Distributions with Fixed Marginals and Related Topics
IMS Lecture Notes – Monograph Series Vol. 28, 1996

DERIVABILITY OF SOME OPERATIONS ON DISTRIBUTION FUNCTIONS

BY ROGER B. NELSEN, JOSÉ JUAN QUESADA-MOLINA,[1]
BERTHOLD SCHWEIZER,[2] AND CARLO SEMPI[1]
Lewis and Clark College, Universidad de Granada,
University of Massachusetts, and Università di Lecce

In this paper we characterize the operations on distribution functions that are both derivable from functions on random variables defined on a common probability space and induced pointwise by functions from $[0,1]^n$ into $[0,1]$. We specify the class of functions on random variables from which the operations are derived and show that it includes all order statistics; and we give a description of the n-place functions from which these operations are induced pointwise. In addition, by way of illustration, we show that mixtures, which are induced pointwise, are not derivable.

1. Preliminary Concepts and Results. We shall denote by $\mathcal{D}$ the space of proper one-dimensional distribution functions (d.f.'s), i.e. the space of functions $F : \overline{\mathbf{R}} := [-\infty, +\infty] \to [0,1]$ that are nondecreasing, left-continuous on $\mathbf{R} := (-\infty, +\infty)$ and such that

$$F(-\infty) = 0 = \lim_{x \to -\infty} F(x) \text{ and } F(+\infty) = 1 = \lim_{x \to +\infty} F(x).$$

An n-operation ϕ on $\mathcal{D}$ is a mapping from $\mathcal{D}^n := \mathcal{D} \times \mathcal{D} \times \cdots \times \mathcal{D}$ into $\mathcal{D}$, i.e., a mapping that assigns a d.f. to every ordered collection of n d.f.'s. If X_i is a random variable (r.v.), we shall denote the distribution function of X_i by F_i, F_{X_i}, or $df(X_i)$, whichever is more convenient.

DEFINITION 1.1. An n-operation ϕ on $\mathcal{D}$ is said to be *derivable* from a function on r.v.'s if there exists a Borel measurable function V from $\overline{\mathbf{R}}^n$ into $\overline{\mathbf{R}}$ that

[1] Research supported by the Universities of Granada and Lecce, the Italian Consiglio delle Richerche and the Italian MURST.

[2] Research supported by ONR Grant N-00014-87-K-0379.

AMS 1991 Subject Classification: Primary 60E05 62E10.

Key words and phrases: Operations derivable from functions on random variables, operations induced pointwise, order statistics, Boolean functions, antichains.

satisfies the following condition: For every collection of n d.f.'s $F_1, F_2, \cdots, F_n$ in $\mathcal{D}$, there exist a probability space $(\Omega, \mathcal{A}, P)$ and an n-dimensional random vector $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)$ on $(\Omega, \mathcal{A}, P)$ whose one-dimensional marginals are $F_1, F_2, \cdots, F_n$, respectively, i.e. $df(X_i) = F_i$, and such that $\phi(F_1, F_2, \cdots, F_n)$ is the d.f. of the r.v. $V(X_1, X_2, \cdots, X_n)$ whose value for any ω in Ω is given by $V(X_1(\omega), X_2(\omega), \cdots, X_n(\omega))$.

Therefore, if ϕ is derivable, then

$$\phi(F_1, F_2, \cdots, F_n) = F_{V(X_1, X_2, \cdots, X_n)} \tag{1.1}$$

for every choice of $F_1, F_2, \cdots, F_n$ in $\mathcal{D}$, where $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)$ has a distribution function belonging to the Fréchet class of $F_1, F_2, \cdots, F_n$.

Definition 1.2. An n-operation ϕ on $\mathcal{D}$ is said to be *induced pointwise* by an n-place function Φ from $[0,1]^n$ into $[0,1]$ if

$$\phi(F_1, F_2, \cdots, F_n)(t) = \Phi[F_1(t), F_2(t), \cdots, F_n(t)]$$

for every choice of $F_1, F_2, \cdots, F_n$ in $\mathcal{D}$ and for every t in $\overline{\mathbf{R}}$.

