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THE MARGINAL PROBLEM IN ARBITRARY PRODUCT SPACES

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Using a property of perfect measures due to Marczewski and Ryll-Nardzewski (1953), we unify the solutions to the marginal problem for two-dimensional products. We then extend that property to arbitrary product spaces and provide a general solution to the marginal problem in arbitrary product spaces. Our results remove the restrictive topological assumptions in earlier works and are valid in spaces where the σ-algebras need not be countably generated. A general result on the existence of simultaneous preimage measures as well as a "converse" to it are derived.

1. Introduction. Let $\{(X_i, \mathcal{A}_i, P_i), i \in I\}$ be a family of probability spaces. The marginal problem is connected to probabilities P on the space $(\prod_{i\in I} X_i, \otimes_{i\in I} \mathcal{A}_i)$ such that P has the given family $\{P_i, i \in I\}$ as marginals, i.e. $P \circ \pi_i^{-1} = P_i$ for every $i \in I$ where $\pi_j : \prod_{i\in I} X_i \to X_j$ is the canonical projection map. It can be formulated as:

> Given $S \subset \prod_{i\in I} X_i$, what are the conditions that would ensure the existence of a probability P on $(\prod_{i\in I} X_i, \otimes_{i\in I}\mathcal{A}_i)$ with marginals $\{P_i, i \in I\}$ such that $P^*(S) = 1$?

For $I = \{1, 2\}$, with underlying spaces being Polish, this problem reduces to Strassen's (1965) marginal problem. Variants of this have been investigated earlier by Banach (1948), Marczewski (1948, 1951) for the case when projections are required to be stochastically independent under P and later for the general case by Kellerer (1964a,b) as well as Maharam (1971). The importance of the above formulation in applications can be found in Hoffman-Jørgensen (1987).

In recent work by Hansel and Troallic (1978, 1986), Shortt (1983) and Kellerer (1984, 1988) solutions to Strassen's version were derived under topological restrictions. Plebanek (1989) has combined the solution for the finitely

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additive case of Strassen's problem due to Hansel and Troallic (1986) with a key result on perfect measures due to Marczewski and Ryll-Nardzewski (1953) to obtain a general solution of the marginal problem for two-dimensional products. In this paper, we first view the two-dimensional problem in a different light using the notion of common extensions. We then obtain a general solution to the marginal problem in arbitrary product spaces. We also derive a general result on the existence of simultaneous preimage measures (see Lembcke (1982) and Shortt (1983)), a "converse" to it and discuss illustrative examples. Perfect measures which play a crucial role in proving these results have also led to the establishment of a general duality theorem for marginal problems (see Ramachandran and Rüschendorf (1995)).

2. Notations and Preliminary Results. We use customary measure theoretic terminology and notation (as, for instance, in Neveu (1965)). A finitely additive probability on an algebra will be called a *charge* and a σ-additive charge will be called a *measure*.

> **Special Notation:** If a script letter such as $\mathcal{A}$ (respectively $\mathcal{A}_i$) is used to denote a special class of subsets like an algebra, then the corresponding capital letter in roman such as A, A_j (respectively A_i, A_{ij}) etc. denote sets from that class; $\sum_{i \in I} G_i$ denotes the union of sets $\{G_i, i \in I\}$ that are pairwise disjoint.

Recall that P on $(X, \mathcal{A})$ is *perfect* iff for every real valued $\mathcal{A}$-measurable function f on X there is a linear Borel set $B_f \subset f(X)$ such that $P(f^{-1}(B_f)) = 1$. For the properties of perfect measures, we refer the reader to Ramachandran (1979).

If $\mathcal{C}$ is a collection of subsets of Ω, then alg($\mathcal{C}$) and $\sigma(\mathcal{C})$ will denote respectively the algebra and the σ-algebra generated by $\mathcal{C}$; $\mathcal{C}^{\cup\delta}$ will denote the smallest class containing $\mathcal{C}$ which is closed under finite unions and countable intersections. Recall that a class $\mathcal{S}$ of subsets of a set Ω is called a *semialgebra* if it satisfies: (a) $\Omega, \emptyset \in \mathcal{S}$ (b) $\mathcal{S}$ is closed under finite intersections and (c) if $S \in \mathcal{S}$ then S^c is a finite disjoint union of members of $\mathcal{S}$.

If $\mathcal{S}$ is a semialgebra then alg($\mathcal{S}$) $= \{A : A$ is a finite disjoint union of members of $\mathcal{S}\}$ (see Proposition I.6.1 in Neveu (1965)) and so we can talk of a charge (measure) defined on $\mathcal{S}$ since it admits a unique extension as a charge (measure) to alg($\mathcal{S}$)). When dealing with the product of given measurable spaces, $\mathcal{R}$ will denote the semialgebra of measurable rectangles depending on finitely many coordinates in the product space under consideration.

