Chapter 9

Asymptotic behavior of interacting system of stochastic differential equations on duals of nuclear spaces

The study of the limit of interacting n-particle diffusions was undertaken by McKean [39] whose work was followed by the papers of Hitsuda and Mitoma [15] and Shiga and Tanaka [49] among others.

This chapter is concerned with "propagation of chaos" problems for stochastic systems with an infinite number of degrees of freedom such as random strings or the fluctuation of voltage potentials of interacting spatially extended neurons. The latter is a more realistic model for large numbers of neurons in close proximity to one another and has provided the motivation for the work presented here which is novel in one respect: we consider Φ'-valued SDE's driven by Poisson random measures. The results for Φ'-valued interacting diffusions can be similarly derived. (See also Chiang, Kallianpur and Sundar [3])

The results obtained in this chapter are of interest beyond the neurophysiological applications that motivated them. It should be mentioned that the interaction considered here is mean field interaction which seems more applicable to phenomena in statistical physics. A type of interaction known as "parallel fiber interaction" has been used in connection with potentials of interacting neurons the consideration of which however, remains outside the scope of this book.

Consider the following stochastic system

$$\begin{aligned} & X_j^n(t) - X_j^n(0) \qquad (9.0.1) \\ = & \int_0^t \left(a(s, X_j^n(s)) + \frac{1}{n}\sum_{i=1}^n b(s, X_j^n(s), X_i^n(s) \right) ds + \int_0^t \int_U \\ & \left(g(s, X_j^n(s-), u) + \frac{1}{n}\sum_{i=1}^n c(s, X_j^n(s-), X_i^n(s-), u) \right) \tilde{N}_j(duds), \end{aligned}$$

where $X_j^n(0)$, $1 \le j \le n$, are Φ'-valued random variables and $\{N_j\}$, $1 \le j \le n$, are independent copies of a Poisson random measure on $\mathbf{R}_+ \times U$ with characteristic measure μ on U, $a : \mathbf{R}_+ \times \Phi' \to \Phi'$, $b : \mathbf{R}_+ \times \Phi' \times \Phi' \to \Phi'$, $g : \mathbf{R}_+ \times \Phi' \times U \to \Phi'$ and $c : \mathbf{R}_+ \times \Phi' \times \Phi' \times U \to \Phi'$ are measurable maps in the corresponding spaces.

In the above model, the coefficients b and c represent the interactions for each pair of particles in the system. The interactions among three (or more) particles are assumed for simplicity, to be negligible.

For the solution $(X_1^n(t), X_2^n(t), \cdots, X_n^n(t))$ of (9.0.1), we consider the limiting behavior as $n \to \infty$ of the following empirical measures

$$\zeta_n(\omega, B) = \frac{1}{n}\sum_{j=1}^n \delta_{X_j^n(\cdot,\omega)}(B), \quad \omega \in \Omega \text{ and } B \in \mathcal{B}(D([0,T], \Phi')). \qquad (9.0.2)$$

We shall prove that the limiting distribution of ζ_n is characterized by the following McKean-Vlasov equation

$$X_t = X_0 + \int_0^t A(s, X_s, \mathcal{D}(X_s))ds + \int_0^t \int_U G(s, X_{s-}, u, \mathcal{D}(X_s))\tilde{N}(duds) \qquad (9.0.3)$$

where $A : R_+ \times \Phi' \times \mathcal{P}(\Phi') \to \Phi'$ and $G : R_+ \times \Phi' \times U \times \mathcal{P}(\Phi') \to \Phi'$ are two measurable maps and $\mathcal{D}(X_t) \in \mathcal{P}(\Phi')$ is the distribution of X_t.

This chapter is organized as follows: In Section 1 we establish the existence and uniqueness of solution for the McKean-Vlasov equation (9.0.3). Then, in Section 2, we show the existence and uniqueness of solution of the system (9.0.1). Also we associate a McKean-Vlasov equation of the form (9.0.3) with the system (9.0.1) and characterize the limit of the sequence of empirical measures $\{\zeta_n\}$ as the solution of this McKean-Vlasov equation. Most of the material of this chapter is taken from Kallianpur and Xiong [29].

9.1 McKean-Vlasov equation

The nonlinear SDE (9.0.3) on Φ' is intended to characterize the limiting behavior of the empirical measure sequence of the system (9.0.1) when n tends to infinity.

To show the existence and uniqueness of the solution of the SDE (9.0.3), we make the following assumptions (MV) which is similar to assumptions (I) and (M) of Chapter 6.
Assumptions (MV): $\forall T > 0$, $\exists p_0 = p_0(T) \in \mathbf{N}^+$ such that $\forall p \geq p_0$, $\exists q \geq p$ and a constant K=K(p,q,T) such that
(MV1) $\forall t \in [0,T]$ and $M > 0$, the maps

$$(v,\rho) \in \Phi_{-p} \times M_{-p} \to A(t,v,\rho) \in \Phi_{-q}$$

and

$$(v,\rho) \in \Phi_{-p} \times M_{-p} \to G(t,v,\cdot,\rho) \in L^2(U,\mu;\Phi_{-p})$$

are continuous, where

$$M_{-p} = \left\{\rho \in \mathcal{P}(\Phi_{-p}) : \int ||x||_{-p}^2 \rho(dx) \leq M\right\}.$$

(MV2) (Coercivity) $\forall t \in [0,T]$, $\phi \in \Phi$ and $\rho \in \mathcal{P}(\Phi_{-p})$,

$$2A(t,\phi,\rho)[\theta_p \phi] \leq K(1 + ||\phi||_{-p}^2).$$

(MV3) (Growth) $\forall t \in [0,T]$, $v \in \Phi_{-p}$ and $\rho \in \mathcal{P}(\Phi_{-p})$,

$$||A(t,v,\rho)||_{-q}^2 \leq K(1 + ||v||_{-p}^2)$$

and

$$\int_U ||G(t,v,u,\rho)||_{-p}^2 \mu(du) \leq K(1 + ||v||_{-p}^2).$$

(MV4) (Monotonicity) $\forall t \in [0,T]$, $v_1, v_2 \in \Phi_{-p}$ and $\rho_1, \rho_2 \in \mathcal{P}(\Phi_{-p})$

$$\begin{aligned} & 2 < A(t,v_1,\rho_1) - A(t,v_2,\rho_2), v_1 - v_2 >_{-q} \\ & + \int_U ||G(t,v_1,u,\rho_1) - G(t,v_2,u,\rho_2)||_{-q}^2 \mu(du) \\ \leq\ & K(||v_1 - v_2||_{-q}^2 + d_q(\rho_1,\rho_2)^2), \end{aligned} \tag{9.1.1}$$

where, for any ρ_1 and $\rho_2 \in \mathcal{P}(\Phi_{-q})$,

$$d_q(\rho_1,\rho_2)^2 = \inf \left\{ \begin{array}{l} \int_{\Phi_{-q}} \int_{\Phi_{-q}} ||x - y||_{-q}^2 R(dxdy) : \\ R(dx, \Phi_{-q}) = \rho_1(dx), R(\Phi_{-q}, dy) = \rho_2(dy) \end{array} \right\}. \tag{9.1.2}$$

Definition 9.1.1 *Let λ_0 be a probability measure on the Borel sets of Φ'. A probability measure λ on $D([0,T],\Phi')$ is called a solution of (9.0.3) with initial measure λ_0 if it is the weak solution of the SDE*

$$X_t = X_0 + \int_0^t A(s,X_s,\lambda(s))ds + \int_0^t \int_U G(s,X_{s-},u,\lambda(s))\tilde{N}(duds)$$

and $\lambda(0) = \lambda_0$, where $\lambda(s)$ is the marginal distribution $\lambda \circ Z_s^{-1}$ of λ. If, furthermore, λ can be regarded as a measure on $D([0,T],\Phi_{-p})$, it is called a Φ_{-p}-valued solution.

