*Distributions with Fixed Marginals and Related Topics*
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# COPULAS, MARGINALS, AND JOINT DISTRIBUTIONS

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Any pair of univariate marginal distributions can be combined with any copula to yield a bivariate distribution with the given marginals. This being the case, it is tempting to conclude that the dependence properties of the distribution can be determined by examination of the copula alone. Unfortunately, this idea is seriously flawed: (i) Copulas exist which yield the Fréchet upper bound for some marginal pairs and the Fréchet lower bound for other marginal pairs. (ii) There is no nonconstant measure of dependence which depends only on the copula. (iii) Weakly convergent sequences of bivariate distributions with continuous marginals exist for which the unique corresponding copulas do not converge. Related issues are considered.

**1. Introduction.** If $C$ is a bivariate distribution function with marginals uniform on $[0, 1]$, and if $F$ and $G$ are univariate distribution functions, then as is well known,

$$H = C(F, G) \tag{1.1}$$

is a bivariate distribution function with marginals $F$ and $G$. In this context, $C$ is variously called a "dependence function" or a "copula", and $C$ is often thought of as a function which "couples" the marginals $F$ and $G$. A number of parametric families of copulas have been proposed in the literature, with at least an implied suggestion that they be used to generate bivariate distributions with given marginals through the formula (1.1). What can one say about the joint distributions so generated if the marginals are unknown?

The marginals $F$ and $G$ can be inserted into any copula, so they carry no direct information about the coupling; at the same time, any pair of marginals can be inserted into $C$ so $C$ carries no direct information about the marginals. This being the case, it may seem reasonable to expect that the connections

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between the marginals of $H$ are determined by $C$ alone, and any question about these connections can be answered with the knowledge of $C$ alone.

Of course, things are not that simple. Some problems stem from the fact that copulas are not unique when at least one marginal is discontinuous. In fact the marginals can sometimes play as significant a role as the copula in determining the way in which they are coupled in $H$; in the extreme case, degenerate marginals by themselves determine the joint distribution and any copula can be used. But interaction between the copula and the marginals is often critical; copulas can look quite different in different parts of their domain, and the relevant part is determined by the range of the marginals.

1.1. EXAMPLE. Consider the copula which has mass uniformly distributed on the line segments $v = u$, $0 \le u \le 1/3$; $v = u + 1/3$, $1/3 \le u < 2/3$; and $v = u - 1/3$, $2/3 \le u \le 1$. This copula coincides with the Fréchet upper bound in the region $0 \le u$, $v < 1/3$ and it coincides with the Fréchet lower bound in the region $2/3 \le u$, $v \le 1$. So if the range of $F$ and $G$ is limited to the region $0 \le u$, $v < 1/3$, then $H(u, v) = C(F(u), G(v))$ will be a Fréchet upper bound; on the other hand, if the range of $F$ and $G$ is limited to the region $2/3 \le u$, $v \le 1$, $H$ will be a Fréchet lower bound. Depending on the marginals, this copula can yield a Fréchet upper bound or a Fréchet lower bound.

This kind of extreme behavior occurs only because discontinous marginals are involved. Moreover, the copula of Example 1.1 has been constructed to show quite different characteristics over different parts of its domain. Because of this, the range of the marginals radically affects the properties of the joint distribution.

Much of the literature regarding copulas has been based upon the assumption that the marginals of $H$ are continuous because this is a necessary and sufficient condition for the copula of $H$ to be unique (Sklar, 1959; Schweizer and Sklar, 1983). In this paper the assumption of continuous marginals is avoided, so that various familiar results no longer apply. Some details are discussed in Section 2 below.

Even though the marginals $F$ and $G$ together carry no direct information about the copula $C$, much of the literature on bivariate distributions with given marginals can be regarded as providing copulas that have a particular affinity for, or are particularly appropriate for the given marginals. So it is not a new idea that marginal pairs at least sometimes carry implicit information or make suggestions about the copula. Perhaps this is so because the probabilistic mechanism that generates $F$ and $G$ may also have a hand in generating the copula. If such linkage exists, then it is to be expected that copulas sometimes carry implicit information about appropriate marginals, but this idea does not seem to have been exploited in the literature. In Section 3, an example is given that may be relevant to this issue.