To illustrate, the convolution $F_1 \star F_2 \star \cdots \star F_n$ of n d.f.'s $F_1, F_2, \cdots, F_n$ is an n-operation on $\mathcal{D}$ which, since it may be viewed as the d.f. of the sum of n independent r.v.'s, is derivable from the operation of addition. However, since the value $(F_1 \star F_2 \star \cdots \star F_n)(t)$ generally depends on more than the values $F_1(t), F_2(t), \cdots, F_n(t)$, this operation is not induced pointwise by any n-place function. In the other direction, the mixture $cF_1 + (1-c)F_2$, $0 < c < 1$, of two d.f.'s F_1 and F_2 is induced pointwise by the two-place function $\Phi(x,y) = cx + (1-c)y$; but, as shown in Alsina and Schweizer (1988), this mixture is not derivable from any binary operation on $\mathcal{D}$.

In Alsina, Nelsen, and Schweizer (1993), the first and third authors of this paper, in collaboration with C. Alsina, characterized the class of those binary operations on $\mathcal{D}$ which are both induced pointwise and derivable from functions on random variables. In this paper we generalize these results to n-operations on $\mathcal{D}$. We provide a complete characterization (see Theorem 2.2) of the functions V from which these n-operations are derived: this class includes the usual order statistics, and, indeed, its elements may be viewed as generalized order statistics. We also give a description (see Theorem 2.5) of the n-place functions from which these n-operations are induced pointwise.

To present our results we need a number of preliminary notions which are combinatorial in nature.

The set of vertices of the unit n-cube $[0,1]^n$ will be denoted by J_n, i.e. $J_n := \{(z_1, z_2, \cdots, z_n) \mid z_i = 0 \text{ or } 1, 1 \leq i \leq n\}$. The set J_n with the usual coordinate-wise partial ordering, given by $(y_1, y_2, \cdots, y_n) \leq (z_1, z_2, \cdots, z_n)$ if and only if $y_i \leq z_i$ for $1 \leq i \leq n$, is a lattice which, as is well-known, is isomorphic to the lattice $\mathcal{P}(\mathbf{n})$ of all subsets of $\mathbf{n} = \{1, 2, \cdots, n\}$ – the vertex $(z_1, z_2, \cdots, z_n)$ corresponding to the set of integers i for which $z_i = 1$. We let $\mathcal{F}_n$ denote the set of nondecreasing Boolean functions f from J_n onto $\{0,1\}$ and, for any $f \in \mathcal{F}_n$, we let S_f denote the set $\{(z_1, z_2, \cdots, z_n) \in J_n \mid f(z_1, z_2, \cdots, z_n) = 1\}$. Note that since f is onto $\{0,1\}$, the set S_f is neither empty nor equal to J_n. Clearly any $f \in \mathcal{F}_n$ is completely determined by the (non-empty) set of minimal elements of S_f. This set is an antichain (any two elements are incomparable) which corresponds to a non-empty antichain in $\mathcal{P}(\mathbf{n})$. Since any non-empty antichain in $J_n \backslash \{(0, 0, \cdots, 0)\}$ can be taken as the set of minimal elements of the set S_f associated with a nondecreasing Boolean function f from J_n onto $\{0,1\}$, it follows that $\mathcal{F}_n$, endowed with the usual pointwise partial ordering, is isomorphic to the set of non-empty antichains in $\mathcal{P}(\mathbf{n}) \backslash \{\emptyset\}$, ordered by set inclusion (for details, see Kleitman (1969)).

While we shall give a complete description of the case $n = 3$ in Section 3, it is instructive to consider an example from that case now. Suppose that h is the nondecreasing Boolean function from J_3 onto $\{0,1\}$ for which $S_h = \{(0,0,1), (0,1,1), (1,0,1), (1,1,0), (1,1,1)\}$. The set of minimal elements of S_h is $\{(0,0,1), (1,1,0)\}$; and the corresponding antichain in $\mathcal{P}(\mathbf{3}) \backslash \{\emptyset\}$ is $\sigma = \{\{3\}, \{1,2\}\}$. Furthermore, it can be shown, Harrison (1965), that h is given by the Boolean sum of products of the variables whose subscripts appear as members of the elements of σ; so that here $h(z_1, z_2, z_3) = z_3 + z_1 z_2$. We shall return to this example throughout this section and the next.

Now let Ψ_n be the set of all the functions Φ from $[0,1]^n$ into $[0,1]$ satisfying

(a) Φ is nondecreasing in each place on $[0,1]^n$ and left-continuous in each place on $(0,1]^n$;

(b) $\Phi(0, 0, \cdots, 0) = 0$ and $\Phi(1, 1, \cdots, 1) = 1$;

(c) $\Phi(z_1, z_2, \cdots, z_n)$ equals either 0 or 1 at every vertex $(z_1, z_2, \cdots, z_n) \in J_n$.

Note that the restriction of Φ to J_n is an element of $\mathcal{F}_n$.

