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THE GAMBLER'S RUIN PROBLEM FOR PERIODIC WALKS

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As a general formula due to T.E. Harris makes evident, the probability, P, that a nearest-neighbor random walk, X, initially at 0, ever increases by a preassigned amount, g, is a rather complicated function of the distribution, D, of X. I happened to make a simple observation – which I hope will elicit one of Dave Blackwell's inimitable grins – that there are cases, herein called periodic, somewhat more general than the classical ones of de Moivre, in which P continues to be a simple function of D. The observation applies, in the discrete case, to periodic, nearest-neighbor walks on the integers or, by projection, to walks on the vertices of a polygon, and, in the continuous case, to periodic diffusions on the line or, by projection, to diffusions on the circle.

1. The polygonal walk. In this section, X is a nearest- neighbor random walk on the set, V, of the $g > 2$ vertices of a polygon, Q. Though Q is arbitrary , possibly knotted or intersecting, it is convenient to assume that it is planar and convex, for then the notions of clockwise and winding angle become available. Let w_i, strictly between 0 and 1, be the probability that X makes a transition from the ith vertex, v_i, to its successor, v_{i+1}, where addition of integers is modulo g, so, v_g is v_O, and v_{-1} is v_{g-1}. Let X_0 be v_0, and, for each positive integer, t, X_t is the position at time t, and X^t designates the partial path of X over the time interval $[0, t]$.

Let: the unit of angle be a full circle (360 degrees); A_t be the winding angle of X^t; T^+ be the first time, t, that A_t equals 1; T^- be the first time, t, that A_t is -1. The focus herein is on the event, C^+, that X completes a winding of the entire polygon that is clockwise before one that is counterclockwise. Plainly, C^+ is the event $T^+ < T^-$.

Let T, the DURATION, be the lesser of T^+ and T^-. There is precisely one path for which $T = g$ and $T^+ < T^-$, namely, f^+, the one for which $X_t = v_t$ for all t in $[0, g]$. Likewise there is a unique path f^- for which $T = g$ and $T^- < T^+$. So, f^+ and f^- are the only paths for which $T = g$. Plainly, the probability that X^T is f^+ is the product of the g w_i, say W, and the probability that X^T is f^- is W^-, the product of the corresponding $[1 - wi]$. So, conditional on the event that X^T is either f^+ or f^-, or, equivalently, that $T = g$, the probability of C^+ is $W/[W + W^-]$. As the next theorem asserts, this is also the unconditional probability of C^+.

Theorem 1. *The probability that a clockwise winding of the polygon occurs before a counterclockwise winding is $W/[W + W^-]$. Moreover, the event is stochastically independent of the occupation measure, M.*

For the benefit of those who, like the author, have barely heard of the OCCUPATION MEASURE, M, it is the finite measure that assigns to each subset B, of Q, the measure of the set of t in $[0, T]$ for which X_t is in B. The last occurrence of "measure" refers, in the present context, to counting measure and, in the continuous context, to Lebesgue measure.

The program of proof for Theorem 1 is to exhibit a pairing of the elements of C^+, that is, the paths for which $T^+ < T^-$, with the elements of C^-, the paths for which $T^- < T^+$, such that, given any pair, the conditional probability that $T^+ < T^-$ is the same as for the pair f^+ and f^-. The existence of such a pairing obviously implies Theorem 1.

Let C be the union of C^- and C^+. A DUALITY is a bijection of C, say f to f^*, that satisfies each of these four conditions:

[a] $f^{**} = f$;
[b] f is in C^+ iff f^* is in C^-;
[c] The occupation measures of f, and of f^*, are identical;
[d] $Pf/P[f^*]$ is constant for f in C^+.

Lemma 1. *Dualities exist.*

Associate to f not only T, but also the last time, z, that f is at the initial vertex, v_0. Then define the one-one map t to t^* of $[0, T]$ onto itself by letting t^* be t or $T+z-t$ according as $t < z$ or not. Define f^* by $f^*t = f[t*]$. Call the map that associates f^* to f the FLIP operation.