**2. Relationships Between $H$ and $C$.**

**Dependence properties of $H$ from $C$.** It is easy to check that if $C$ is positive quadrant dependent, negative quadrant dependent, or associated in the sense of Esary, Proschan, and Walkup (1967), then $H$ has the same property. A similar statement can be made about positive dependence by mixture (Shaked, 1977) although that concept requires equal marginals. Various other of the notions of positive or negative dependence in the literature also have this nice property. (See, e.g., Nelsen, 1991). Because copulas are not unique for discontinuous marginals, the converse of these results are all false. However, the following proposition holds.

2.1. PROPOSITION. *If $H$ is positive (negative) quadrant dependent, then among the various copulas of $H$, there is at least one that is positive (negative) quadrant dependent.*

PROOF. If $C$ is a copula for $H$, then $C$ is uniquely defined on (range $F$) $\times$ (range $G$), and can be extended to closure of this set by taking limits. The part of the unit square outside this closed set is a union of a countable number of sets of the form $(a,b) \times [0,1]$ and sets of the form $[0,1] \times (c,d)$. Suppose that $(u,v)$ is in $(a,b) \times$ (range $G$), and define

$$C(u,v) = \frac{b-u}{b-a}C(a,v) + \frac{u-a}{b-a}C(b,v).$$

Similarly, if $(u,v)$ is in (range $F$) $\times$ $(c,d)$, let

$$C(u,v) = \frac{d-v}{d-c}C(u,c) + \frac{v-c}{d-c}C(u,d).$$

For $(u,v)$ in $(a,b) \times (c,d)$, take

$$\begin{aligned} C(u,v) = [&(b-u)(d-v)C(a,c) + (b-u)(v-c)C(a,d) \\ &+ (u-a)(d-v)C(b,c) + (u-a)(v-c)C(b,d)]/(b-a)(d-c). \end{aligned}$$

It is routine to check that this extension of $C$ is a copula (in fact, it is a standard extension); positive (negative) quadrant dependence follows immediately from the positive (negative) quadrant dependence of $C$ on the closure of (range $F$) $\times$ (range $G$). ∎

**Weak Convergence.** Suppose that $H_1, H_2, \cdots$ is a sequence of distributions converging weakly to $H$. Suppose further that the marginals of each $H_n$ are continuous, so that $H_n$ has a unique copula $C_n$. Must $C_n$ converge to some $C$ that is a copula of $H$? The following example shows that the answer to this question is "no"; the copulas $C_n$ need not converge to anything.

2.2. EXAMPLE. Let $F_1, F_2, \cdots$ and $G_1, G_2, \cdots$ be sequences of continuous distributions that converge weakly to $F$ and $G$ respectively. Suppose further that $F$ and $G$ have discrete parts so that range $F$ and range $G$ are proper subsets of $[0,1]$. Let $C$ be a subcopula defined on (range $F$) $\times$ (range $G$) and for each $n$, let $C_n$ be a copula extending $C$. Then $H_n = C_n(F_n, G_n)$ converges weakly to $H = C(F,G)$, but $C_n$ need not converge.

**Measures of dependence.** Pearson's correlation coefficient, Spearman's $\rho$, and Kendall's $\tau$ are perhaps the most widely known measures of dependence. Since they are intended to measure dependence, and since the copula couples the marginals, it is natural to ask if these measures can be determined from the copula. For Pearson's correlation coefficient, the answer is clearly "no." On the other hand, with the assumption that the marginals are continuous, Nelsen (1991) provides nice expressions for both Spearman's $\rho$ and Kendall's $\tau$ in terms of the copula alone. But *the assumption of continuous marginals is critical*; in general, none of these measures can be determined from the copula. A stronger statement can be made.

2.3. PROPOSITION. *Let $\mathcal{H}$ be a set of bivariate distributions which includes those with Bernoulli marginals, and let $\rho : \mathcal{H} \to (-\infty, \infty)$. Suppose that $H \in \mathcal{H}$ and $H = C(F,G)$ implies $\rho(H) = \rho(C)$, i.e., $\rho$ depends only on a copula of $H$. Then $\rho$ is a constant.*

PROOF. Let $F$ and $G$ be Bernoulli distributions with respective ranges $\{0, r, 1\}$ and $\{0, s, 1\}$. If $C$ is a copula for the joint distribution $H$ and $C^*(r,s) = C(r,s)$, then $C^*$ is also a copula for $H$. So $\rho(C) = \rho(C^*)$ depends only on the number $C(r,s)$. The idea of the proof is to define $C^*$ in such a way that not only is $C^*(r,s) = C(r,s)$, but also $C^*(r^*, s^*) = r^* s^*$ for some $r^*, s^*$ in (0,1). Then with another pair of Bernoulli marginals, it will follow that $\rho(C^*) = \rho(\Pi)$, where $\Pi$ is the independent copula. This would mean that for any $H$, $\rho(H) = \rho(\Pi)$, and so $\rho$ is a constant.

