Prospects of further development

In the development of the theory outlined here the main effort has been spent on the description of the mathematical properties of analytic models. There are several 'loose ends' which it would be desirable to have tied up, and the asymptotic theory based on these models is only briefly sketched. Some open problems of a more or less concrete nature, and possible lines of further research, are listed below. The order of the items listed corresponds to the order of the corresponding material in the previous chapters, and it has not been attempted to assign priorities to the particular points.

(1) A few 'global' results are given as Lemma 3.7 and Corollary 5.8. By 'global' is meant a result that applies to the entire parameter space B on which the model is analytic, as opposed to the local results that apply to some neighbourhood of a fixed point only. As is seen, e.g., in the proof of Theorem 5.4.1, more such global results are available and it would be desirable to stretch this part of the theory as far as possible. Notice that all proofs of such results go as follows. Assume that the model is analytic on the open connected set B. Let A be the set of points at which some property holds. Then prove that A is open as well as closed relative to B. If A is non-empty then $A = B$.

(2) By use of Lemma 2.6.5, or otherwise, it might be possible to derive conditions for models obtained by conditioning to be analytic, for a general (non-ancillary) conditioning variable. This would, i.a., give some hope of a further development of asymptotic results in relation to approximate ancillarity, outside the framework of independent replications.

(3) There is ample room for improvement of the bounds obtained in the approximation results in Section 2.7. Whether it is of interest to obtain such improvements is not clear, because the order of magnitude of the bound would still be the same in connection with common schemes of asymptotics. However, in the other sections in Chapter 2, bounds turned out to be 'close to optimal' in most cases, whereas those in Theorem 2.7.3 are, admittedly, not quite satisfactory.

(4) Another matter of interest in relation to Section 2.7 is the concept of higher order asymptotic sufficiency, cf., e.g., Michel (1978). Theorem 2.7.4 may be interpreted as a result on local higher order asymptotic sufficiency; to turn it into a global result an estimator should be substituted for the score statistic $D_1(\beta_0)$.

(5) A quite specific open problem in Section 2.8 is whether the statistic $\mathbf{T}$ belongs to the space

$$\left\{ \mathbf{T} : \sup_k \{ k!^{-1} \|T_k\| \, \lambda^{k-1} \} < \infty \right\}$$

with probability one. This space is the dual of Θ_λ defined in (2.8.14), and since it is shown in Lemma 2.8.2 that for all $\boldsymbol{\theta} \in \Theta_\lambda$ the inequality

$$|\boldsymbol{\theta}(\mathbf{T})| < \infty$$

holds almost surely, the conjecture seems natural. The problem is that the null-set on which the inequality may fail, may depend on $\boldsymbol{\theta}$.

(6) A more vague, and much broader, problem related to Section 2.8, is whether it is possible to develop an asymptotic theory by working within the locally defined infinite-dimensional family. For example, it might be possible to derive asymptotic expansions (of the Edgeworth or of the saddlepoint type) of the density of the infinite sequence $\mathbf{T} = (T_1, T_2, \ldots)$ of bias-corrected log-likelihood differentials. In that case (infinite) expansions for various statistics might be obtained directly by transformation of such expansions.

(7) A particularly promising line of development is to consider the class of analytic transformation models, i.e., analytic models generated by a group of transformations. In Sections 3.4 and 3.6 the particular examples of location models and location-scale models were considered. The group structure was not exploited explicitly here, although it was instrumental in parts of the proofs, in particular in the proofs that the index is constant for these models. It would be illuminating to know the generality of this result, and the extent to which it is tied to the parametrization. Furthermore, since some structure is imposed on the sample space through the group of transformations, it may well be feasible to derive higher order asymptotic results (or results of the saddlepoint type) for sequences of transformation models, outside the framework of independent replications.

(8) The Weibull distribution model in Section 3.10 provides the 'most regular' type of model which is not analytic. The log-likelihood function is analytic and all its differentials exist and have finite moments of all orders, but they fail to have finite exponential moments. Parts of the theory do not require the existence of finite exponential moments, and it would be desirable to see how far the theory could be carried, if the exponential moments condition on M from (iv) in Definition 2.2.1 were replaced by the requirement that all (or some) moments of M are finite.

(9) The asymptotic theory developed in Sections 4.1–4.4 applies to any sequence of analytic models. Thus, there should be possibilities of applications of this theory to asymptotic theory for stochastic processes, or to other types of models for dependent observations. The difficulty is to develop methods for obtaining bounds on the index for such models, because essentially the only condition that needs to be verified is that the index tends to zero. As mentioned above, cf. (2), the problem may be closely related to the consideration of mixtures of models.

(10) The higher order asymptotic theory outlined in Chapter 5 is confined to sequences of independent replications. As discussed in the introductions to Chapters 4 and 5, the problem of derivation of higher order asymptotic results (or results of the saddlepoint type) is tied to the Cramér type condition

on the characteristic function, cf. (5.2.10). Within the class of generalized linear models, cf. Section 4.5, some crucial simplification of this condition ought to be obtainable, because the densities of the individual observations belong to the same family of densities. Also for analytic transformation models, cf. (8) above, there might be possibilities for the derivation of higher order asymptotic results.