Lemma 2. *The flip operation is a duality.*

Proof. That the flip operation satisfies the three conditions, [a], [b] and [c] is easily verified. As for [d], Pf is plainly some product of w_i's and q_i's, where q_i is an abbreviation for $[1 - wi]$. The number, U_i, of w_i that occur in the product is the number of upcrossings by f of the edge, $E_i = (i, i+1)$; and the number, D_i, of q_{i+1} that occur in the product is the number of downcrossings of E_i by f. Plainly, for f in C^+,

(i) $\quad U_i = 1 + D_i.$

Let U^* and D^*, functions of f and i, be the number of up and down crossings of E_i by f^*, the flip of f, or, as is equivalent, the number of w_i and of q_{i+1} that occur when $P[f*]$ is expressed as a product of w_i's and q_i's. So $Pf/P[f*]$, say Rf, is a product of $U_i - U_i^*$ factors w_i divided by the product of $D_i^* - D_i$ factors q_{i+1}. As is easily verified, $U^* = D$, and $D^* = U$. So $U - U^* = U - D = 1$ by (i), and $D^* - D = U - D = 1$. Therefore, Rf is the product of w_i/q_{i+1} for i in $[1, g]$, which plainly equals the product of the g odds ratios w_i/q_i, so [d] holds. ∎

Plainly, Lemma 2 implies Lemma 1 and Lemma 1 implies Theorem 1.

Remark 1. The flip operation is universal in that it serves as a duality for all nearest-neighbor walks on Q. Whether there are others, and if so which, is not studied herein.

Subsets of C, and functions defined on C, are FLIP INVARIANT if, upon replacing f by f^*, their value does not change. As is evident from Lemma 2:

Corollary 1. *The event, C^+, that a clockwise winding occurs before a counterclockwise one, is stochastically independent of the flip-invariant random variables and events, and, therefore, of the occupation measure, and, a fortiori, of the duration.*

Adapting an argument well-known in the classical de Moivre case of constant w, one obtains from Theorem 1:

Corollary 2. *The probability that a random walk on the vertices of a polygon, Q, ever achieves a full clockwise winding of Q is the minimum of 1 and W/W^-.*

2. Periodic walks on the integers. For a slight modification of Theorem 1 and Corollary 2, transfer the walk on the g vertices of the polygon, Q, to the integers, where it becomes a nearest- neighbor walk, X, of period g, meaning $w_i = w_{i+g}$ for all integers i. Let X_0 be 0, and call the first moment, T, that X reaches an endpoint of the interval $[-g, g]$ the DURATION, and let C^+ be the event that X at time T is g.

Theorem 1A. *The probability of C^+ is $W/[W + W^-]$, which is the same as its conditional probability given that $T = g$; the probability that X ever reaches g is the minimum of 1 and the ratio W/W^-; C^+ is stochastically independent of the duration.*

That, for walks with constant w, C^+ is stochastically independent of the duration was observed by S. Samuels and reported in [3].

Remark 2. In contrast to Theorem 1, C^+ is not stochastically independent of the occupation measure, M. This is evident since the probability of C^+ is positive but its conditional probability given that $M([0, g])$ is less than g, is zero.

Of course, Theorem 1A can be obtained either by imitating the observations made for the polygonal walk or by deriving it as a corollary to those observations.

3. Diffusions X on the line, L. This section and the next concern applications to the continuous case of the observations made above for the discrete case. Suppose that, for each x on L, and each finite, open interval, J, containing x, the expectation of the time, $t[x, J]$, say $T[x, J]$, for X, initially at x, to exit from J, is finite. Let $P[x, J]$ be the probability that this exit occurs at the right-hand end-point of J.

The next lemma and proof must surely be known, and in the literature, but aware of no reference, I record it here.

Lemma 3. *The hitting times $T[x, J]$ determine the hitting probabilities, $P[x, J]$.*

Proof. Let $b < c$ be the end points of J. Verify that J is a subinterval of some finite interval, K, for which the difference, D, between $T[c, K]$ and $T[b, K]$ is not 0. Verify, too, that $T[x, K]$ minus $T[x, J]$ is equal to the sum of two terms, say F and G, where F is the product of $P[x, J]$ with $T[c, K]$, and G is the product of $(1 - P[x, J])$ with $T[b, K]$. Since D, the coefficient of $P[x, J]$ in this equality, is not 0, $P[x, J]$ is the ratio of $T[x, K] - T[x, J] - T[b, K]$ to D. ∎

If $T[x, J]$ remains unchanged whenever both x and J are translated to the right by g, then X is of PERIOD g. If the weaker condition that $P[x, J]$ remains unchanged by those translations, then X is of QUASI-PERIOD g. By the argument for Lemma 3, one may verify:

Lemma 4. *Diffusions of period g are also of quasi-period g.*

Remark 3. The converse does not hold. For example, each unbounded martingale diffusion, X, is quasi-periodic for all g, but, it is not of period 1 if, for instance, it behaves like Brownian motion, of speed 1 for $X < 0$, and of some other constant speed for $X > 0$.