Theorem 9.1.1 *Under assumptions (MV), if we have an index* r_0 *such that* $E\|X_0\|_{-r_0}^2 < \infty$, *then the SDE (9.0.3) has a unique* Φ_{-p_3}*-valued solution where* $p_3(T) \geq p_2(T) \geq p_1(T)$ *such that the canonical injections* $\Phi_{-p_1} \to \Phi_{-p_2} \to \Phi_{-p_3}$ *are Hilbert-Schmidt and* $p_1(T)$ *is as in Section 6.1.*

Proof: We prove the theorem in five steps.
Step 1: We construct a sequence of probability measures λ^n on $D([0,T], \Phi_{-p_1})$ by induction such that

$$E^{\lambda^n} \sup_{0\leq t\leq T} \|Z_t\|_{-p}^2 \leq \tilde{K} < \infty, \tag{9.1.3}$$

where $\tilde{K}$ is the constant given in Theorem 6.2.2.

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ be a stochastic basis and X_0 an $\mathcal{F}_0$ -measurable Φ_{-p_1}-valued random variable with distribution λ_0. Let X^0 be a Φ_{-p_1}-valued process defined by $X_t^0 = X_0$ and let $\lambda^0 \in \mathcal{P}(D([0,T], \Phi_{-p_1}))$ be the distribution of X^0. It is obvious that (9.1.3) holds for λ^0. Now, suppose that $\lambda^n \in \mathcal{P}(D([0,T], \Phi_{-p_1}))$ is given and satisfies (9.1.3). Consider the following SDE

$$X_t^{n+1} = X_0 + \int_0^t B^{n+1}(s, X_s^{n+1})ds + \int_0^t \int_U C^{n+1}(s, X_{s-}^{n+1}, u)\tilde{N}(duds) \tag{9.1.4}$$

where

$$B^{n+1}(s,v) = A(s,v,\lambda^n(s)) \quad \text{and} \quad c^{n+1}(s,v,u) = G(s,v,u,\lambda^n(s)).$$

To solve this SDE, we verify that (B^{n+1}, C^{n+1}, μ) satisfies assumptions (I) and (M) of Chapter 6 with $p_0(T)$ replaced by $p(T)$. It follows from (9.1.3) that, for $p \geq p(T)$ and $t \in [0,T]$, $\lambda^n(s) \in M_{-p}$ with $M = \tilde{K}$. Hence, by (MV1), $B^{n+1}(t,v) : \Phi_{-p} \to \Phi_{-q}$ and $C^{n+1}(t,v,\cdot) : \Phi_{-p} \to L^2(U, \mu; \Phi_{-p})$ are continuous in v. This proves (I1). The assumptions (I2), (I3) and (M) follow from (MV) directly. Hence, by Theorem 6.3.1, (9.1.4) has a unique solution X^{n+1}. Let λ^{n+1} be the distribution of X^{n+1}. Then, by Theorem 6.2.2, $\lambda^{n+1} \in \mathcal{P}(D([0,T], \Phi_{-p_1}))$ and satisfies (9.1.3).
Step 2: For any $t \in [0,T]$, $d_{q_1}(\lambda^n(t), \lambda^{n-1}(t))$ tends to zero as n tends to ∞, where q_1 is determined by p_1 through Assumption (MV).

For any $\phi \in \Phi$, by Itô's formula, we have

$$\begin{aligned}
& E(X_t^{n+1} - X_t^n)[\phi]^2 \\
= \; & E\int_0^t 2(X_s^{n+1} - X_s^n)[\phi](A(s, X_s^{n+1}, \lambda^n(s)) - A(s, X_s^n, \lambda^{n-1}(s)))[\phi]ds \\
& + E\int_0^t \int_U (G(s, X_s^{n+1}, u, \lambda^n(s)) - G(s, X_s^n, u, \lambda^{n-1}(s)))[\phi]^2\mu(du)ds
\end{aligned}$$

Taking $\phi = h_j^{-q_1}$, $j \in \mathbf{N}$ and adding, we have

$$\begin{aligned}
& E\|X_t^{n+1} - X_t^n\|^2_{-q_1} \\
= \; & E\int_0^t 2 < X_s^{n+1} - X_s^n, A(s, X_s^{n+1}, \lambda^n(s)) - A(s, X_s^n, \lambda^{n-1}(s)) >_{-q_1} ds \\
& + E\int_0^t \int_U \|G(s, X_s^{n+1}, u, \lambda^n(s)) - G(s, X_s^n, u, \lambda^{n-1}(s))\|^2_{-q_1} \mu(du) ds \\
\leq \; & E\int_0^t K\left(\|X_s^{n+1} - X_s^n\|^2_{-q_1} + d_{q_1}(\lambda^n(s), \lambda^{n-1}(s)\right) ds \\
\leq \; & K\int_0^t (E\|X_s^{n+1} - X_s^n\|^2_{-q_1} + E\|X_s^n - X_s^{n-1}\|^2_{-q_1}) ds
\end{aligned}$$

from (9.1.1) and (9.1.2). Hence

$$E\|X_t^{n+1} - X_t^n\|^2_{-q_1} \leq Ke^{Kt}\int_0^t e^{-Ks} E\|X_s^n - X_s^{n-1}\|^2_{-q_1} ds.$$

So

$$\begin{aligned}
& d_{q_1}(\lambda^{n+1}(t), \lambda^n(t)) \leq E\|X_t^{n+1} - X_t^n\|^2_{-q_1} \\
\leq \; & Ke^{Kt}\int_0^t \frac{(K(t-s))^{n-1}}{(n-1)!} e^{-Ks} E\|X_s^1 - X_s^0\|^2_{-q_1} ds \to 0.
\end{aligned}$$

Step 3: $\{\lambda^n\}$ is tight in $\mathcal{P}(D([0,T], \Phi_{-p_2}))$.