We unify the known results by showing that the solution to the marginal problem consists of the three steps: (i) obtaining a charge on $\mathcal{R}$ with the given marginals (ii) forcing the charge to be σ-additive and (iii) ensuring that $P^*(S) = 1$. To achieve this goal we first look at the common extension problem.

Let Ω be an abstract set and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two algebras of subsets of Ω. Let $\mathcal{C}_1 \vee \mathcal{C}_2 = alg(\mathcal{C}_1 \cup \mathcal{C}_2) = \{\sum_{k=1}^{n}(C_{1k} \cap C_{2k}) : n \geq 1\}$. If μ_1 and μ_2 are two charges on $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively, then a charge (measure) μ_0 on $\mathcal{C}_1 \vee \mathcal{C}_2$ such that $\mu_0 \mid \mathcal{C}_i = \mu_i$, $i = 1, 2$ will be called a common extension charge (measure) of μ_1 and μ_2. We quote the following result due to Guy (1961) (An error in Guy's proof has been corrected and the result has been extended to vector valued charges by Schmidt and Waldschaks (1991)):

PROPOSITION 1. *There is a common extension charge of μ_1 and μ_2 iff*

$$C_1 \subset C_2 \Rightarrow \mu_1(C_1) \leq \mu_2(C_2). \tag{1}$$

We need the

LEMMA 1. *Let $\mathcal{S}$ be a semialgebra of subsets of Ω. A charge μ_0 on $\mathcal{S}$ is a measure iff*

$$\Omega = \bigcup_{n=1}^{\infty} S_n \Rightarrow \sum_{n=1}^{\infty} \mu_0(S_n) \geq 1. \tag{2}$$

PROOF. Necessity is obvious. To prove the sufficiency, first note that μ_0 is countably superadditive since it is a charge. Let $F = \sum_{n=1}^{\infty} F_n$, where $\{F, F_n, n \geq 1\} \subset \mathcal{S}$. Then $\Omega = \sum_{n=1}^{\infty} F_n + F^c = \sum_{n=1}^{\infty} F_n + \sum_{i=1}^{m} S_i$. Hence $(2) \Rightarrow \sum_{n=1}^{\infty} \mu_0(F_n) + \sum_{i=1}^{m} \mu_0(S_i) \geq 1 \Rightarrow \sum_{n=1}^{\infty} \mu_0(F_n) \geq 1 - \sum_{i=1}^{m} \mu_0(S_i) = 1 - \mu_0(F^c) = \mu_0(F)$. Thus, μ_0 is countably subadditive as well. ∎

PROPOSITION 2. *There exists a common extension measure Q on $\mathcal{C}_1 \vee \mathcal{C}_2$ extending the measures Q_i on $\mathcal{C}_i$, i=1,2 iff there exists a common extension charge Q_0 of $Q_1$and Q_2 such that*

$$\Omega = \bigcup_{n=1}^{\infty}(C_{1n} \cap C_{2n}) \Rightarrow \sum_{n=1}^{\infty} Q_0(C_{1n} \cap C_{2n}) \geq 1. \tag{3}$$

PROOF. Necessity is clear. If the given condition holds then, by Lemma 1, Q_0 is a measure on the semialgebra $\mathcal{S} = \{C_1 \cap C_2\}$. Hence Q_0 extends as a measure Q to $\text{alg}(\mathcal{S}) = \mathcal{C}_1 \vee \mathcal{C}_2$. ∎

REMARK 1. In pursuit of a problem of Marczewski (1951), Stroock (1976) defined a measure η on $\mathcal{C}_1 \vee \mathcal{C}_2$ to be a *splicing* of measures μ_i on $\mathcal{C}_i$, i=1,2 if

$$\eta(C_1 \cap C_2) = \mu_1(C_1) \ \mu_2(C_2) \quad \text{for every } C_1, C_2.$$

Marczewski had shown that

$$C_1 \cap C_2 = \phi \Rightarrow \mu_1(C_1) \ \mu_2(C_2) = 0 \tag{4}$$

is necessary and sufficient for a finitely additive splicing to exist (see also Proposition 2 in Kallianpur and Ramachandran (1983)). Stroock's main theorem proves that

$$\Omega = \bigcup_{n=1}^{\infty}(C_{1n} \cap C_{2n}) \Rightarrow \sum_{n=1}^{\infty} \mu_1(C_{1n})\ \ \mu_2(C_{2n})\ \ \geq\ \ 1 \tag{5}$$

is necessary and sufficient for η to be a splicing of μ_1 and μ_2. We have a short proof of this result using Lemma 1: Writing

$$\Omega = (C_1 \cap C_2) \cup (C_1 \cap C_2^c) \cup (C_1^c \cap C_2) \cup (C_1^c \cap C_2^c)$$

we can check that (5) $\Rightarrow$ (4) and hence η is a charge. By Lemma 1, (5) implies that η is a measure. Thus Proposition 2 is a generalization of Stroock's result. For further extension of Stroock's results we refer the reader to Hackenbroch (1992).