I. Suppose first that $rs \le C(r,s) \le \min(r,s)$. Let $r^* = C(r,s)/s$, so that $r \le r^* \le 1$. It is readily verified that the quantities $p_{11} = C(r,s)$, $p_{12} = r - C(r,s)$, $p_{21} = 0$, $p_{22} = r^* - r$, $p_{31} = s - C(r,s)$, and $p_{32} = C(r,s) - r^* + 1 - s$ are all nonnegative. Let $R_{11} = \{(u,v) : 0 \le u < r,\ \ 0 \le v < s\}$, $R_{12} = \{(u,v) : 0 \le u \le r, s \le v \le 1\}$, $R_{21} = \{(u,v) : r \le u < r^*, 0 \le v < s\}$, $R_{22} = \{(u,v) : r \le u < r^*, s \le v \le 1\}$, $R_{31} = \{(u,v) : r^* \le u \le 1, 0 \le v < s\}$, $R_{32} = \{(u,v) : r^* \le u \le 1, s \le v \le 1\}$, and let $C^*$ be the copula that distributes mass $p_{ij}$ uniformly on the rectangle $R_{ij}$. Then $C^*(r^*, s) = r^* s = \Pi(r^*, s)$. Thus $\rho(C) = \rho(\Pi)$.

II. Suppose that $\max(0, r+s-1) \le C(r,s) \le rs$, so that $0 \le r^* = C(r,s)/s \le r$. Then $q_{11} = C(r,s)$, $q_{12} = r^* - C(r,s)$, $q_{21} = 0$, $q_{22} = r - r^*$,

$p_{31} = s - C(r,s)$, and $p_{32} = C(r,s) - r + 1 - s$ are all nonnegative. Let $C^*$ be the copula which puts mass $q_{ij}$ uniformly on the rectangle $R_{ij}$ redefined by interchanging $r$ and $r^*$. Then $C^*$ is a copula and again $C^*(r^*, s) = r^*s = \Pi(r^*, s)$. ∎

Suppose that $H_n$ converges weakly to $H$, so that the corresponding marginals $F_n$ and $G_n$ converge weakly to $F$ and $G$ respectively. Suppose further that all of the $H_n$ have a common copula $C$, so that $H = C(F, G)$. Still, it need not be true for Spearman's $\rho$ that $\rho(H_n)$ converges to $\rho(H)$. A similar statement can be made about Kendall's $\tau$. To make this important observation clear, suppose that $F_n = G_n$ for all $n$, $F_n(x) = x$, $0 \le x \le 1/2$, and $\lim F_n(x) = 1$, $x > 1/2$. Then $H$ and hence $\rho(H)$ depend upon $C(u,v)$ only for $0 \le u$, $v \le 1/2$. Any copula which coincides with $C$ on this square gives the same limit $H$, and these various copulas will have various $\rho$.

**Correlations possible with a given copula.** Let $C$ be a copula and let

$$R_C = \{\rho : \text{for some pair of marginals } F \text{ and } G,\ H = C(F,G) \text{ has correlation } \rho\}.$$

Here are two open questions:

1) For a given copula $C$, how can $R_C$ be determined?

2) What kinds of sets can play the role of $R_C$ for some $C$?

Although these are open questions, some partial answers can be given.

2.4. EXAMPLE. If $C(u,v) = \min[u,v]$ is the Fréchet upper bound, then $R_C = (0,1]$. To see this suppose that $F$ and $G$ are Bernoulli distributions, with respective expectations $p$ and $1-p$. If $H = C(F,G)$, then $H$ has covariance $[\min(p, 1-p)] - p(1-p)$ and correlation

$$\min\left(\frac{1}{1-p}, \frac{1}{p}\right) - 1.$$

This quantity is 1 for $p = 1/2$, and it approaches 0 as $p$ approaches 0 or 1.