As $\lambda^{n-1}(s) \in M_{-p}$, $\forall s \in [0,T]$ and $n \geq 1$, and M_{-p} is compact in M_{-p_1}, it follows from (MV1) that $\forall r \geq p_1$ and $s \in [0,T], B^n(s,v)$ and $C^n(s,v,\cdot)$ are continuous in $v \in \Phi_{-r}$ uniform for $n \geq 1$. The rest of the condition (A1) for (B^n, C^n, μ) with p_0 replaced by p_1 follows directly from (MV). Hence, by Lemma 6.1.2, $\{\lambda^n\}$ is tight in $\mathcal{P}(D([0,T], \Phi_{-p_2}))$.

Step 4: Existence.

Let λ be a cluster point of $\{\lambda^n\}$ and let $\{\lambda^{n_k}\}$ be a subsequence which converges to λ as k tends to ∞. Now, we verify the condition (A2) for $(B^{n_k+1}, C^{n_k+1}, \mu, \lambda_0)$ with $p_0(T)$ replaced by $p_2(T)$ and

$$B(s,v) = A(s,v,\lambda(s)) \quad \text{and} \quad C(s,v,u) = G(s,v,u,\lambda(s)).$$

In fact, for $s \notin \mathcal{N}$, where

$$\mathcal{N} \equiv \{t \in [0,T] : \lambda(Z \in D([0,T], \Phi_{-p_2}) : Z_{t-} \neq Z_t) > 0\},$$

the sequence $\lambda^{n_k}(s)$ converges weakly to $\lambda(s)$ as measures on Φ_{-p_2} and hence, by (MV1) and (9.1.3), we have

$$A(s,v,\lambda^{n_k}(s))[\phi] \to A(s,v,\lambda(s))[\phi] \tag{9.1.5}$$

and

$$\int_U \|G(s,v,u,\lambda^{n_k}(s)) - G(s,v,u,\lambda(s))\|_{-p}^2 \mu(du) \to 0. \tag{9.1.6}$$

On the other hand, $\mathcal{N}$ is a countable set and we can modify the definition of $A(s,v,\lambda^{n_k}(s))$ and $G(s,v,u,\lambda^{n_k}(s))$ such that (9.1.5) and (9.1.6) still hold for $s \in \mathcal{N}$ without changing the SDE (9.1.4). This proves the condition (A2). Hence, by the results of Chapter 6, λ^{n_k+1} converges in $\mathcal{P}(D([0,T],\Phi_{-p_3})$ to the distribution λ' of the unique solution of the SDE

$$X_t = X_0 + \int_0^t A(s,X_s,\lambda(s))ds + \int_0^t \int_U G(s,X_{s-},u,\lambda(s))\tilde{N}(duds).$$

It follows from Step 2 that, for any $s \in [0,T]$, $\lambda'(s) = \lambda(s)$. So that λ' is a solution of the McKean-Vlasov equation (9.0.3).
Step 5: Uniqueness.

Let λ'' be another solution of (9.0.3). From the proof of the existence we can assume that λ' and λ'' are the distributions of X', X'' based on the same stochastic basis (as X' can be chosen on a pre-determined stochastic basis). Then there exists an index p such that X' and $X'' \in D([0,T],\Phi_{-p})$. It follows from the same arguments as in Step 2 that

$$E\|X'_t - X''_t\|_{-q}^2 \le 2K \int_0^t E\|X'_s - X''_s\|_{-q}^2 ds.$$

Hence, $E\|X'_t - X''_t\|_{-q}^2 = 0$. By the right-continuity, we have $X = X'$ a.s. and so $\lambda'' = \lambda$. ∎

9.2 Interacting systems

We first establish the existence and uniqueness of the solution of (9.0.1) by making use of the results in Chapter 6. Then we define a McKean-Vlasov equation corresponding to (9.0.1) and prove that the sequence of empirical measures (9.0.2) converges to the unique solution of this equation.

To apply the results of Chapter 6 to the system (9.0.1), we need the following
Assumptions (C): $\forall T > 0$, $\exists p_0 = p_0(T) \in \mathbf{N}^+$ such that $\forall p \ge p_0$, $\exists q \ge p$ and a constant $K = K(p,q,T)$ such that
(C1) (Continuity) $\forall t \in [0,T]$, the maps

$$v \in \Phi_{-p} \to a(t,v) \in \Phi_{-q},$$

$$(v_1,v_2) \in \Phi_{-p} \times \Phi_{-p} \to b(t,v_1,v_2) \in \Phi_{-q},$$

$$v \in \Phi_{-p} \to g(t, v, \cdot) \in L^2(U, \mu; \Phi_{-p})$$

and

$$(v_1, v_2) \in \Phi_{-p} \times \Phi_{-p} \to c(t, v_1, v_2, \cdot) \in L^2(U, \mu; \Phi_{-p})$$

are continuous.
(C2) (Coercivity) $\forall t \in [0, T]$, $\phi \in \Phi$ and $\psi \in \Phi$,

$$2a(t, \phi)[\theta_p \phi] \leq K(1 + \|\phi\|_{-p}^2)$$

and

$$2b(t, \phi, \psi)[\theta_p \phi] \leq K(1 + \|\phi\|_{-p}^2 + \|\psi\|_{-p}^2).$$

(C3) (Growth) $\forall t \in [0, T]$ and $v, w \in \Phi_{-p}$,

$$\|a(t, v)\|_{-q}^2 \leq K(1 + \|v\|_{-p}^2),$$

$$\int_U \|g(t, v, u)\|_{-p}^2 \mu(du) \leq K(1 + \|v\|_{-p}^2),$$

$$\|b(t, v, w)\|_{-q}^2 \leq K(1 + \|v\|_{-p}^2 + \|w\|_{-p}^2) \tag{9.2.1}$$

and

$$\int_U \|c(t, v, w, u)\|_{-p}^2 \mu(du) \leq K(1 + \|v\|_{-p}^2 + \|w\|_{-p}^2). \tag{9.2.2}$$

(C4) (Monotonicity) $\forall t \in [0, T]$, $v_1, v_2, w_1, w_2 \in \Phi_{-p}$,

$$\begin{aligned} & < a(t, v_1) - a(t, v_2), v_1 - v_2 >_{-q} \\ & + \int_U \|g(t, v_1, u) - g(t, v_2, u)\|_{-q}^2 \mu(du) \\ \leq \;& K\|v_1 - v_2\|_{-q}^2, \end{aligned}$$

and

$$\begin{aligned} & < b(t, v_1, w_1) - b(t, v_2, w_2), v_1 - v_2 >_{-q} \\ & + \int_U \|c(t, v_1, w_1, u) - c(t, v_2, w_2, u)\|_{-q}^2 \mu(du) \\ \leq \;& K(\|v_1 - v_2\|_{-q}^2 + \|w_1 - w_2\|_{-q}^2). \end{aligned}$$

(C5) For any index p, there exists a positive constant $M(p)$ such that if $v, w \in \Phi_{-p}$ and $\|v - w\|_{-p} > M(p)$, then $b(t, v, w) = 0$ in Φ_{-q} and $c(t, v, w, \cdot) = 0$ in $L^2(U, \mu; \Phi_{-p})$.

The assumption (C5) makes physical sense that there is no interaction between a pair of particles far apart. It is desirable to relax this condition so that our results can be applied to more circumstances.