3. Main Results. Let $\{(X_i, \mathcal{A}_i, P_i), i \in I\}$ be a family of probability spaces and let $S \subset \prod_{i \in I} X_i$. Let $\mathcal{R}$ denote the semialgebra of measurable rectangles in $\otimes_{i \in I} \mathcal{A}_i$.

DEFINITION 1. S is called marginalizable if there exists a probability P on $(\prod_{i \in I} X_i, \otimes_{i \in I} \mathcal{A}_i)$ with marginals $\{P_i, i \in I\}$ such that $P^*(S) = 1$.

THEOREM 1. *S is marginalizable iff there is a charge P_0 on $\mathcal{R}$ with marginals $\{P_i, i \in I\}$ such that*

$$S \subset \bigcup_{n=1}^{\infty} R_n \Rightarrow \sum_{n=1}^{\infty} P_0(R_n) \geq 1. \tag{6}$$

PROOF. (6) is clearly necessary. To prove the sufficiency, let (6) hold. It is easy to check that $\mu_0(R \cap S) = P_0(R), R \in \mathcal{R}$ defines unambiguously a charge on $\mathcal{R} \cap S$. By Lemma 1, μ_0 is a measure on $\mathcal{R} \cap S$ and hence extends as a measure μ on $(\otimes_{i \in I} \mathcal{A}_i) \cap S$. Let $P(E) = \mu(E \cap S), E \in \otimes_{i \in I} \mathcal{A}_i$. ∎

Although Theorem 1 solves the marginal problem, it may not be easy to check whether (6) holds. This has been noted by Lembcke (1981) as well. At this stage, in the other works cited, topological conditions were imposed on the underlying spaces to make the charge countably additive. Then an additional condition was imposed on S to ensure that the obtained measure is supported by S. We avoid the topological restrictions by requiring all but perhaps one of the measures to be perfect and impose an analogous condition on S in order to unify the known results. A similar approach can be found in Plebanek (1989) for two dimensional products.

DEFINITION 2. Let $\mathcal{S}$ be a semialgebra of subsets of Ω. A set $S \subset \Omega$ is called $\mathcal{S}$-enclosable if

$$S^* = (\cap\{E \in \sigma(\mathcal{S}) : E \supset S\}) \in \mathcal{S}^{\cup\delta}.$$

In the context of a product space and the semialgebra $\mathcal{R}$ of measurable rectangles we shall simply use the term *enclosable* instead of *$\mathcal{R}$-enclosable.* In the case when I is a countable, nonempty index set and, for each $i \in I$, $\mathcal{A}_i$ is generated by a countable family which separates the points of X_i a subset $S \subset \prod_{i\in I} X_i$ is enclosable iff it is possible to choose metrics d_i for X_i, $i \in I$ making $\mathcal{A}_i$ the Borel σ-algebra under d_i such that S is closed in the corresponding product of these metric topologies (see Shortt (1983), p. 314).

The following result due to Marczewski and Ryll-Nardzewski (1953) is crucial for reaching our goal (see Corollary 3.2.2. of Ramachandran(1979)). Later, we extend this result in Theorem 6 to arbitrary products in order to obtain a general solution to the marginal problem.

THEOREM 2. *A charge on the semialgebra of measurable rectangles of the product of two measurable spaces with countably additive marginals is countably additive if at least one of the marginals is perfect.*

THEOREM 3. *Let $(X_i, \mathcal{A}_i, P_i)$, i=1,2 be two probability spaces where at least one of the measures P_1 and P_2 is perfect. A subset $S \subset X_1 \times X_2$ is marginalizable if*

(a) S is enclosable, and

(b) $(A_1 \times X_2) \cap S \subset (X_1 \times A_2) \cap S \Rightarrow P_1(A_1) \leq P_2(A_2)$.

