $\{q_i^{(1)}\}$ are functionally independent, and similarly for the $\{q_i^{(2)}\}$.

Suppose that $f(x,y)$ is a bivariate density for which

(i) For every y for which $f(x \mid y)$ is defined, this density is a member of the family (5) for some $\underline{\theta}$ which may depend on y.

(ii) For every x for which $f(y \mid x)$ is defined, this conditional density is a member of the family (6) for some $\underline{\tau}$ which may depend on x.

Arnold and Strauss show that if (i) and (ii) hold, then $f(x,y)$ must be of the form

$$f(x,y) = r_1(x) r_2(y) \exp\{\underline{q}^{(1)}(x)' M \underline{q}^{(2)}(y) + \underline{a}' \underline{q}^{(1)}(x) + \underline{b}' \underline{q}^{(2)}(y) + c\} \tag{7}$$

for suitable choices of $M, \underline{a}, \underline{b}$, and c (subject to the requirement that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dx dy = 1$). Here

$$\underline{q}^{(1)}(x) = (q_1^{(1)}(x), \ldots, q_k^{(1)}(x))$$

and

$$\underline{q}^{(2)}(y) = (q_1^{(2)}(y), \ldots, q_\ell^{(2)}(y))$$

and the dimensions of M, $\underline{a}$ and $\underline{b}$ are appropriately selected. (Note that the case of independence is included; it corresponds to $M \equiv 0$. The parameters included in M, $\underline{a}$ and $\underline{b}$ are constrained to be such that

$$\psi(M, \underline{a}, \underline{b}) = \int_{S_x} \int_{S_y} e^{-c} f(x,y) dx dy < \infty.$$

The remaining constant c is then necessarily given by

$$e^c = \frac{1}{\psi(M, \underline{a}, \underline{b})}$$

so that the joint density integrates to 1. Here $S_x = \{x : r_1(x) > 0\}$ and $S_y = \{y : r_2(y) > 0\}$.

Under what conditions can we determine that $f(x,y)$ given by (7) is positive quadrant dependent and hence that $\rho(X,Y) \geq 0$? A convenient sufficient condition for such positive dependence is that the density be totally positive of order 2, i.e., that

$$\begin{vmatrix} f(x_1,y_1) & f(x_1,y_2) \\ f(x_2,y_1) & f(x_2,y_2) \end{vmatrix} \geq 0 \tag{8}$$

for every $x_1 < x_2$, $y_1 < y_2$ in S_x and S_y respectively. (This is part of Barlow and Proschan's (1981) Theorem 5.4.2). However the determinant (8) assumes a particularly simple form if the joint density is of the form (7). Substitution into (8) yields the following sufficient condition for total positivity of order 2:

$$[\underline{q}^{(1)}(x_1) - q^{(1)}(x_2)]' M [\underline{q}^{(2)}(y_1) - \underline{q}^{(2)}(y_2)] \geq 0 \tag{9}$$

for every $x_1 < x_2$ in S_x and $y_1 < y_2$ in S_y. For example if $\underline{q}^{(1)}(x) \uparrow$ as $x \uparrow$ and $\underline{q}^{(2)}(y) \uparrow$ as $y \uparrow$, then a sufficient condition for TP_2 and consequently for nonnegative correlation is $M \geq 0$. If $\underline{q}^{(1)}$ and $q^{(2)}$ are not monotone, it is unlikely that any choice for M will yield a TP_2 density. If the determinant in (9) is always ≤ 0, then negative correlation is guaranteed. Consider the following examples abstracted from Arnold and Strauss (1987).

EXAMPLE 2.1. Bivariate exponential conditionals distribution. Here $k = \ell = 1$, $r_1(t) = r_2(t) = I(t > 0)$, $q^{(1)}(t) = q^{(2)}(t) = -t$. The joint densities are of the form

$$f(x, y) = \exp[m_{11}xy - ax - by + c], \quad x > 0, y > 0.$$

For convergence we require $m_{11} \leq 0$, $a > 0$, $b > 0$. This guarantees that expression (9) is ≤ 0 and consequently $\rho(X, Y) \leq 0$.

EXAMPLE 2.2. Bivariate Poisson conditionals distribution. Here

$$f(x, y) = \exp(m_{11}xy + ax + by + c)/x!y!$$

for $x = 0, 1, 2, \ldots, y = 0, 1, 2, \ldots$. For convergence we need $m_{11} \leq 0$. Again (9) is ≤ 0 and negative correlation is assured.

EXAMPLE 2.3. Bivariate geometric conditionals distribution. Here

$$f(x, y) = \exp(m_{11}xy + ax + by + c)$$

for $x = 0, 1, 2, \ldots, y = 0, 1, 2, \ldots$. For convergence we require $a < 0$, $b < 0$ and $m_{11} \leq 0$. Consequently (9) is ≤ 0 and negative correlation is encountered.

EXAMPLE 2.4. Normal conditionals with variance 1. In this case we find

$$f(x, y) = e^{-(x^2+y^2)/2} \exp(m_{11}xy + ax + by + c)$$

which yields a valid density provided $| m_{11} | \leq 1$. It is the classical bivariate normal. The correlation is determined by the sign of m_{11}.