Theorem 9.2.1 *Let r_0 be such that*

$$\sum_{j=1}^{n} E\|X_j^n(0)\|_{-r_0}^2 < \infty.$$

Then under assumptions (C1)-(C4) the system (9.0.1) has a unique Φ_{-p_1}-valued strong solution $\{X_j^n, j = 1, \cdots, n\}$.

Proof: Let $U_1, \cdots, U_n$ be n copies of the measurable space U. Define a measurable space $\hat{U}$ by $\hat{U} = U_1 \oplus \cdots \oplus U_n$ with σ-field

$$\hat{\mathcal{E}} \equiv \{A_1 \oplus \cdots \oplus A_n : A_j \in \mathcal{E}_j, j = 1, \cdots, n\}$$

and a random measure N on $\mathbf{R}_+ \times \hat{U}$ given by

$$N(\omega, [0,t] \times (A_1 \oplus \cdots \oplus A_n)) = \sum_{j=1}^{n} N_j(\omega, [0,t] \times A_j).$$

It is easy to see that N is a Poisson random measure on $\mathbf{R}_+ \times \hat{U}$ with respect to the stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ with characteristic measure $\hat{\mu}$ given by

$$\hat{\mu}(A_1 \oplus \cdots \oplus A_n) = \sum_{j=1}^{n} \mu(A_j).$$

Let

$$\Phi^{(n)} = \{\phi = (\phi_1, \cdots, \phi_n) : \phi_j \in \Phi, 1 \le j \le n\}$$

and

$$\|\phi\|_p^{(n)} = \sum_{j=1}^{n} \|\phi_j\|_p^2.$$

Then $\Phi^{(n)}$ is a countably Hilbertian nuclear space. Let $B^{(n)} : \mathbf{R}_+ \times \Phi^{(n)\prime} \to \Phi^{(n)\prime}$ and $C^{(n)} : \mathbf{R}_+ \times \Phi^{(n)\prime} \times \hat{U} \to \Phi^{(n)\prime}$ be defined by

$$B^{(n)}(t,v)_i = a(t, v_i) + \frac{1}{n}\sum_{j=1}^{n} b(t, v_i, v_j),$$

and

$$C^{(n)}(t,v,u)_i = \left(g(t, v_i, u) + \frac{1}{n}\sum_{j=1}^{n} c(t, v_i, v_j, u) \right) 1_{U_i}(u),\ i = 1, \cdots, n.$$

Let $(X_0^{(n)})_i = X_i^n(0)$, $i = 1, \cdots, n$. Then (9.0.1) is equivalent to the following SDE on $\Phi^{(n)\prime}$

$$X_t^{(n)} = X_0^{(n)} + \int_0^t B^{(n)}(s, X_s^{(n)})ds + \int_0^t \int_{\hat{U}} C^{(n)}(s, X_{s-}^{(n)}, u)\tilde{N}(duds). \quad (9.2.3)$$

Under assumptions (C1)-(C4), it is easy to verify that assumptions (I) and (M) hold for $B^{(n)}$ and $C^{(n)}$ with K replaced by $3nK$. Hence the SDE (9.2.3), i.e. the system (9.0.1), has a unique strong solution. ∎

Note that Lemma 6.1.4 yields

$$E \sup_{0\leq t\leq T} \sum_{j=1}^{n} \|X_j^n(t)\|_{-p}^2 \leq nK_1 \qquad \forall n \geq 1 \tag{9.2.4}$$

where K_1 is a constant.

To consider the limiting behavior of the system (9.0.1), we need the estimate (9.2.6) below which is stronger that (9.2.4). The exchangeability defined below for the initial random variables is needed to get the estimate (9.2.6).

Definition 9.2.1 *$(X_1^n(0), \cdots, X_n^n(0))$ are exchangeable random variables if, for any permutation τ of $\{1, \cdots, n\}$,*

$$(X_{\tau(1)}^n(0), \cdots, X_{\tau(n)}^n(0)) = (X_1^n(0), \cdots, X_n^n(0))$$

in distribution.

Lemma 9.2.1 *Under assumptions (C1)-(C4), if $(X_1^n(0), \cdots, X_n^n(0))$ are exchangeable and*

$$\sup_n E\|X_j^n(0)\|_{-r_0}^2 < \infty, \tag{9.2.5}$$

then there exists a finite constant K_1 such that

$$\sup_n E \sup_{0\leq t\leq T} \|X_j^n(t)\|_{-p}^2 \leq K_1. \tag{9.2.6}$$

Proof: First of all, we prove the estimate (9.2.6) when $\Phi = \mathbf{R}^d$. In this case, (9.2.6) becomes

$$\sup_n E \sup_{0\leq t\leq T} |X_j^n(t)|^2 \leq K_1$$

where $|\cdot|$ is the Euclidian norm on $\mathbf{R}^d$.

Using Itô's formula to (9.0.1), we have

$$|X_j^n(t)|^2 = |X_j^n(0)|^2 + R_t + \gamma_t + M_t \tag{9.2.7}$$

where

$$\begin{aligned} R_t \;=\; & \int_0^t 2\left\langle X_j^n(s), a(s, X_j^n(s)) + \frac{1}{n}\sum_{i=1}^n b(s, X_j^n(s), X_i^n(s)\right\rangle ds \\ & + \int_0^t \int_U \left| g(s, X_j^n(s), u) + \frac{1}{n}\sum_{i=1}^n c(s, X_j^n(s), X_i^n(s), u)\right|^2 \mu(du)ds, \end{aligned}$$

$$\begin{aligned} M_t &= 2\int_0^t \int_U \Big\langle X_j^n(s-), g(s, X_j^n(s-), u) \\ &\quad + \frac{1}{n}\sum_{i=1}^n c(s, X_j^n(s-), X_i^n(s-), u)\Big\rangle \tilde{N}_j(duds) \end{aligned}$$

and

$$\begin{aligned} \gamma_t &= \int_0^t \int_U \left| g(s, X_j^n(s-), u) + \frac{1}{n}\sum_{i=1}^n c(s, X_j^n(s-), X_i^n(s-), u)\right|^2 \\ &\quad \tilde{N}_j(duds). \end{aligned}$$