PROOF. Let $\Omega = \mathcal{S}, \mathcal{C}_1 = (\mathcal{A}_1 \times X_2) \cap S, \mu_1((A_1 \times X_2) \cap S) = P_1(A_1), \mathcal{C}_2 = (X_1 \times \mathcal{A}_2) \cap S$ and let $\mu_2((X_1 \times A_2) \cap S) = P_2(A_2)$. μ_1 is well-defined because $(A_1 \times X_2) \cap S = (A_1' \times X_2) \cap S$ implies that $((A_1 \Delta A_1') \times X_2) \cap S = \emptyset = (X_1 \times \emptyset) \cap S$ and so by (b), $P_1(A_1 \Delta A_1') = 0$; similarly μ_2 is well-defined. By Proposition 1, there is a common extension charge μ_0 on $\mathcal{C}_1 \vee \mathcal{C}_2 = \text{alg}(\mathcal{R}) \cap S$. Let $\mu(E) = \mu_0(E \cap S)$, $E \in \text{alg}(\mathcal{R})$. Then μ is a charge on alg($\mathcal{R}$) concentrated on S (i.e., $E \in \text{alg}(\mathcal{R})$, $E \supset S \Rightarrow \mu(E) = 1$). Since μ has P_1 and P_2 as marginals of which P_2 is perfect, by Theorem 2, μ is a measure. Since S is enclosable, $P^*(S) = 1$ where P is the extension of μ to $\mathcal{A}_1 \otimes \mathcal{A}_2$ as a measure. ∎

Theorem 3 unifies and extends beyond the context of analytic and separable spaces the results contained in Theorem 1 of Shortt (1983) and Proposition (3.8) of Kellerer (1984). Recently Plebanek (1989) has used the above approach together with a result concerning charges by Hansel and Troallic (1986) to establish the following general solution of the marginal problem for two-dimensional products:

Let $(X_i, \mathcal{A}_i, P_i), i = 1,2$ be two probability spaces. For $E \subset X_1 \times X_2$ define

$$\eta(E) = \sup\{P_1(A_1) + P_2(A_2) : (A_1 \times A_2) \cap E = \emptyset\}.$$

THEOREM 4. *Let $I = \{1,2\}$ and let at least one of P_1 and P_2 be perfect. Let $d \geq 0$. If $D \in \mathcal{A}_1 \otimes \mathcal{A}_2$ is enclosable then the existence of a probability P with marginals P_1 and P_2 such that $P(D) \geq 1-d$ is equivalent to the condition $\eta(D) \leq 1+d$.*

In view of Definition 2, using Theorem 4, we can recast Theorem 3 as the following

COROLLARY 1. *Let $I = \{1,2\}$ and let at least one of P_1 and P_2 be perfect. A subset $S \subset X_1 \times X_2$ is marginalizable if*

(a) S is enclosable, and

(b) $\eta(S^*) \leq 1$.

We now discuss two examples.

EXAMPLE 1. Let X_1 be a subset of $[0,1]$ with outer Lebesgue measure $P^*(X_1) = 1 = P^*(X_1^c)$. Let $X_2 = X_1^c$, $\mathcal{A}_1 = \{B \cap X_1 : B$ is a Borel subset of $[0,1]\}$ = the trace of the Borel σ-algebra on X_1, $\mathcal{A}_2$ = the trace of the Borel σ-algebra on $X_2, P_1 = P_2 = P^*$ and let $S = \{(x_1,x_2) : x_1 \geq x_2\}$. Then, S is not marginalizable noting that $(X_1 \times X_2) \cap$ (Diagonal in $[0,1] \times [0,1]$) is empty. However,

$$S = \cap_{n=1}^{\infty} \cup_{k=0}^{n-1} ([\frac{k}{n}, \frac{k+1}{n}) \times [0, \frac{k+1}{n}) \cap (X_1 \times X_2)) \in \mathcal{R}^{\cup\delta}$$

is enclosable and it can be checked that $\eta(S) = 1$.

If both the measures P_1 and P_2 are not perfect then the above example due to Shortt (1983) shows that (a) and (b) do not guarantee that S is marginalizable. This is not surprising if we look at the following characterization of perfect measures due to Pachl (1979) (see Theorem 12.2.1 in Ramachandran (1979)):

A measure P on $(X, \mathcal{A})$ is perfect iff for every probability space $(Y, \mathcal{B}, Q)$, every charge on the class of measurable rectangles with marginals P and Q is countably additive.