Definition 1.3. Two functions Φ_1 and Φ_2 in Ψ_n are said to be *vertex-equivalent*, and we write $\Phi_1 \mathcal{E} \Phi_2$, if they take the same value at every vertex of $[0,1]^n$.

Clearly $\mathcal{E}$ is an equivalence relation on the set Ψ_n; and it follows at once from the above discussion that the quotient set $\Psi_n/\mathcal{E}$ is in one-to-one correspondence with $\mathcal{F}_n$. Thus the preceding discussion yields:

LEMMA 1.4. *The quotient set $\Psi_n/\mathcal{E}$ is in one-to-one correspondence with the set of non-empty antichains of $\mathcal{P}(\mathbf{n})\backslash\{\phi\}$.*

2. Derivable Operations. We begin with the following:

LEMMA 2.1. *If ϕ is an n-operation on $\mathcal{D}$ which is derivable from $V : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}$ and induced pointwise by $\Phi : [0,1]^n \to [0,1]$ then Φ belongs to Ψ_n.*

PROOF. Since ϕ is induced pointwise by Φ, for all $F_1, F_2, \cdots, F_n$ in $\mathcal{D}$, and all t in $\overline{\mathbf{R}}$, we have

$$\phi(F_1, F_2, \cdots, F_n)(t) = \Phi[F_1(t), F_2(t), \cdots, F_n(t)];$$

and since $\phi(F_1, F_2, \cdots, F_n)$ is in $\mathcal{D}$, setting, respectively, $t = -\infty$ and $t = +\infty$ yields $\Phi(0, 0, \cdots, 0) = 0$ and $\Phi(1, 1, \cdots, 1) = 1$. It is also easy to see that Φ is nondecreasing and left-continuous in each place.

Next, for any x in $\mathbf{R}$, let ε_x be the unit step function in $\mathcal{D}$ defined by

$$\varepsilon_x(t) = \begin{cases} 0, & t \le x, \\ 1, & t > x, \end{cases}$$

and for any $x_1, x_2, \cdots, x_n$ in $\mathbf{R}$, consider the d.f.'s $\varepsilon_{x_1}, \varepsilon_{x_2}, \cdots, \varepsilon_{x_n}$. Since ϕ is derivable from V, there is a probability space $(\Omega, \mathcal{A}, P)$ and a random vector $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ on $(\Omega, \mathcal{A}, P)$ such that $df(X_i) = \varepsilon_{x_i}$, whence $X_i = x_i$ P-a.s. $(i = 1, 2, \cdots, n)$. It follows that $V(X_1, X_2, \cdots, X_n)$ is a r.v. which is equal to $V(x_1, x_2, \cdots, x_n)$ P-a.s. Thus, for every $t \in \mathbf{R}$,

$$\Phi(\varepsilon_{x_1}(t), \varepsilon_{x_2}(t), \cdots, \varepsilon_{x_n}(t)) = F_{V(x_1, x_2, \cdots, x_n)}(t) = \varepsilon_{V(x_1, x_2, \cdots, x_n)}(t). \tag{2.1}$$

Now let $(z_1, z_2, \cdots, z_n)$ be an arbitrary vertex in J_n. Then it is clear that for an appropriate choice of $x_1, x_2, \cdots, x_n$ and t we have $\varepsilon_{x_i}(t) = z_i$, $i = 1, 2, \cdots, n$. (Indeed, we may let $x_i = 1 - z_i$ and $t = 1/2$.) Thus, since $\varepsilon_{V(x_1, x_2, \cdots, x_n)}(t)$ is either 0 or 1, it follows that Φ belongs to Ψ_n. ∎

Using Lemma 2.1, we can now give a representation of the Borel measurable functions V from which the ϕ's are derived.

THEOREM 2.2. *Let ϕ, V and Φ be as in Lemma 2.1. Let $\Phi/\mathcal{E}$ be the equivalence class of Φ, let f be the nondecreasing Boolean function in $\mathcal{F}_n$ that* *corresponds to $\Phi/\mathcal{E}$, and let σ be the corresponding antichain in $\mathcal{P}(\mathbf{n})\backslash\{\emptyset\}$. Then, for any $x_1, x_2, \cdots, x_n$ in $\overline{\mathbf{R}}$, we have*

$$V(x_1, x_2, \cdots, x_n) = \min\{\max\{x_j \mid j \in s\} \mid s \in \sigma\}. \tag{2.2}$$

For example, if $\sigma = \{\{3\}, \{1,2\}\}$, then $V(x_1, x_2, x_3) = \min\{x_3, \max\{x_1, x_2\}\}$.