Let

$$f(r) = E \sup_{0\le t\le r} |X_j^n(t)|^2. \tag{9.2.8}$$

By the uniqueness of the solution of (9.0.1) and the exchangeability of $(X_1^n(0), \cdots, X_n^n(0))$, we have that $(X_1^n, \cdots, X_n^n)$ are exchangeable $D([0,T], \mathbf{R}^d)$-valued random variables. Hence, the definition of (9.2.8) does not depend on j. By assumptions (C2), (C3) and the exchangeability, it is easy to see that

$$\begin{aligned} & E \sup_{0\le t\le r} R_t \\ \le\ & E\left(6Kr + 6K\int_0^r |X_j^n(s)|^2 ds + \frac{3K}{n}\sum_{i=1}^n \int_0^t |X_i^n(s)|^2 ds\right) \\ \le\ & 6Kr + 9K\int_0^r f(s)ds \end{aligned} \tag{9.2.9}$$

and

$$\begin{aligned} & E \sup_{0\le t\le r} \gamma_t \\ \le\ & E \sup_{0\le t\le r} \int_0^t \int_U \left| g(s, X_j^n(s-), u) + \frac{1}{n}\sum_{i=1}^n c(s, X_j^n(s-), X_i^n(s-), u)\right|^2 \\ & \qquad (N_j(duds) + \mu(du)ds) \\ =\ & 2E\int_0^r \int_U \Big| g(s, X_j^n(s-), u) \\ & \qquad + \frac{1}{n}\sum_{i=1}^n c(s, X_j^n(s-), X_i^n(s-), u)\Big|^2 \mu(du)ds \\ \le\ & 8Kr + 12K\int_0^r f(s)ds. \end{aligned} \tag{9.2.10}$$

Furthermore, since M is a square integrable martingale with quadratic variation process

$$[M]_t = 4\int_0^t \int_U \Big\langle X_j^n(s-), g(s, X_j^n(s-), u)$$

$$+\frac{1}{n}\sum_{i=1}^{n} c(s, X_j^n(s-), X_i^n(s-), u)\Big\rangle^2 N_j(duds),$$

it follows from the Burkholder-Davis-Gundy inequality that

$$\begin{aligned}
& E\sup_{0\le t\le r} M_t \le 4E[M]_r^{\frac{1}{2}} \\
\le\ & 8E\left\{\int_0^r\int_U |X_j^n(s-)|^2\Big|g(s, X_j^n(s-), u)\right. \\
& \left.+\frac{1}{n}\sum_{i=1}^{n} c(s, X_j^n(s-), X_i^n(s-), u)\Big|^2 N_j(duds)\right\}^{1/2} \\
\le\ & 8E\sup_{0\le t\le r}|X_j^n(t)|\left\{\int_0^r\int_U \Big|g(s, X_j^n(s-), u)\right. \\
& \left.+\frac{1}{n}\sum_{i=1}^{n} c(s, X_j^n(s-), X_i^n(s-), u)\Big|^2 N_j(duds)\right\}^{1/2} \\
\le\ & \frac{1}{2}E\sup_{0\le t\le r}|X_j^n(t)|^2 + 8E\int_0^r\int_U \Big|g(s, X_j^n(s-), u) \\
& +\frac{1}{n}\sum_{i=1}^{n} c(s, X_j^n(s-), X_i^n(s-), u)\Big|^2 N_j(duds) \\
\le\ & \frac{1}{2}f(r) + 32Kr + 48K\int_0^r f(s)ds. \qquad (9.2.11)
\end{aligned}$$

By (9.2.7), (9.2.9), (9.2.10) and (9.2.11), we have

$$f(r) \le f(0) + 46Kr + 69K\int_0^r f(s)ds + \frac{1}{2}f(r).$$

Then

$$f(r) \le (f(0) + 46KT)\left(1 + \int_0^T e^{69K(T-s)}ds\right) \equiv K_1 < \infty. \qquad (9.2.12)$$

The estimate (9.2.6) for general Φ follows from finite dimensional approximation by the same type of arguments as in Section 6.1. ∎

Now, we introduce two supplemental sequences of measures to show that ζ_n converges in distribution. For each n, let $\eta_n \in \mathcal{P}(\mathcal{P}(D([0,T], \Phi_{-p_1})))$ be the distribution of ζ_n and $\mathcal{V}_n = E\zeta_n \in \mathcal{P}(D([0,T], \Phi_{-p_1}))$, i.e. $\eta_n = P\circ\zeta_n^{-1}$ and

$$\mathcal{V}_n(B) = E\zeta_n(\omega, B),\ \forall B \in \mathcal{B}(D([0,T], \Phi_{-p_1})).$$

Theorem 9.2.2 *Under the conditions of Lemma 9.2.1,*
(a) $\mathcal{V}_n$ *is tight in* $\mathcal{P}(D([0,T], \Phi_{-p_1}))$.
(b) η_n *is tight in* $\mathcal{P}(\mathcal{P}(D([0,T], \Phi_{-p_1})))$.

Proof: (a) For any $B \in \mathcal{B}(D([0,T],\Phi_{-p_1}))$, we have

$$\mathcal{V}_n(B) = E\zeta_n(\omega,B) = \frac{1}{n}\sum_{j=1}^n P(X_j^n \in B) = P(X_1^n \in B).$$

So, we only need to show that $\{X_1^n\}$ is tight in $\mathcal{P}(D([0,T],\Phi_{-p_1}))$. This can be proved by using the same arguments as in the proof of Lemma 6.1.2.
(b) By (a), $\forall j \geq 1$, there exists a compact set $K_j \subset D([0,T],\Phi_{-p_1})$ such that

$$\sup_n \mathcal{V}_n(K_j^c) \leq \frac{\epsilon}{j^3}.$$

Let

$$K = \{\rho \in \mathcal{P}(D([0,T],\Phi_{-p_1})) : \rho(K_j) \geq 1 - j^{-1}, \forall j\}.$$

Then K is a compact subset of $\mathcal{P}(D([0,T],\Phi_{-p_1}))$ and

$$\begin{aligned}\eta_n(K^c) &= P(\zeta_n \in K^c) \leq \sum_{j=1}^\infty P(\zeta_n(\omega,K_j^c) \geq j^{-1}) \\ &\leq \sum_{j=1}^\infty j\mathcal{V}_n(K_j^c) \leq \sum_{j=1}^\infty j^{-2}\epsilon < 2\epsilon.\end{aligned}$$

Hence $\{\eta_n\}$ is tight. ∎

Next, we introduce the McKean-Vlasov equation corresponding to the system (9.0.1) and show that this equation has a unique solution by verifying the conditions of Theorem 9.1.1. For $t \in [0,T]$, $v \in \Phi'$, $\rho \in \mathcal{P}(\Phi')$ and $u \in U$, let

$$\begin{aligned}A(t,v,\rho) &= a(t,v) + \int_{\Phi'} b(t,v,y)\rho(dy), \\ G(t,v,u,\rho) &= g(t,v,u) + \int_{\Phi'} c(t,v,y,u)\rho(dy).\end{aligned} \tag{9.2.13}$$

Lemma 9.2.2 *Under assumptions (C), A and G of (9.2.13) satisfy assumptions (MV) with K replaced by $4(2K + M(p)^2)$.*