The following example due to Kellerer (1964a, p. 196) is illustrative:

EXAMPLE 2. Let $X_1 = X_2 = [0,1]$, $\mathcal{A}_1 = \mathcal{A}_2$ = Borel σ-algebra on $[0,1]$, $P_1 = P_2$ = Lebesgue measure and let $S = \{(x_1,x_2) : x_1 > x_2\}$. Since S is open it is not enclosable. The measures are perfect and condition (b) of Theorem

3 holds. Hence there is a charge μ_0 on S with the Lebesgue measure as the marginals. For every such charge μ_0 $((A_1 \times A_2) \cap S) = 0$ if $A_1 \cap A_2 = \phi$. Since

$$S = \bigcup_{n=1}^{\infty} \sum_{k=1}^{n-1} [\frac{k}{n}, \frac{k+1}{n}) \times [0, \frac{k}{n}),$$

for every such charge μ_0 we have

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \mu_0 ([\frac{k}{n}, \frac{k+1}{n}) \times [0, \frac{k}{n})) = 0.$$

So Theorem 1 implies that S is not marginalizable.

Although in the above example S is not enclosable we emphasize that it is the failure of (6) that prevents S from being marginalizable. For, if we take $S_1 = \{(x_1, x_2) : x_1 \neq x_2\}$ then S_1 is not enclosable either. But S_1 supports the product Lebesgue measure. Thus being enclosable is sufficient but not necessary for being marginalizable.

We now turn our attention to the existence of simultaneous preimage measures (See Lembcke (1982), Theorems 2, 3 and 4 of Shortt (1983)) for two-dimensional products.

THEOREM 5. *Let $f_i : (S, \mathcal{C}) \longrightarrow (X_i, \mathcal{A}_i, P_i)$, $i = 1, 2$ be two measurable mappings. Then the following conditions are sufficient for the existence of a probability Q on $(S, \mathcal{C})$ such that $Qf_i^{-1} = P_i$ for $i = 1, 2$:*

(a) At least one of P_1 and P_2 is perfect,

(b) $\mathcal{C} = \sigma(f_1, f_2)$,

(c) $f_1^{-1}(A_1) \subset f_2^{-1}(A_2) \Rightarrow P_1(A_1) \leq P_2(A_2)$, and

(d) $\{(x_1, x_2) : f_1^{-1}(x_1) \cap f_2^{-1}(x_2) \neq \emptyset\}$ is enclosable.

PROOF. Let $F(s) = (f_1(s), f_2(s))$ define a map $F : S \longrightarrow X_1 \times X_2$. By (b), $F^{-1}((\mathcal{A}_1 \otimes \mathcal{A}_2) \cap F(S)) = \mathcal{C}$. If $(A_1 \times X_2) \cap F(S) \subset (X_1 \times A_2) \cap F(S)$ then applying F^{-1} we get $f_1^{-1}(A_1) \subset f_2^{-1}(A_2)$; by (c), $P_1(A_1) \leq P_2(A_2)$. The set in (d) is precisely $F(S)$. By Theorem 3, $F(S)$ is marginalizable. Take $Q = P^* F$. ∎

Examples 1 and 2 show respectively that if (a) or (d) does not hold then the conclusion of Theorem 5 may not hold. Conditions (b) and (c) are easily seen to be natural in the context.

We now establish general results concerning the marginal problem and the simultaneous preimage measure problem for the case of arbitrary products of probability spaces. We first prove the following generalization of Theorem 2.

THEOREM 6. *Let $\{(X_i, \mathcal{A}_i), i \in I\}$ be a family of measurable spaces. If μ is a charge on the semialgebra $\mathcal{R}$ with countably additive marginals such that all but perhaps one of its marginals are perfect then μ is countably additive.*