PROOF. First consider the case when σ is a singleton, say $\sigma = \{s\}$. Then, since $f(z_1, z_2, \cdots, z_n) = 0$ unless all the z_i having subscripts in s are equal to 1, we have that

$$\Phi(\varepsilon_{x_1}(t), \varepsilon_{x_2}(t), \cdots, \varepsilon_{x_n}(t)) = \begin{cases} 0, & \text{if } t \le \max\{x_j \mid j \in s\}, \\ 1, & \text{if } t > \max\{x_j \mid j \in s\}, \end{cases}$$

from which, using (2.1), it follows that $V(x_1, x_2, \cdots, x_n) = \max\{x_j \mid j \in s\}$. Now suppose that σ has two elements, say $\sigma = \{s_1, s_2\}$. Then $f(z_1, z_2, \cdots, z_n) = 0$ unless all the z_i having subscripts in s_1 are 1, or all the z_i having subscripts in s_2 are 1, whence $\Phi(\varepsilon_{x_1}(t), \varepsilon_{x_2}(t), \cdots, \varepsilon_{x_n}(t))$ is 0 as long as t is less than or equal to the smaller of $\max\{x_j \mid j \in s_1\}$, $\max\{x_j \mid j \in s_2\}$ and $\Phi(\varepsilon_{x_1}(t), \varepsilon_{x_2}(t), \cdots, \varepsilon_{x_n}(t))$ is 1 for any larger t. It follows that in this case $V = (x_1, x_2, \cdots, x_n) = \min\{\max\{x_j \mid j \in s_1\}, \max\{x_j \mid j \in s_2\}\}$. Continuing in the same fashion yields (2.2). ∎

Note: It is convenient to view the composite function $\Phi(\varepsilon_{x_1}(t), \varepsilon_{x_2}(t), \cdots, \varepsilon_{x_n}(t))$ as a "two-stage binary counter" and to consider the operation of this counter as t increases from $-\infty$ to $+\infty$. For $t \le \min\{x_1, x_2, \cdots, x_n\}$, we have $\varepsilon_{x_1}(t) = \varepsilon_{x_2}(t) = \cdots = \varepsilon_{x_n}(t) = 0$, whence $\Phi(\varepsilon_{x_1}(t), \varepsilon_{x_2}(t), \cdots, \varepsilon_{x_n}(t)) = 0$. As t increases the ε_{x_i}'s begin to jump from 0 to 1 in an order which is determined by the ordering of the x_i. The resulting arguments of Φ form a chain from $(0, 0, \cdots, 0)$ to $(1, 1, \cdots, 1)$ in J_n – which has maximal length whenever the x_i are distinct. Finally $\Phi(\varepsilon_{x_1}(t), \varepsilon_{x_2}(t), \cdots, \varepsilon_{x_n}(t))$ jumps from 0 to 1 when this chain first hits the set S_f.

COROLLARY 2.3. *When $S_f = \{(1, 1, \cdots, 1)\}$, i.e., when $\sigma = \{\{1, 2, \cdots, n\}\}$, we have $V(x_1, x_2, \cdots, x_n) = \max\{x_1, x_2, \cdots, x_n\}$; when $S_f = J_n\backslash\{(0, 0, \cdots, 0)\}$, i.e., when $\sigma = \{\{1\}, \{2\}, \cdots, \{n\}\}$, we have $V(x_1, x_2, \cdots, x_n) = \min\{x_1, x_2, \cdots, x_n\}$; and when $S_f = \{(z_1, z_2, \cdots, z_n) \in J_n \mid z_j = 1\}$, i.e., when $\sigma = \{\{j\}\}$, we have $V(x_1, x_2, \cdots, x_n) = x_j$.*

COROLLARY 2.4. *When S_f consists of precisely the ordered n-tuples in J_n with k or more 1's, $1 \le k \le n$, i.e., when σ consists solely of the $\binom{n}{k}$*

subsets of $\mathbf{n}$ *of cardinality* k *(so that* $s \in \sigma$ *if and only if card*$(s) = k$*), then* $V(x_1, x_2, \cdots, x_n)$ *is the* k*th order statistic* $x_{[k]}$*. That is,*

$$V(x_1, x_2, \cdots, x_n) = \min\{\max\{x_j \mid j \in s\} \mid \text{card}(s) = k\} = x_{[k]}.$$