EXAMPLE 2.5. Bivariate normal conditionals distribution. Here $k = \ell = 2$, $r_1(t) = r_2(t) = 1$, $\underline{q}^{(1)}(t) = \underline{q}^{(2)}(t) = \binom{t^2}{t}$. The resulting bivariate density is

$$f(x, y) = \exp\left\{ \begin{pmatrix} x^2 \\ x \end{pmatrix}' M \begin{pmatrix} y^2 \\ y \end{pmatrix} + \underline{a}' \begin{pmatrix} x^2 \\ x \end{pmatrix} + \underline{b}' \begin{pmatrix} y^2 \\ y \end{pmatrix} + c \right\}.$$

Since $\underline{q}^{(1)}$ and $\underline{q}^{(2)}$ are not monotone, we cannot expect total positivity. The exceptional case occurs when $M_{11} = M_{12} = M_{21} = 0$ in which case the density reduces to a standard bivariate normal with correlation determined by the sign of M_{22}.

3. Other Conditionally Specified Bivariate Distributions.

(a) **Bivariate Pareto** (α) **conditionals.** Suppose $f(x, y)$ is such that each $f(x \mid y)$ and each $f(y \mid x)$ is a member of the Pareto(α) ($\alpha > 0$) family of densities, i.e., of the form

$$f(u) = \frac{\alpha}{\sigma}\left(1 + \frac{u}{\sigma}\right)^{-(\alpha+1)}, \quad u > 0. \tag{10}$$

Arnold (1987) showed that, necessarily, we then have

$$f(x, y) = [\lambda_{00} + \lambda_{10}x + \lambda_{01}y + \lambda_{11}xy]^{-(\alpha+1)}, \quad x > 0,\ y > 0. \tag{11}$$

In order to have a valid density we must have $\lambda_{00} > 0$, $\lambda_{01} > 0$, $\lambda_{10} > 0$, and $\lambda_{11} \geq 0(\lambda_{11} > 0$ if $0 < \alpha \leq 1)$. For such a density we find

$$P(X > x \mid Y = y) = \left(1 + \frac{\lambda_{10} + \lambda_{11}y}{\lambda_{00} + \lambda_{01}y}\, x\right)^{-\alpha}.$$

If we compute $\frac{d}{dy}P(X > x \mid Y = y)$ we find that its sign depends on the sign of $\lambda_{00}\lambda_{11} - \lambda_{10}\lambda_{01}$ and so X is either stochastically increasing or decreasing in Y. Consequently, when $\rho(X, Y)$ exists,

$$\text{sign } \rho(X, Y) = \text{sign}(\lambda_{00}\lambda_{11} - \lambda_{01}\lambda_{10}).$$

(b) **Bivariate uniform conditionals.** Suppose $f(x, y)$ is such that $f(x \mid y)$ is uniform $(0, \phi(y))$ for each $y \in (0, 1)$ and $f(y \mid x)$ is uniform $(0, \psi(x))$ for each $x \in (0, 1)$. By writing $f(x, y)$ as a product of marginal and conditional densities in both possible manners, we conclude that $\phi(y) = \psi^{-1}(y)$ and that ψ is an arbitrary nonincreasing function defined on (0,1) with $\psi(0) = 1$ and $\psi(1) = 0$. For such a function ψ, the joint density assumes the form

$$f(x, y) = I(0 < x < 1, 0 < y < \psi(x))/c(\psi), \tag{12}$$

where

$$c(\psi) = \int_0^1 \psi(x)dx.$$

For such a bivariate distribution we have $P(Y > y \mid X = x)$ nonincreasing in x for every y, so that (X, Y) are negatively correlated.

4. Multivariate Conditionally Specified Distributions. There is in principle no difficulty in extending the discussion of Sections 2 and 3 to higher dimensions. Arnold and Strauss (1987, 1988) provide several examples. Two such examples will be described here.

(1) Multivariate exponential conditionals distributions. Here we assume X is a k-dimensional random variable with nonnegative coordinate random variables.

For $i = 1, 2, \ldots, k, X^{(i)}$ denotes the vector X with the i^{th} coordinate deleted. If for each i and for each $x^{(i)} \in \mathbf{R}_+^{(k-1)}$, the conditional distribution of X_i given $X^{(i)} = x^{(i)}$ is exponential $\mu_i(x^{(i)})$ for some functions $\mu_i(\cdot)$, it may be verified that the joint density of X must be of the form

$$f(\underline{x}) = \exp\left[-\sum_{s\in\xi_k} \lambda_s \left(\prod_{i=1}^{k} x_i^{s_i}\right)\right], \quad \underline{x} > \underline{0} \tag{13}$$

where ξ_k is the set of all vectors of 0's and 1's of dimension k. The parameters $\lambda_s (s \not\equiv \underline{0})$ are nonnegative, those for which $\sum_{i=1}^{k} s_i = 1$ are positive, and $\lambda_{\underline{0}}$ is such that the density integrates to 1. It is evident that X_1 is stochastically decreasing in $(X_2, \ldots, X_k)$ (more generally X_i is stochastically decreasing in $X^{(i)}$). Negative correlations are thus assured.

(2) Multivariate uniform conditionals distibution. Suppose that $X = (X_1, \ldots, X_k)$ is such that each X_i has the interval (0,1) as its set of possible values. Assume that for each i, the conditional distribution of X_i given $X^{(i)} = x^{(i)}$ is uniform $(0, \psi_i(x^{(i)}))$. In such a case the joint density must be uniform over a region in the k-dimensional positive orthant under some surface passing through the k points

$$(1, 0, \ldots, 0), (0, 1, 0, \ldots), \ldots, (0, \ldots, 0, 1).$$

Negative dependence and negative correlation are consequently present.

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