PROOF. Let i_0, if it exists, be such that P_{i_0} is not perfect. Now $\mu_0 = \mu \mid \mathcal{R}_0$ where $\mathcal{R}_0 = \{R : proj_{i_0}(R) = X_{i_0}\}$ is a charge such that every marginal is perfect. By a theorem of Ryll-Nardzewski's (see Theorem 3.1.2 in Ramachandran (1979)) μ_0 is countably additive on $\mathcal{R}_0$ and its unique extension $\overline{\mu}_0$ to $\sigma(\mathcal{R}_0)$ is a perfect measure. Now let $\Omega = \prod_{i \in I} X_i$, $\mathcal{C}_1 = alg(\mathcal{R})$, $\mu_1 =$ the unique charge on $alg(\mathcal{R})$ induced by μ, $\mathcal{C}_2 = \sigma(\mathcal{R}_0)$, $\mu_2 = \overline{\mu}_0$; so $\mathcal{C}_1 \vee \mathcal{C}_2 = alg(\mathcal{R}')$ where $\mathcal{R}'$ is the class of measurable rectangles in $(X_{i_0}, \mathcal{A}_{i_0}) \times (\prod_{i \neq i_0} X_i, \otimes_{i \neq i_0} \mathcal{A}_i)$. Suppose $C_1 \subset C_2$; then $C_1 = \sum_{j=1}^n R_j$ and $C_2 = X_{i_0} \times A$ where $A \in \otimes_{i \neq i_0} \mathcal{A}_i$. Hence $C_1 = \sum_{j=1}^n R_j \subset X_{i_0} \times (\cup_{j=1}^n proj_{I-\{i_0\}} R_j) \subset X_{i_0} \times A = C_2$ whereby $\mu_1(C_1) \leq \mu_1(X_{i_0} \times (\cup_{j=1}^n proj_{I-\{i_0\}} R_j)) = \mu_0(X_{i_0} \times (\cup_{j=1}^n proj_{I-\{i_0\}} R_j)) \leq \overline{\mu}_0(X_{i_0} \times A) = \mu_2(C_2)$. By Proposition 1, there is a common extension charge P_0 on $\mathcal{C}_1 \vee \mathcal{C}_2$ of μ_1 and μ_2. Since $\mathcal{C}_1 \vee \mathcal{C}_2 = alg(\mathcal{R}')$ and μ_2 is perfect, by Theorem 2, P_0 is a measure on $\mathcal{C}_1 \vee \mathcal{C}_2$. Hence $P_0 \mid \mathcal{R} = \mu$ is countably additive. ∎

Let $(X_i, \mathcal{A}_i, P_i)$, $i \in I$ be a family of probability spaces. Consider the product space $(\prod_{i \in I} X_i, \otimes_{i \in I} \mathcal{A}_i)$. Analogous to Theorem 3 we have

THEOREM 7. *A subset $S \subset \prod_{i \in I} X_i$ is marginalizable if the following conditions hold:*

(a) All but perhaps one of the P_i's are perfect.

(b) For each finite subset $J \subset I$ and for every choice $\{g_j, j \in J\}$ where g_j on X_j is $\mathcal{A}_j$-measurable and bounded for each $j \in J$

$$\sum_{j \in J} g_j \circ proj_j 1_S \geq 0 \Rightarrow \sum_{j \in J} \int_{X_j} g_j \, dP_j \geq 0.$$

(c) S is enclosable.

PROOF. By using Theorem 6.1 from Maharam (1971), (b) implies the existence of a charge μ on $\mathcal{R}$ concentrated on S with $\{P_i, i \in I\}$ as the marginals. By (a) and Theorem (6), μ is σ-additive and extends as a measure P to $\sigma(\mathcal{R})$. By (c), $P^*(S) = 1$. ∎

Theorem 5 has a corresponding version in

THEOREM 8. *Let $(S, \mathcal{C})$ be a measurable space and let, for each $i \in I$, $f_i : (S, \mathcal{C}) \longrightarrow (X_i, \mathcal{A}_i, P_i)$ be a measurable map . The following conditions are* *sufficient to ensure the existence of a measure Q on $(S,\mathcal{C})$ such that $Qf_i^{-1} = P_i$ for each $i \in I$:*

(a) *All but perhaps one of the P_i's are perfect.*

(b) $\mathcal{C} = \sigma(\{f_i, i \in I\})$.

(c) $\sum_{j\in J} g_j \circ f_j \geq 0 \Rightarrow \sum_{j\in J} \int g_j\, dP_j \geq 0$ *for each finite subset $J \subset I$ and for every choice $\{g_j, j \in J\}$ where g_j on X_j is $\mathcal{A}_j$-measurable and bounded for each $j \in J$, and*

(d) *The set $\{\{x_i\}_{i\in I} : \cap_{i\in I} f_i^{-1}(x_i) \neq \phi\}$ is enclosable.*

PROOF. Let $F(s) = \{f_i(s)\}_{i\in I}$ define a map $F : S \longrightarrow \prod_{i\in I} X_i$. By (b), $F(\mathcal{C}) = \otimes_{i\in I}\mathcal{A}_i \cap F(S)$. By (c) and Theorem 6.1 of Maharam (1971) there is a charge μ on $\mathcal{R}$ concentrated on $F(S)$ with $\{P_i, i \in I\}$ as marginals. $F(S)$ is the set in (d) and, by Theorem 7, it is marginalizable. Take $Q = P^*F$ to complete the proof. ∎

COROLLARY 2. *By taking $S = \prod_{i\in I} X_i$ and letting f_J be projections of S to finite partial products $\prod_{j\in J} X_j$ we obtain the Daniell-Kolmogorov consistency theorem for products of standard Borel spaces.*

Examples 4 and 6 of Shortt (1983) show respectively that if (a) or (d) does not hold then the conclusion of Theorem 8 may fail even on separable spaces.