2.5. EXAMPLE. If $C(u,v) = \max(u+v-1, 0)$, $0 \le u$, $v \le 1$ is the Fréchet lower bound, then $R_C = [-1, 0)$. To see this, let $F$ and $G$ be Bernoulli distributions with expectation $p \le 1/2$. If $H = C(F,G)$, then $H$ has correlation $-p/(1-p)$, which is $-1$ at $p = 1/2$, and which approaches 0 as $p$ approaches 0.

2.6. PROPOSITION. *$R_C$ is an interval.*

PROOF. Supose that $H_1 = C(F,G)$ has correlation $\rho_1$ and $H_2 = C(F^*, G^*)$ has correlation $\rho_2$. The result follows from the fact that $H_\alpha = C(\alpha F + (1-\alpha)F^*, \alpha G + (1-\alpha)G^*)$ has a correlation continuous in $\alpha$. ∎

2.7. PROPOSITION. *If $\rho > 0$ is in $R_C$, then $(0, \rho] \subset R_C$. If $\rho < 0$ is in $R_C$, then $[\rho, 0) \subset R_C$.*

PROOF. Suppose $\rho > 0$. Because $C(F, G) \leq \min[F, G]$, the corresponding correlations are ordered in the same way; but according to Example 2.4, this upper bound can be arbitrarily close to 0. Finally, the result follows from Proposition 2.6. The proof for $\rho < 0$ is similar. ∎

2.8. EXAMPLE. Suppose $C$ has a density 1 on $[0, 1/3)^2 \cup [1/3, 2/3)^2$ and 2 on $[0, 1/3) \times [1/3, 2/3) \cup [1/3, 2/3) \times [0, 1/3)$. Suppose also that $C(u, v) = \min(u, v)$ elsewhere. Then $C$ is positive quadrant dependent so that $R_C$ contains no negative values. If $F$ and $G$ take no value in $[1/3, 1)$, then $H(x, y) = C(F(x), G(y)) = F(x)G(y)$ is the case of independence. If $F = G$ takes no value in $(0, 2/3)$, then $H$ is an upper Fréchet bound with correlation 1. Thus $R_C = [0, 1]$.

2.9. EXAMPLE. Suppose that $C$ has mass uniformly distrbuted on the line segments $u = v$, $0 \leq u < a$, and $v = a + 1 - u$, $a < u \leq 1$. Then Corr $C = (2a^3 - 6a^2 + 6a - 1)$, which is arbitrarily small when $a$ is near $-1$. On the other hand, if $F = G$ and $F$ takes no value in the interval $(a, 1)$, then Corr $C(F, G) = 1$. So this example shows that $R_C$ can include intervals of the form $[-1 + \epsilon, 1]$ for arbitrarily small $\epsilon$. A similar example shows that $R_C$ can include intervals of the form $[-1, 1 - \epsilon]$. Whether or not $R_C$ can be the full interval $[-1, 1]$ is not known.

Conditions under which correlations of $H = C(F, G)$ are always nonnegative are given by the following proposition.

2.10. PROPOSITION. *$C(F, G)$ has a nonnegative correlation for all $F$ and $G$ (for which a correlation exists) if and only if $C$ is positive quadrant dependent.*

This result is an immediate consequence of Proposition 2.11.

2.11. PROPOSITION.

*(i)* Corr $C(F, G) \geq$ Corr $C^*(F, G)$ *for all marginal pairs $F, G$ for which the correlations exist if and only if*

*(ii)* $C(u, v) \geq C^*(u, v)$ *for all $u, v$.*

PROOF. That (ii) implies (i) is immediate from the fact that

$$\text{Cov } C(F, G) = -\int [C(F(x), G(y)) - F(x)G(y)] dx\, dy.$$

To prove that (i) implies (ii), let $F$ and $G$ be Bernoulli distributions with respective values $u$ and $v$ on $(0, 1)$.

$$\text{Cov } C(F, G) - \text{Cov } C^*(F, G) = C(r, s) - C^*(r, s) \geq 0. \qquad ∎$$

**Inclusion of the Fréchet bound and/or independence.** Example 1.1 shows that with proper choices of $F$ and $G$, it is possible for one copula to yield both of the Fréchet bounds. There are simple criteria for $C$ to admit these possibilities.