{*A Digression Concerning Existence.* That the existence of such X is a consequence of general results in the diffusion literature has been persuasively assured to me by Steve Evans and Jim Pitman independently. Exchanges with Gideon Schwarz led to a seemingly more direct approach to existence, probably also known to cogniscienti. Namely, consider a discrete version of X: that unique martingale, nearest-neighbor, random walk, Y, with $Y_0 = 0$, that has for its state space, the set theoretic union, J, of the negative integers and the non-negative even integers, and show, via a limiting argument, that the discrete Y leads to a continuous X. These two related approaches to existence may be supplemented by an equally related third: modify standard Brownian motion, B, by an appropriate, path-dependent change of the time scale: if, up to time t, B spent time, n, below 0, and time, p, above 0, let $h[t]$ be $n + cp$, where c is a positive constant, and let X at time $h[t]$ be B_t.}

Assume henceforth that the scale function of X has an everywhere positive derivative, s', and that $X_0 = 0$.

Theorem 1B. *If X is of quasi-period g, and, a fortiori, if it is of period g, then the probability that X reaches g before $-g$ is the ratio of s'_0 to the sum of s'_0 and s'_g.*

Proof. Without real loss, let the quasi-period be 1. For a fixed positive integer, g, let e_j be the sign of the jth change by X of size $1/g$, and let Z be the sequence of the partial sums of the e_j. As is easily verified, Z is a nearest-neighbor random walk of period g. Moreover, Z reaches g before $-g$ iff X reaches 1 before -1. As Theorem 1A implies, P, the probability of success, satisfies $P/[1-P] = W/W^-$, where W is the product of the g w_i, the probability that, when at i/g, X gains $1/g$ before losing $1/g$, and W^- is the product of the $[1-w_i]$. Express, w_i, $1-w_i$, and their ratio, r_i, in terms of the increments of the scale function, s. That is, each point i/g is the end point of two intervals, L_i and R_i, of length $1/g$, where L_i is to the left of R_i, and the ratio of the increment of s over L_i to its increment over R_i is r_i. So, in calculating the product of the r_i, say R, most increments of s cancel, and R is seen to equal the ratio of the increment of s over the interval of length $1/g$ whose right-hand end point is 0 to the increment of s over the interval of length $1/g$ whose right-hand end point is 1. This ratio has the same value as the ratio of the slopes of the chords of s over these two intervals. The conclusion follows by letting g approach infinity. ∎

4. Diffusions X on a circle S. Each process, Y, on the line projects, via the exponential map, to a process, $Y\tilde{}$, on the circle, S. As is easily verified, if Y is a diffusion of period equal to the perimeter of S, then $Y\tilde{}$ is a diffusion on S, and each diffusion, X, on S is such a $Y\tilde{}$, for some essentially unique periodic diffusion, Y.

In particular, the scale function, s, for Y is determined by X. There should therefore be no confusion if s is referred to as the scale function of X. For simplicity of exposition, the regularity assumption that the scale function for X has an everywhere positive derivative, s', continues to be in force.

Because of the relationship between a diffusion X on the circle and its sister diffusion, Y, on L, from Theorem 1B easily follows:

Theorem 1C. *The probability that a diffusion, X, on a circle completes a clockwise winding of the entire circle before one that is counterclockwise, is the ratio of s'_0 to $[s'_0 + s'_g]$, where g is the perimeter of the circle. The probability that X will ever attain a positive winding of the circle is the minimum of 1 and the ratio of s'_0 to s'_g.*

Query. Can one describe interesting events associated with non-periodic nearest-neighbor walks, and/or some, beside C^+, associated with periodic, nearest-neighbor walks, whose probabilities are simple functions of the transition probabilities?

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