Finally, we investigate the "converse" direction to obtain results that will encompass Corollary 3 of Hansel and Troallic (1978) and Theorem 5 of Shortt (1983). The following theorem shows how to lift measures on products of sub σ-algebras with given marginals to the entire product σ-algebra. It is similar in spirit to Theorem 4 of Plebanek (1989) but neither result contains the other and the proofs use different techniques. Theorem 9 plays a major role in the proof of a general duality theorem (see Ramachandran and Rüschendorf (1995)).

THEOREM 9. *Let $(X_1, \mathcal{A}_1, P_1)$ and $(X_2, \mathcal{A}_2, P_2)$ be two probability spaces of which P_2 is perfect. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are sub σ-algebras of $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively and if λ is a measure on $\mathcal{D}_1 \otimes \mathcal{D}_2$ with $P_1 \mid \mathcal{D}_1$ and $P_2 \mid \mathcal{D}_2$ as the marginals then λ admits an extension $\overline{\lambda}$ to $\mathcal{A}_1 \otimes \mathcal{A}_2$ with marginals P_1 and P_2.*

PROOF. Let $\Omega = X_1 \times X_2$, $\mathcal{C}_1$ = algebra generated by measurable rectangles in $\mathcal{D}_1 \otimes \mathcal{D}_2$, $\mu_1 = \lambda \mid \mathcal{C}_1$, $\mathcal{C}_2 = \mathcal{A}_1 \times X_2$ and let μ_2 be defined by $\mu_2(A_1 \times X_2) = P_1(A_1)$. Suppose that $C_1 \subset C_2$. Then $C_1 = \sum_{k=1}^n D_{1k} \times D_{2k} \subset A_1 \times X_2 = C_2$; hence $\sum_{k=1}^n (D_{1k} \times D_{2k}) \subset (\cup_{k=1}^n D_{1k}) \times X_2 \subset A_1 \times X_2$ and so $\mu_1(C_1) = \sum_{k=1}^n \lambda(D_{1k} \times D_{2k}) \leq P_1(\cup_{k=1}^n D_{1k}) \leq P_1(A_1) = \mu_2(C_2)$. By Proposition 1, there exists a common extension charge μ_0 extending μ_1 and μ_2

to $\mathcal{C}_1 \vee \mathcal{C}_2$ which is the algebra generated by measurable rectangles in $\mathcal{A}_1 \otimes \mathcal{D}_2$. Since $P_2 \mid \mathcal{D}_2$ is perfect (see property P2 in Ramachandran (1979)), by Theorem 2, μ_0 is countably additive and extends as a measure $\overline{\mu}_0$ to $\mathcal{A}_1 \otimes \mathcal{D}_2$. By construction, the marginals of μ_0 are P_1 and $P_2 \mid \mathcal{D}_2$ and $\overline{\mu}_0 \mid \mathcal{D}_1 \otimes \mathcal{D}_2 = \lambda$. Repeating the argument starting with $\overline{\mu}_0$ on $\mathcal{A}_1 \otimes \mathcal{D}_2$ and the measure induced by P_2 on $X_1 \times \mathcal{A}_2$ we get the desired extension $\overline{\lambda}$ to $\mathcal{A}_1 \otimes \mathcal{A}_2$ with marginals P_1 and P_2. ∎

COROLLARY 3. *In the setup of Theorem 9, if $S \in \mathcal{D}_1 \otimes \mathcal{D}_2$ with $\lambda(S) = 1$, then there is a measure $\overline{\lambda}$ on $\mathcal{A}_1 \otimes \mathcal{A}_2$ with marginals P_1 and P_2 such that $\overline{\lambda}(S) = 1$.*

The sought "converse" is

THEOREM 10. *Let $(Z, \mathcal{C}, Q)$ be a probability space such that the diagonal $\triangle$ in Z^∞ is $\mathcal{C}^\infty$-measurable. Let $f_i : (X_i, \mathcal{A}_i, P_i) \longrightarrow (Z, \mathcal{C}, Q)$ be a measurable surjection such that $Q = P_i f_i^{-1}$ for $i = 1, 2, \ldots$ Let P_i be perfect for $i \geq 2$. Then there is a measure P on $(X, \mathcal{A}) = (\prod_{i=1}^\infty X_i, \otimes_{i=1}^\infty \mathcal{A}_i)$ with marginals P_i for all i such that $P(S) = 1$ where $S = \{(x_1, x_2, \ldots) : f_1(x_1) = f_2(x_2) = \ldots\}$.*