Proof: Let $t \in [0,T]$, $v \in \Phi_{-p}$ and $\rho \in M_{-p}$ be fixed, $v_n \in \Phi_{-p}$ and $\rho_n \in M_{-p}$ such that v_n tends to v in Φ_{-p} and ρ_n tends to ρ with respect to the weak topology of $\mathcal{P}(\Phi_{-p})$. Then, by Skorohod's theorem, there exist a probability space $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde P)$ and Φ_{-p}-valued random variables Y_n and Y such that ρ_n and ρ are the distributions of Y_n and Y respectively and Y_n converges to Y, $\tilde P$-a.s. As

$$\begin{aligned}&\sup_n E^{\tilde P}\|b(t,v_n,Y_n) - b(t,v,Y)\|_{-q}^2 \\ \leq\ & 2K\sup_n E^{\tilde P}(2 + \|v_n\|_{-p}^2 + \|v\|_{-p}^2 + \|Y_n\|_{-p}^2 + \|Y\|_{-p}^2) \\ \leq\ & 2K\sup_n(2 + \|v_n\|_{-p}^2 + \|v\|_{-p}^2 + 2M^2) < \infty,\end{aligned}$$

$\|b(t,v_n,Y_n)-b(t,v,Y)\|_{-q}$ is uniformly integrable. Hence, by the continuity of a and b,

$$\begin{aligned}
&\|A(t,v_n,\rho_n)-A(t,v,\rho)\|_{-q}\\
\le\ &\|a(t,v_n)-a(t,v)\|_{-q}\\
&+\left\|\int_{\Phi_{-p}}b(t,v_n,y)\rho_n(dy)-\int_{\Phi_{-p}}b(t,v,y))\rho(dy)\right\|_{-q}\\
=\ &\|a(t,v_n)-a(t,v)\|_{-q}+\|E^{\tilde{P}}(b(t,v_n,Y_n)-b(t,v,Y))\|_{-q}\to 0.
\end{aligned}$$

The continuity of G is proved similarly. Hence (MV1) holds. For (MV2), note that

$$\begin{aligned}
&2A(t,\phi,\rho)[\theta_p\phi]=2a(t,\phi)[\theta_p\phi]+\int_{\Phi_{-p}}2b(t,\phi,y)[\theta_p\phi]\rho(dy)\\
\le\ &K(1+\|\phi\|_{-p}^2)+\int_{\Phi_{-p}}K(1+\|\phi\|_{-p}^2+\|y\|_{-p}^2)1_{\{\|\phi-y\|_{-p}\le M(p)\}}\rho(dy)\\
\le\ &(4K+2M(p)^2)(1+\|\phi\|_{-p}^2).
\end{aligned}$$

Proceeding similarly, we can prove (MV3). Finally, we verify (MV4). For any $\rho_1,\rho_2\in\mathcal{P}(\Phi_{-p})$ and $\epsilon>0$, there exists a probability measure R on $\Phi_{-q}\times\Phi_{-q}$ such that $R(dx,\Phi_{-q})=\rho_1(dx)$, $R(\Phi_{-q},dy)=\rho_2(dy)$ and

$$\int_{\Phi_{-q}}\int_{\Phi_{-q}}\|x-y\|_{-q}^2R(dxdy)\le d_q(\rho_1,\rho_2)^2+\epsilon.$$

It is clear that

$$R(\Phi_{-p}\times\Phi_{-p})=1.$$

Then

$$\begin{aligned}
&2<A(t,v_1,\rho_1)-A(t,v_2,\rho_2),v_1-v_2>_{-q}\\
&+\int_U\|G(t,v_1,u,\rho_1)-G(t,v_2,u,\rho_2)\|_{-q}^2\mu(du)\\
\le\ &2<a(t,v_1)-a(t,v_2),v_1-v_2>_{-q}\\
&+2\int_U\|g(t,v_1,u)-g(t,v_2,u)\|_{-q}^2\mu(du)\\
&+2\left\langle\int_{\Phi_{-p}}b(t,v_1,y)\rho_1(dy)-\int_{\Phi_{-p}}b(t,v_2,y)\rho_2(dy),v_1-v_2\right\rangle_{-q}\\
&+2\int_U\left\|\int_{\Phi_{-p}}c(t,v_1,y,u)\rho(dy)-\int_{\Phi_{-p}}c(t,v_1,y,u)\rho(dy)\right\|_{-q}^2\mu(du)\\
\le\ &2K\|v_1-v_2\|_{-q}^2
\end{aligned}$$

$$
\begin{aligned}
& +2\left\langle \int_{\Phi_{-p}}\int_{\Phi_{-p}} (b(t,v_1,y_1)-b(t,v_2,y_2))R(dy_1dy_2), v_1-v_2\right\rangle_{-q} \\
& +2\int_U \left\| \int_{\Phi_{-p}}\int_{\Phi_{-p}} (c(t,v_1,y_1,u)-c(t,v_2,y_2,u))R(dy_1dy_2)\right\|_{-q}^2 \mu(du) \\
\leq\ & 2K\|v_1-v_2\|_{-q}^2 \\
& +2\int_{\Phi_{-p}}\int_{\Phi_{-p}} < b(t,v_1,y_1)-b(t,v_2,y_2), v_1-v_2 >_{-q} R(dy_1dy_2) \\
& +2\int_{\Phi_{-p}}\int_{\Phi_{-p}}\int_U \|c(t,v_1,y_1,u)-c(t,v_2,y_2,u)\|_{-q}^2\mu(du)R(dy_1dy_2) \\
\leq\ & 2K\|v_1-v_2\|_{-q}^2 + 2\int_{\Phi_{-p}}\int_{\Phi_{-p}} K(\|v_1-v_2\|_{-q}^2+\|y_1-y_2\|_{-q}^2)R(dy_1dy_2) \\
\leq\ & 4K\|v_1-v_2\|_{-q}^2 + 2K(d_q(\rho_1,\rho_2)^2+\epsilon).
\end{aligned}
$$

(MV4) follows since ϵ is arbitrary. ∎

Finally, we come to our main result of this chapter which establishes the relationship between the limit point of the empirical measure sequence (9.0.2) and the McKean-Vlasov equation corresponding to (9.0.1). The idea is to prove that for any cluster point $\eta \in \mathcal{P}(\mathcal{P}(D([0,T],\Phi_{-p_1})))$ of the sequence $\{\eta_n\}$,

$$\eta\left\{\rho \in \mathcal{P}(D([0,T],\Phi_{-p_1})) : \rho \text{ is a solution of } (9.0.3)\right\} = 1.$$

As the McKean-Vlasov equation (9.0.3) has a unique solution λ, we see that $\eta = \delta_\lambda$.

Theorem 9.2.3 (Propagation of chaos) *Suppose that assumptions (C) holds, $(X_1^n(0),\cdots,X_n^n(0))$ are exchangeable such that (9.2.5) holds and*

$$\frac{1}{n}\sum_{j=1}^n \delta_{X_j^n(0)}(\cdot)$$

tends to a measure λ_0 in $\mathcal{P}(\Phi')$, then $\lambda \in \mathcal{P}(D([0,T],\Phi_{-p_1}))$ and

$$\eta_n \to \delta_\lambda \text{ in } \mathcal{P}(\mathcal{P}(D([0,T],\Phi_{-p_1}))),$$

where λ is the unique probability measure which solves the McKean-Vlasov equation corresponding to the system (9.0.1) and such that $\lambda \circ Z_0^{-1} = \lambda_0$.