2.12. PROPOSITION. *There exist nondegenerate marginals $F$ and $G$ for which $H(x,y) = C(F(x), G(y)) = \min[F(x), G(y)]$ if and only if there exists a square $[0,a]\times[1-a,1]$ or a square $[1-a,1]\times[0,a]$ where $C$ puts no mass. There exist nondegenerate marginals $F$ and $G$ for which $H(x,y) = C(F(x), G(y)) = \max[0, x+y-1]$ if and only if some square of the form $[0,a]^2$ or of the form $[1-a,1]^2$ is given no mass by $C$.*

PROOF. First, suppose that $H = C(F,G)$ is the Fréchet upper bound for marginals $F$ and $G$ and let $(u,v)$ be a point in (range $F$) $\times$ (range $G$), $0 < u$, $v < 1$. Then $C(u,v) = \min(u,v)$. If $u \leq v$, then $C(u,v) = C(u,1) = u$, so there is no mass in the rectangle in the unit square above and to the left of $(u,v)$. If $u \geq v$, then $C(u,v) = v$, so there is no mass in the rectangle in the unit square below and to the right of $(u,v)$.

Suppose that $C$ puts no mass above and to the left of $(u,v)$, where $0 < u$, $v < 1$. Then $C(u,v) = u = \min(u,v)$. Let $F$ be a distribution taking no values in the open interval $(u,1)$, and let $G$ be a distribution taking no values in $(0,v)$. Then $C(F,G) = \min(F,G)$.

The proof for the lower bound is similar. ∎

2.13. PROPOSITION. *There exist nondegenerate marginals $F$ and $G$ for which $H = FG = C(F,G)$ is the case of independence if and only if there exists a point $(u,v)$ in the interior of the unit square such that $C(u,v) = uv$.*

PROOF. Suppose that $(u,v)$ is in the interior of the unit square and $C(u,v) = uv$. If $F$ and $G$ are Bernoulli distributions such that $F(x) = u$, $0 < x < 1$ and $G(y) = v$, $0 < y < 1$, then $C(F,G) = FG$ for all $x, y$.

Suppose that $C(F,G) = FG$ where $F$ and $G$ are not degenerate, and let $u$ be a point in range $F, v$ be a point in range $G$. Then $C(u,v) = uv$. ∎

2.14. REMARK. The conditions of Propositions 2.12 and 2.13 can all be simultaneously satisfied. Example 1.1 is of this kind because there, $C(u,v) = uv$ when $u = v = 1/\sqrt{3}$.

Example 1.1 shows that with different marginal pairs, there are copulas that can produce both Fréchet bounds. It is possible to generalize this result by replacing the Fréchet bounds by any other pair of symmetric copulas.

2.15. PROPOSITION. *If $C_1$ and $C_2$ are symmetric copulas, then there exists a copula $C$ and marginal pairs $F_1, G_1$ and $F_2, G_2$ such that $C(F_1, G_1) = C_1(F_1, G_1)$, $C(F_2, G_2) = C_2(F_2, G_2)$.*

PROOF. To construct the required copula $C$, divide the unit square into nine subsquares

$$A_{ij} = \{u, v\} : (i-1)/3 \le u < i/3, (j-1)/3 \le v < j/3\}, \quad i, j = 1, 2, 3.$$

Let

$$\begin{aligned}
f_{11}(u) &= \partial\{C_1(u, 1/3)\}/\partial u \\
f_{33}(u) &= \partial\{C_2(u+2/3, 1) - C_2(u+2/3, 2/3)\}/\partial u, \\
f_{31}(u) &= f_{13}(u) = \min\{1 - f_{11}(u), 1 - f_{33}(u)\}, \\
f_{21}(u) &= f_{12}(u) = 1 - f_{13}(u) - f_{11}(u), \\
f_{32}(u) &= f_{23}(u) = 1 - f_{13}(u) - f_{33}(u), \\
f_{22}(u) &= 1 - f_{12}(u) - f_{32}(u) = 1 - f_{23}(u) - f_{21}(u).
\end{aligned}$$

On $A_{11}$, set $C = C_1$, and on $A_{33}$, set $C = C_2$. On the diagonal of the other squares $A_{ij}$, put mass which projects onto the $u$-axis with density $f_{ij}$. Now, if (range $F_1$) $\times$ (range $G_1$) contains points off the boundary of the unit square only in $A_{11}$, then $C(F_1, G_1) = C_1(F_1, G_1)$. If (range $F_2$) $\times$ (range $G_2$) contains points off the boundary of the unit square only in $A_{33}$ then $C(F_2, G_2) = C_2(F_2, G_2)$. ∎

**3. Copulas Suggesting Particular Marginals.** In this section, a rather large class of copulas that take on particular interest with particular marginals is presented. Of course what is interesting depends upon the eye of the beholder; in this case, "interesting" means "supported by a probabilistic model that has proven to be useful in certain contexts."