PROOF. Let $g : Z \longrightarrow Z^\infty$ be defined by $g(z) = (z, z, \ldots)$. Then $(Z, \mathcal{C})$ is isomorphic to $(\triangle, \mathcal{C}^\infty \cap \triangle)$ under g. Let μ on $\mathcal{C}^\infty \cap \triangle$ be defined by $\mu = Qg^{-1}$. Now let $F : \prod_{i=1}^\infty X_i \longrightarrow Z^\infty$ be defined by $F(x_1, x_2, \ldots) = (f_1(x_1), f_2(x_2), \ldots)$. Let $\mathcal{D}_i = f_i^{-1}(\mathcal{C})$ for $i \geq 1$. Since $\otimes_{i=1}^\infty \mathcal{D}_i$ is σ-isomorphic under F to $\mathcal{C}^\infty$, define $\lambda(\overline{D}) = \mu(F(\overline{D}) \cap \triangle)$, $\overline{D} \in \otimes_{i=1}^\infty \mathcal{D}_i$. Note that $F^{-1}(\triangle) = S$ and so $\lambda(S) = \mu(\triangle) = 1$. λ has marginals $P_i \mid \mathcal{D}_i$ for $i \geq 1$. Starting from λ on $\otimes_{i=1}^\infty \mathcal{D}_i$, $P_1 \circ (proj_1^{-1})$ on $\mathcal{A}_1 \times \prod_{i=2}^\infty X_i$ and applying the argument in the proof of Theorem 9 we extend λ to $\mathcal{A}_1 \otimes (\otimes_{i=2}^\infty \mathcal{D}_i)$ with marginals P_1 and $P_i \mid \mathcal{D}_i$ for $i \geq 2$. Repeating the argument with λ on $\mathcal{A}_1 \otimes (\otimes_{i=2}^\infty \mathcal{D}_i)$ and $P_2 \circ (proj_2^{-1})$ on $\mathcal{A}_2 \times (\prod_{i \neq 2} X_i)$ and so on we can extend λ as a charge to $\mathcal{R} \subset \otimes_{i=1}^\infty \mathcal{A}_i$ with marginals P_i for all $i \geq 1$. By Theorem 6, λ is countably additive and hence extends as a measure P on $\otimes_{i=1}^\infty \mathcal{A}_i$ with marginals P_i for all i. By construction, $P(S) = \lambda(S) = 1$. ∎

Theorem 10 subsumes Theorems 5 and 6 of Shortt (1983). The following example shows that the conclusion of Theorem 10 may not hold if two of the measures are not perfect.

EXAMPLE 3. Let $X_1 \subset [0,1], X_2 = [0,1] - X_1$ be such that (i) $\lambda^*(X_1) = \lambda^*(X_2) = 1$ and (ii) there exist Borel sets $N_i \subset X_i, i = 1, 2$ with each having cardinality c. Let $\mathcal{A}_i = \mathcal{L} \cap X_i, P_i = \lambda^*_{X_i}, i = 1, 2$ where λ is the Lebesgue measure and $\mathcal{L}$ is the σ-algebra of Lebesgue-measurable sets. Let $h_1 : N_1 \rightarrow$

$X_2 \cup N_1$ and $h_2 : N_2 \to X_1 \cup N_2$ be 1-1, onto maps. Let, for $i = 1, 2$,

$$f_i(x_i) = \begin{cases} x_i & \text{on } X_i - N_i, \\ h_i(x_i) & \text{on } N_i. \end{cases}$$

Then, $f_i : X_i \to [0,1]$ is 1-1, onto and if $B \in \mathcal{B}_{[0,1]}$ then

$$f_i^{-1}(B) = (B \cap (X_i - N_i)) \cup h_i^{-1}(B \cap (N_i \cup ([0,1] - X_i)))$$

is in $\mathcal{L} \cap X_i$ with $P_i f_i^{-1}(B) = \lambda(B)$. Taking $Z = [0,1], \mathcal{C} = \mathcal{B}_{[0,1]}, Q = \lambda$ we get the setup in Theorem 10 with both marginals nonperfect. The set $S = \{(x_1, x_2) : f_1(x_1) = f_2(x_2)\} \subset (N_1 \times X_2) \cup (X_1 \times N_2) \in \mathcal{A}_1 \otimes \mathcal{A}_2$ and so $P(S) = 0$ for every probability P on $(X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ with marginals P_1 and P_2.

In the above construction, we can also take $\mathcal{A}_i = \sigma(\mathcal{B} \cap X_i, h_I^{-1}(\mathcal{B} \cap (N_i \cup ([0,1] - X_i))))$ to get $\mathcal{A}_i$ which are countably generated.

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