Proof: By Theorem 9.2.2 (b), without loss of generality, we assume that the sequence η_n converges to η weakly in $\mathcal{P}(\mathcal{P}(D([0,T],\Phi_{-p_1})))$. Then again by Skorohod's Theorem, there exists a probability space $(\Omega',\mathcal{F}',P')$ and

$\mathcal{P}(D([0,T],\Phi_{-p_1}))$-valued random variables ξ_n and ξ with distributions η_n and η respectively such that ξ_n converges to ξ P'-a.s. We prove the theorem in four steps.

Step 1: Let $F \in \mathcal{D}_0^\infty(\Phi')$ be given by $F(v) = h(v[\phi])$. For $\omega \in \Omega$ and $\omega' \in \Omega'$, we define

$$\begin{array}{ll} B^n_{\omega'}(s,v) = A(s,v,\xi_n(s,\omega')), & C^n_{\omega'}(s,v,u) = G(s,v,u,\xi_n(s,\omega')) \\ B_{\omega'}(s,v) = A(s,v,\xi(s,\omega')), & C_{\omega'}(s,v,u) = G(s,v,u,\xi(s,\omega')) \\ B^n_{\omega}(s,v) = A(s,v,\zeta_n(s,\omega)) \quad \text{and} & C^n_{\omega}(s,v,u) = G(s,v,u,\zeta_n(s,\omega)) \end{array}$$

Define $\mathcal{L}^n_{\omega'}$, $\mathcal{L}_{\omega'}$, $\mathcal{L}^n_{\omega}$, $M^F_{n,\omega'}(Z)$, $M^F_{\omega'}(Z)$ and $M^F_{n,\omega}(Z)$ as in (6.1.4).

Let f be a $\mathcal{B}_r$-measurable continuous function ($r < t$) on $D([0,T],\Phi_{-p_1})$ with compact support C. Applying Itô's formula to (9.0.1), we have

$$\begin{aligned} & F(X^n_j(t)) - F(X^n_j(r)) \qquad\qquad (9.2.14)\\ = & \int_r^t A(s,X^n_j(s),\zeta_n(s))[\phi]h'(X^n_j(s)[\phi])ds \\ & + \int_r^t \int_U \{F(X^n_j(s-) + G(s,X^n_j(s-),u,\zeta_n(s-))) \\ & \qquad -F(X^n_j(s-))\}\tilde{N}_j(duds) \\ & + \int_r^t \int_U \{F(X^n_j(s) + G(s,X^n_j(s),u,\zeta_n(s))) - F(X^n_j(s)) \\ & \qquad -G(s,X^n_j(s),u,\zeta_n(s))[\phi]h'(X^n_j(s)[\phi])\}\mu(du)ds. \end{aligned}$$

Let

$$\mathcal{M}(n,\omega) = \int_{D([0,T],\Phi_{-p_1})} f(Z)\left\{M^F_{n,\omega}(Z)_t - M^F_{n,\omega}(Z)_r\right\}\zeta_n(\omega,dZ).$$

Let $\mathcal{M}(n,\omega')$ and $\mathcal{M}(\omega')$ be defined similarly. It follows from (9.2.14) that

$$\begin{aligned} \mathcal{M}(n,\omega) = & \frac{1}{n}\sum_{j=1}^n \int_r^t \int_U \{F(X^n_j(s-) + G(s,X^n_j(s-),u,\zeta_n(s-))) \\ & \qquad -F(X^n_j(s-))\}\tilde{N}_j(duds)f(X^n_j). \end{aligned}$$

Step 2: $E\mathcal{M}(n,\omega)^2$ tends to 0 as n tends to ∞.

$$\begin{aligned} & E\mathcal{M}(n,\omega)^2 \\ = & E\Big\{\frac{1}{n}\sum_{j=1}^n \int_r^t \int_U \{F(X^n_j(s-) + G(s,X^n_j(s-),u,\zeta_n(s-))) \\ & \qquad -F(X^n_j(s-))\}\tilde{N}_j(duds)f(X^n_j)\Big\}^2 \\ = & n^{-2}\sum_{j=1}^n E\int_r^t \int_U \{F(X^n_j(s) + G(s,X^n_j(s),u,\zeta_n(s))) \end{aligned}$$

$$
\begin{aligned}
& -F(X_j^n(s))\}^2\mu(du)dsf(X_j^n)^2 \\
\leq\ & n^{-1}\|f\|_\infty^2 E\int_r^t\int_U \|h'\|_\infty^2\|\phi\|_{-p}^2\|G(s,X_j^n(s),u,\zeta_n(s))\|_{-p}^2\mu(du)ds \\
\leq\ & 2n^{-1}\|f\|_\infty^2\|h'\|_\infty^2\|\phi\|_p^2 \\
& E\int_r^t K\left(2+2\|X_j^n(s)\|_{-p}^2+\frac{1}{n}\sum_{i=1}^n\|X_i^n(s)\|_{-p}^2\right)ds \\
\leq\ & 2n^{-1}\|f\|_\infty^2\|h'\|_\infty^2\|\phi\|_p^2K(2+3K_1)(t-r)\to 0,
\end{aligned}
\tag{9.2.15}
$$

as $n\to\infty$, where K_1 is given by Lemma 9.2.1.
Step 3: Let $\omega'\in\Omega'$ be fixed such that $\xi_n(\omega',dZ)$ converges to $\xi(\omega',dZ)$ and let

$$\mathcal{N}(\omega')=\{t\in[0,T]:\xi(\omega')\{Z\in D([0,T],\Phi_{-p_1}):Z_{t-}\neq Z_t\}>0\}.$$

Then $\mathcal{N}(\omega')$ is a countable set. As C is a compact subset of $D([0,T],\Phi_{-p_1})$, there exists a compact subset C_0 of Φ_{-p_1} and a constant M such that $Z\in C$ implies that $Z_t\in C_0$ for all $t\in[0,T]$ and $v\in C_0$ implies that $\|v\|_{-p_1}\leq M$. We now show that, if $r,t\notin\mathcal{N}(\omega')$, then

$$\lim_{n\to\infty}\mathcal{M}(n,\omega')=\mathcal{M}(\omega'). \tag{9.2.16}$$

Since it is easy to see that, for $r,t\notin\mathcal{N}(\omega')$

$$\int_C f(Z)\left\{M_{\omega'}^F(Z)_t-M_{\omega'}^F(Z)_r\right\}\xi_n(\omega',dZ)\to\mathcal{M}(\omega'),$$

we only need to show that

$$\mathcal{M}(n,\omega')-\int_C f(Z)\left\{M_{\omega'}^F(Z)_t-M_{\omega'}^F(Z)_r\right\}\xi_n(\omega',dZ)\to 0.$$