3.1. PROPOSITION. *Suppose that $\phi$ and $\psi$ are increasing functions defined on the interval $[0, 1]$ such that $\phi(0) = \psi(0) = 0$, $\phi(1) = \psi(1) = 1$, and the functions $\phi^*(u) = \phi(u)/u$, $\psi^*(v) = \psi(v)/v$ are both decreasing. Then the function $C$ defined for $0 \le u$, $v \le 1$ by*

$$C(u, v) = uv \min[\phi(u)/u, \psi(v)/v] \tag{3.1}$$

*is a copula.*

3.2. PROPOSITION. *If $C$ is the copula given by (3.1) and $H = C(F, G)$, then the following are equivalent:*

*(i) Random variables $X, Y$ with joint distribution $H$ have a representation of the form*

$$X = \max(R, W), Y = \max(S, W)$$

*where $R$, $S$, and $W$ are independent random variables.*

*(ii) $H$ has the form $H(x,y) = F_R(x)F_S(y)F_W(\min[x,y])$ where $F_R$, $F_S$, and $F_W$ are distribution functions.*

*(iii) $\phi^* F(x) = \psi^* G(x)$, i.e., $G(x) = \psi^{*-1}(\phi^* F(x))$.*

Of course, (iii) says that in order for (i) or (ii) to hold, $F$ and $G$ must be paired in a critical way. One can also take the view that for given $F$ and $G$, a copula of the form (3.1) is particularly appropriate if $\phi$ and $\psi$ are chosen in such a way that (iii) holds.

Proof of Proposition 3.2. The equivalence of (i) and (ii) is immediate. Suppose that $H = C(F,G)$ where $C$ is given by (3.1), and suppose that (ii) holds. Then

$$F = F_R F_W \text{ and } G = F_S F_X, \tag{3.2}$$

and

$$\begin{aligned} H(x,y) = C(F(x),G(y)) &= \min[F(x)\psi(G(y)), G(y)\phi(F(x))] \\ &= \min[F_R(x)F_S(y)F_W(x), F_R(x)F_S(y)F_W(y)]. \end{aligned}$$

It follows that $F_R = \phi(F)$, $F_S = \psi(G)$. Using this in (3.2) yields $F_W = F/F_R = G/F_S = F/\phi(F) = G/\psi(G)$, and this is (iii).

Now, suppose (iii) holds, and $H = C(F,G)$ where $C$ is given by (3.1). Let $F_R = \phi(F)$, $F_S = \psi(G)$, and $F_W = F/\phi(F)$. It can be easily verified that under the conditions on $\phi$ and $\psi$ of the proposition, these functions are distribution functions, and moreover (ii) holds. ∎

*Notation*: For any random variable with distribution function $F$, denote the corresponding survival function by $\bar{F} = 1 - F$. Similarly, if $X$ and $Y$ have joint distribution function $H$, then the joint survival function is denoted by $\bar{H}$; that is, $\bar{H}(x,y) = P\{X > x, Y > y\}$.

3.3. Proposition. *If $C$ is the copula given by (3.1) and $\bar{H}(x,y) = C(\bar{F},\bar{G})$, then the following are equivalent:*

*(i) Random variables $X, Y$ with joint distribution $H$ have a representation of the form*

$$X = \min(R,W), \quad Y = \min(S,W)$$

*where $R$, $S$, and $W$ are independent random variables.*

*(ii) $\bar{H}$ has the form $\bar{H}(x,y) = \bar{F}_R(x)\bar{F}_S(y)\bar{F}_W(\max[x,y])$ where $F_R$, $F_S$, and $F_W$ are distribution functions.*

*(iii) $\phi^* \bar{F}(x) = \psi^* \bar{G}(x)$, i.e., $\bar{G}(x) = \psi^{*-1}(\phi^* \bar{F}(x))$.*

The proof of this result is similar to the proof of the preceding proposition.

Of course, the structure indicated in (i) above is just the structure of the bivariate exponential distribution of Marshall and Olkin (1967), who give motivations in terms of probabilistic models; because of this structure, this distribution has proved useful in diverse applications such as the study of nuclear reactor safety and the study of cancer metastasis. Similarly, the structure of (i), Proposition 3.2, can be expected to be of particular interest when $R$, $S$, and $W$ have extreme value distributions for maxima of the same type.

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