We note that $V(x_1, x_2, \cdots, x_n)$ is an order statistic for the set $\{x_1, x_2, \cdots, x_n\}$ whenever σ is invariant under a permutation of the elements of $\mathbf{n}$, e.g., in cases (1), (11), and (18) of Table I. Whenever this holds for a proper subset $\mathbf{m}$ of $\mathbf{n}$, then we obtain the order statistics for the proper subset of $\{x_1, x_2, \cdots, x_n\}$ whose elements have subscripts in $\mathbf{m}$, e.g., in cases (2) and (15) of Table I, where $\mathbf{m} = \{2, 3\}$. However, not all of the V's given in (2.2) are order statistics for $\{x_1, x_2, \cdots, x_n\}$ or one of its subsets; for example, $V(x_1, x_2, x_3) = \min\{x_3, \max\{x_1, x_2\}\}$ is not such an order statistic.

Using Theorem 2.2, we can obtain further information about the structure of the n-place functions Φ. To this end, recall that if ϕ is derivable from V and induced pointwise by Φ, then for every n-tuple of d.f.'s $F_1, F_2, \cdots, F_n$ in $\mathcal{D}$ and every t in $\overline{\mathbf{R}}$, we have

$$\Phi[F_1(t), F_2(t), \cdots, F_n(t)] = F_{V(X_1, X_2, \cdots, X_n)}(t),$$

where $X_1, X_2, \cdots, X_n$ are random variables defined on a common probability space and such that $df(X_i) = F_i$, $i = 1, 2, \cdots, n$. Considering the functions V listed in Corollary 2.3, we first have – trivially – that:

$$\text{If } V(x_1, x_2, \cdots, x_n) = x_j, \text{ then } \Phi[F_1(t), F_2(t), \cdots, F_n(t)] = F_j(t).$$

If $V(x_1, x_2, \cdots, x_n) = \max\{x_1, x_2, \cdots, x_n\}$, then as is well-known, $\Phi[F_1(t), F_2(t), \cdots, F_n(t)] = H_n(t, t, \cdots, t)$, where H_n is the n-dimensional joint d.f. of $X_1, X_2, \cdots, X_n$. Next, by Sklar's Theorem, Schweizer and Sklar (1983), Sklar (1959), we have that

$$H_n(u_1, u_2, \cdots, u_n) = C_n(F_1(u_1), F_2(u_2), \cdots, F_n(u_n)), \qquad (2.3)$$

where C_n is the n-copula of $X_1, X_2, \cdots, X_n$, whence

$$\Phi[F_1(t), F_2(t), \cdots, F_n(t)] = C_n(F_1(t), F_2(t), \cdots, F_n(t)).$$

Consequently, for any n-tuple of d.f.'s $F_1, F_2, \cdots, F_n$, there exists an n-copula C_n having the property that Φ agrees with C_n on a "*track* in $[0,1]^n$ from $(0, 0, \cdots, 0)$ to $(1, 1, \cdots, 1)$," specifically, on the set $\{(F_1(t), F_2(t), \cdots, F_n(t)) \mid t \in \overline{\mathbf{R}}\}$. Thus, following the terminology introduced in Alsina, Nelsen, and

Schweizer (1993), we define an *n-dimensional* quasi-copula (briefly, an *$n-q$-copula*) to be a function on $[0,1]^n$ that agrees with some copula on every track in $[0,1]^n$ from $(0,0,\cdots,0)$ to $(1,1,\cdots,1)$. Then we have that:

If $V(x_1,x_2,\cdots,x_n)=\max\{x_1,x_2,\cdots,x_n\}$, then Φ is an $n-q$-copula.

In the case $V(x_1,x_2,\cdots,x_n)=\min\{x_1,x_2,\cdots,x_n\}$, indeed, in all the remaining cases, we need to appeal to the Inclusion-Exclusion Principle. Specifically, we have

THEOREM 2.5. *Let ϕ be an n-operation on $\mathcal{D}$ which is derivable from $V:\overline{\mathbf{R}}^n\to\overline{\mathbf{R}}$ and induced pointwise by $\Phi:[0,1]^n\to[0,1]$. Let $\Phi/\mathcal{E}$ be the equivalence class of Φ, let f be the nondecreasing Boolean function in $\mathcal{F}_n$ that corresponds to $\Phi/\mathcal{E}$, and let σ be the corresponding antichain in $\mathcal{P}(\mathbf{n})\backslash\{\emptyset\}$. Let $F_1,F_2,\cdots,F_n$ be any collection of n d.f.'s in $\mathcal{D}$, and for $s\subseteq\mathbf{n}$, let $v_i(s)=F_i(t)$ if $i\in s$ and $v_i(s)=1$ if $i\notin s$. Finally, let $U_k(\sigma)$ be the collection of all $\binom{|\sigma|}{k}$ unions of k elements of σ. Then there exists an n-dimensional copula C_n such that*

$$\Phi[F_1(t),F_2(t),\cdots,F_n(t)]=\sum_{k=1}^{|\sigma|}(-1)^{k+1}\sum_{s\in U_k(\sigma)}C_n(v_1(s),v_2(s),\cdots,v_n(s)). \tag{2.4}$$

Thus, on the track $\{(F_1(t),F_2(t),\cdots,F_n(t))\mid t\in\overline{\mathbf{R}}\}$, Φ agrees with the linear combination of C_n and its margins given by the right-hand side of (2.4).

Note that the collection $U_k(\sigma)$ is a *multiset*, that is, it may have repeated elements; e.g., if $\sigma=\{\{1,2\},\{1,3\},\{2,3\}\}$, then $U_2(\sigma)=\{\{1,2,3\},\{1,2,3\},\{1,2,3\}\}$.

PROOF: From Definition 1.1, we know that there exist a probability space $(\Omega,\mathcal{A},P)$ and an n-dimensional random vector $\mathbf{X}=(X_1,X_2,\cdots,X_n)$ on $(\Omega,\mathcal{A},P)$ whose one-dimensional marginals are $F_1,F_2,\cdots,F_n$ respectively. Let H_n denote the n-dimensional d.f. given by $H_n(x_1,x_2,\cdots,x_n)=P[X_1<x_1,X_2<x_2,\cdots,X_n<x_n]$. Since $\Phi[F_1(t),F_2(t),\cdots,F_n(t)]=F_{V(X_1,X_2,\cdots,X_n)}(t)$ and $V(X_1,X_2,\cdots,X_n)=\min\{\max\{X_j\mid j\in s\}\mid s\in\sigma\}$, we have $\Phi[F_1(t),F_2(t)$ $\cdots,F_n(t)]=P[\bigvee_{s\in\sigma}\{\bigwedge_{j\in s}[X_j^{-1}(-\infty,t)]\}]$. Applying the Inclusion-Exclusion Principle now yields

$$\Phi[F_1(t),F_2(t),\cdots,F_n(t)]=\sum_{k=1}^{|\sigma|}(-1)^{k+1}\sum_{s\in U_k(\sigma)}H_n(u_1(s),u_2(s),\cdots,u_n(s)),$$

where $u_i(s) = t$ if $i \in s$ and $u_i(s) = +\infty$ if $i \notin s$. Invoking (2.3) now yields (2.4). ∎

To illustrate (2.4), if $\sigma = \{\{3\},\{1,2\}\}$, then $|\sigma| = 2$, so that $U_1(\sigma) = \sigma$ and $U_2(\sigma) = \{\{1,2,3\}\}$. Thus

$$\Phi[F_1(t), F_2(t), F_3(t)] = C_3(1,1,F_3(t)) + C_3(F_1(t),F_2(t),1) - C_3(F_1(t),F_2(t),F_3(t)),$$

which is equal to $P[X_3 < t \text{ or } (X_1 < t \text{ and } X_2 < t)]$, as it should be, since $V(x_1,x_2,x_3) = \min\{x_3, \max\{x_1,x_2\}\}$. (See case (14) in Table I.)

On specializing Theorems 2.2 and 2.5 to the case $n = 2$, we obtain the principal result of Alsina and Schweizer (1988) as the following:

COROLLARY 2.6. *Suppose that ϕ is a binary operation on $\mathcal{D}$ which is derivable from $V : \overline{\mathbf{R}}^2 \to \overline{\mathbf{R}}$ and induced pointwise by $\Phi : [0,1]^2 \to [0,1]$. Then precisely one of the following holds:*

(a) $V(x,y) = \max\{x,y\}$ and Φ is a quasi-copula, i.e., for any $F_1, F_2 \in \mathcal{D}$, there exists a copula C_2 such that $\Phi(F_1(t),F_2(t)) = C_2(F_1(t),F_2(t))$ for all t in $\mathbf{R}$;

(b) $V(x,y) = \min\{x,y\}$ and Φ is the dual of a quasi-copula, i.e., for any $F_1, F_2 \in \mathcal{D}$, there exists a copula C_2 such that $\Phi(F_1(t),F_2(t)) = F_1(t) + F_2(t) - C_2(F_1(t),F_2(t))$ for all t in $\mathbf{R}$;

(c) $V(x,y) = x$ and $\Phi(u,v) = u$; or

(d) $V(x,y) = y$ and $\Phi(u,v) = v$.

3. The Case $n = 3$. In this section, to illustrate the situation relating the antichains σ, the Borel-measurable functions $V : \overline{\mathbf{R}}^n \to \overline{R}$, and the n-place functions $\Phi : [0,1]^n \to [0,1]$, we present the case $n = 3$ in detail. The results for the 18 non-empty antichains of $\mathcal{P}(\mathbf{3})\backslash\{\emptyset\}$ appear in Table I. In the first column, we list the antichains; in the second column the functions V; and in the third column, for any F_1, F_2, F_3 in $\mathcal{D}$, we give the values $\Phi(F_1(t),F_2(t),F_3(t))$ in terms of the 3-copula C_3 with which Φ agrees on the track $\{(F_1(t),F_2(t),F_3(t)) \mid t \in \overline{\mathbf{R}}\}$. Furthermore, $x \vee y = \max\{x,y\}$ and $x \wedge y = \min\{x,y\}$.

Table I. The case $n = 3$

case	antichain	$V(x,y,z)$	$\Phi(F_1(t), F_2(t), F_3(t)) = \Phi(a,b,c)$
1	$\{\{1,2,3\}\}$	$x \vee y \vee z$	$C_3(a,b,c)$
2	$\{\{2,3\}\}$	$y \vee z$	$C_3(1,b,c)$
3	$\{\{1,3\}\}$	$x \vee z$	$C_3(a,1,c)$
4	$\{\{1,2\}\}$	$x \vee y$	$C_3(a,b,1)$
5	$\{\{1,2\},\{1,3\}\}$	$(x \vee y) \wedge (x \vee z)$	$C_3(a,b,1) + C_3(a,1,c) - C_3(a,b,c)$
6	$\{\{1,2\},\{2,3\}\}$	$(x \vee y) \wedge (y \vee z)$	$C_3(a,b,1) + C_3(1,b,c) - C_3(a,b,c)$
7	$\{\{1,3\},\{2,3\}\}$	$(x \vee z) \wedge (y \vee z)$	$C_3(a,1,c) + C_3(1,b,c) - C_3(a,b,c)$
8	$\{\{1\}\}$	x	a
9	$\{\{2\}\}$	y	b
10	$\{\{3\}\}$	z	c
11	$\{\{1,2\},\{1,3\},$ $\{2,3\}\}$	$(x \vee y) \wedge (x \vee z)$ $\wedge (y \vee z)$	$C_3(a,b,1) + C_3(a,1,c) +$ $C_3(1,b,c) - 2C_3(a,b,c)$
12	$\{\{1\},\{2,3\}\}$	$x \wedge (y \vee z)$	$a + C_3(1,b,c) - C_3(a,b,c)$
13	$\{\{2\},\{1,3\}\}$	$y \wedge (x \vee z)$	$b + C_3(a,1,c) - C_3(a,b,c)$
14	$\{\{3\},\{1,2\}\}$	$z \wedge (x \vee y)$	$c + C_3(a,b,1) - C_3(a,b,c)$
15	$\{\{2\},\{3\}\}$	$y \wedge z$	$b + c - C_3(1,b,c)$
16	$\{\{1\},\{3\}\}$	$x \wedge z$	$a + c - C_3(a,1,c)$
17	$\{\{1\},\{2\}\}$	$x \wedge y$	$a + b - C_3(a,b,1)$
18	$\{\{1\},\{2\},\{3\}\}$	$x \wedge y \wedge z$	$a + b + c - C_3(a,b,1) - C_3(a,1,c) -$ $C_3(1,b,c) + C_3(a,b,c)$

4. Concluding Remarks and Open Problems.

1. Let $c_1, c_2, \cdots, c_n$ be such that $0 < c_i < 1$ and $c_1 + c_2 + \cdots + c_n = 1$, and let $M_n : [0,1]^n \to [0,1]$ be defined by

$$M_n(z_1, z_2, \cdots, z_n) = c_1 z_1 + c_2 z_2 + \cdots + c_n z_n.$$

It is not true that $M_n(z_1, z_2, \cdots, z_n)$ equals either 0 or 1 at every vertex of $(z_1, z_2, \cdots, z_n) \in J_n$. And, by Lemma 2.1, it follows from this simple observation that mixtures of d.f.'s are not derivable from operations on random variables defined on a common probability space.

2. In Alsina and Schweizer (1988) a two-dimensional quasi-copula that is not a two-dimensional copula was explicitly exhibited. Using this quasi-copula, it is easy to construct n-dimensional quasi-copulas that are not copulas. Consider first the case $n = 3$. Let Φ be the two-dimensional quasi-copula

constructed in Alsina and Schweizer (1988), let Φ' be the mapping from $[0,1]^3$ onto $[0,1]$ given by

$$\Phi'(x,y,z) = z\Phi(x,y),$$

and note that, since $\Phi'(x,y,1) = \Phi(x,y)$, Φ' is not a 3-copula. Now let B be any track in $[0,1]^3$ from $(0,0,0)$ to $(1,1,1)$. The projection β of B onto the xy-plane is a track in $[0,1]^2$ from (0,0) to (1,1); and since Φ is a quasi-copula, there exists a copula C_β that coincides with Φ on β. By Theorem 6.6.3 of Schweizer and Sklar (1983), the mapping C_B from $[0,1]^3$ onto $[0,1]$ given by

$$C_B(x,y,z) = zC_\beta(x,y)$$

is a 3-copula. Clearly C_B and Φ' agree on B, hence Φ' is a 3-quasi-copula.

In general if, for any positive integers m and n, C_m is an m-copula and C_n is an n-copula, then again by Theorem 6.6.3 of Schweizer and Sklar (1983), the mapping C_{m+n} from $[0,1]^{m+n}$ onto $[0,1]$ given by

$$C_{m+n}(x_1,\cdots,x_{m+n}) = C_m(x_1,\cdots,x_m)C_n(x_{m+1},\cdots,x_{m+n})$$

is an $(m+n)$-copula. A simple extension of the above argument then shows that if Φ_n is an n-quasi-copula ($n \geq 2$) and C_m is an m-copula, then the mapping Φ_{m+n} from $[0,1]^{m+n}$ onto $[0,1]$ given by

$$\Phi_{m+n}(x_1,\cdots,x_{m+n}) = C_m(x_1,\cdots,x_m)\Phi_n(x_{m+1},\cdots,x_{m+n})$$

is an $(m+n)$-quasi-copula but not an $(m+n)$-copula. (Note that if $m = 1$, then necessarily $C_m(z) = z$.)

3. The number of order statistics of $\{X_1, X_2, \cdots, X_n\}$ and its subsets is known to be $n2^{n-1}$. The problem of determining the number of derivable n-operations on $\mathcal{D}$ is equivalent to the problem of determining the number of nondecreasing Boolean functions from J_n onto $[0,1]$. This problem dates back to Dedekind. It is known that for $n = 1$ through $n = 7$ these numbers are 1, 4, 18, 166, 7579, 7828352 and 2414682040996, respectively. Asymptotic results are also known. For details, see Kleitman (1969) and Wegener (1987).

References

ALSINA, C., NELSEN, R. B., and SCHWEIZER, B. (1993). On the characterization of a class of binary operations on distribution functions, *Statist. Probab. Lett.* **17**, 85–89.

ALSINA, C. and SCHWEIZER, B. (1988). Mixtures are not derivable. *Found. Phys. Lett.* **1**, 171–174.

HARRISON, M. A. (1965). *Introduction to Switching and Automata Theory.* McGraw-Hill, New York.

KLEITMAN, D. (1969). On Dedekind's problem: the number of monotone Boolean functions. *Proc. Amer. Math. Soc.* **21**, 677–682.

SCHWEIZER, B. and SKLAR, A. (1983). *Probabilistic Metric Spaces.* Elsevier-North Holland, New York.

SKLAR, A. (1959). Fonctions de répartition à n dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris* **8**, 229–231.

WEGENER, I. (1987). *The Complexity of Boolean Functions.* John Wiley & Sons, New York.

DEPARTMENT OF MATHEMATICAL SCIENCES
LEWIS AND CLARK COLLEGE
PORTLAND, OR 97219-7899
nelsen@lclark.edu

DEPARTIMENTO DE MATEMÁTICA APLICADA
UNIVERSIDAD DE GRANADA
18071 GRANADA, SPAIN
jquesada@goliat.ugr.es

DEPARTMENT OF MATHEMATICS & STATISTICS
UNIVERSITY OF MASSACHUSETTS
AMHERST, MA 01003
bert@math.umass.edu

DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI LECCE, C.P. 193
73100 LECCE, ITALY
sempi@ilenic.unile.it