Let H be given in Lemma 6.1.5. Then

$$
\begin{aligned}
& \mathcal{M}(n,\omega')-\int_C f(Z)\left\{M_{\omega'}^F(Z)_t-M_{\omega'}^F(Z)_r\right\}\xi_n(\omega',dZ) \\
=\ & \Bigg|\int_C f(Z)\Big\{\int_r^t(A(s,Z_s,\xi_n(s,\omega'))[\phi]-A(s,Z_s,\xi(s,\omega'))[\phi]) \\
& h'(Z_s[\phi])ds+\int_r^t\int_U\{H(Z_s[\phi],G(s,Z_s,u,\xi_n(s,\omega'))[\phi]) \\
& -H(Z_s[\phi],G(s,Z_s,u,\xi(s,\omega'))[\phi])\Big\}\mu(du)ds\}\xi_n(\omega',dZ)\Bigg| \\
\leq\ & \|f\|_\infty\|h'\|_\infty\|\phi\|_{q_1}\int_r^t I_1(n,s,\omega')ds \\
& +\|f\|_\infty\|h''\|_\infty\|\phi\|_{p_1}^2\int_r^t I_2(n,s,\omega')ds
\end{aligned}
$$

where

$$I_1(n,s,\omega') = \sup_{v\in C_0} \left\| \int_{\Phi_{-p_1}} b(s,v,y)(\xi_n(s,\omega',dy) - \xi(s,\omega',dy)) \right\|_{-q_1}$$

and

$$\begin{aligned} I_2(n,s,\omega') &= \sup_{v\in C_0} \int_U (\|G(s,v,u,\xi_n(s,\omega'))\|_{-p_1} + \|G(s,v,u,\xi(s,\omega'))\|_{-p_1}) \\ &\qquad \|G(s,v,u,\xi_n(s,\omega')) - G(s,v,u,\xi(s,\omega'))\|_{-p_1}\mu(du). \end{aligned}$$

Now we prove that $\int_r^t I_2(n,s,\omega')ds \to 0$ as $n\to\infty$ ($\int_r^t I_1(n,s,\omega')ds \to 0$ can be verified similarly). It follows from Hölder's inequality that

$$I_2(n,s,\omega') \le I_{21}(n,s,\omega') I_{22}(n,s,\omega'),$$

where

$$\begin{aligned} & I_{21}(n,s,\omega')^2 \\ = \;& \sup_{v\in C_0} \int_U (\|G(s,v,u,\xi_n(s,\omega'))\|_{-p_1} + \|G(s,v,u,\xi(s,\omega'))\|_{-p_1})^2 \mu(du) \end{aligned}$$

and

$$\begin{aligned} & I_{22}(n,s,\omega')^2 \\ = \;& \sup_{v\in C_0} \int_U \left\| \int_{\Phi_{-p_1}} c(s,v,y,u)(\xi_n(s,\omega',dy) - \xi(s,\omega',dy)) \right\|_{-p_1}^2 \mu(du). \end{aligned}$$

Note that, by Lemma 9.2.2, we have

$$\begin{aligned} & I_{22}(n,s,\omega')^2 \le I_{21}(n,s,\omega')^2 \\ \le \;& 2\sup_{v\in C_0} \int_U (\|G(s,v,u,\xi_n(s,\omega')\|_{-p_1}^2 + \|G(s,v,u,\xi(s,\omega')\|_{-p_1}^2)\mu(du) \\ \le \;& 4\sup_{v\in C_0} 4(2K + M(p)^2)(1 + \|v\|_{-p_1}^2) \\ \le \;& 16(2K + M(p)^2)(1 + M^2). \end{aligned}$$

Hence by the dominate convergence theorem we only need to prove that $I_{22}(n,s,\omega') \to 0$ for $s \notin \mathcal{N}(\omega')$ fixed.

As $\xi_n(s,\omega';dy)$ converges to $\xi(s,\omega';dy)$ in $\mathcal{P}(\Phi_{-p_1})$, it follows from Skorohod's theorem again that there exists a probability space $(\Omega'', \mathcal{F}'', P'')$ and Φ_{-p_1}-valued random variables $\tilde{\xi}_n$ and $\tilde{\xi}$ on Ω'' such that $\mathcal{D}(\tilde{\xi}_n) = \xi_n(s,\omega';dy)$, $\mathcal{D}(\tilde{\xi}) = \xi_n(s,\omega';dy)$ and $\tilde{\xi}_n$ converges to $\tilde{\xi}$ P''-a.s. By assumptions (C1), (C3) and (C5), we have

$$\sup_{v\in C_0} \int_U \|c(s,v,\tilde{\xi}_n,u) - c(s,v,\tilde{\xi},u)\|_{-p_1}^2 \mu(du) \to 0, \; a.s.$$

and

$$\begin{aligned}
&\sup_{v\in C_0}\int_U \|c(s,v,\tilde{\xi}_n,u)-c(s,v,\tilde{\xi},u)\|^2_{-p_1}\mu(du)\\
\le\ & 2\sup_{v\in C_0} K(2+2\|v\|^2_{-p_1}+\|\tilde{\xi}_n\|^2_{-p_1}1_{\|\tilde{\xi}_n-v\|_{-p_1}\le M(p)}\\
&\qquad +\|\tilde{\xi}\|^2_{-p_1}1_{\|\tilde{\xi}-v\|_{-p_1}\le M(p)})\\
\le\ & 2K(2+2M^2+2(M+M(p))^2).
\end{aligned}$$

By the dominate convergence theorem, we have

$$\begin{aligned}
I_{22}(n,s,\omega')^2 &= \sup_{v\in C_0}\int_U \|E^{P''}(c(s,v,\tilde{\xi}_n,u)-c(s,v,\tilde{\xi},u))\|^2_{-p_1}\mu(du)\\
\le\ E^{P''}&\sup_{v\in C_0}\int_U \|c(s,v,\tilde{\xi}_n,u)-c(s,v,\tilde{\xi},u)\|^2_{-p_1}\mu(du)\to 0.
\end{aligned}$$

Step 4: For any $\epsilon>0$, it follows from (9.2.15) and (9.2.16) that

$$\begin{aligned}
&P'(\omega':t,r\notin\mathcal{N}(\omega'),|\mathcal{M}(\omega')|>\epsilon)\\
\le\ &\lim_{n\to\infty}P'(\omega':t,r\notin\mathcal{N}(\omega'),|\mathcal{M}(n,\omega')|\ge\epsilon)\\
\le\ &\lim_{n\to\infty}P(\omega:|\mathcal{M}(n,\omega)|\ge\epsilon)\\
\le\ &\lim_{n\to\infty}\frac{1}{\epsilon^2}E^P|\mathcal{M}(n,\omega)|^2=0.
\end{aligned}$$

Thus, for P'-a.s. ω', $\forall r<t$ such that $r,t\notin\mathcal{N}(\omega')$, we have that $\mathcal{M}(\omega')=0$. As $\mathcal{N}(\omega')$ is countable, it is easy to see that $\mathcal{M}(\omega')=0$ still holds for any $r<t$ by taking two sequences $r_n<t_n$ not in $\mathcal{N}(\omega')$ such that r_n and t_n decrease to r and t respectively and passing to the limit.

Now, let ω' and $r<t$ be fixed and define a signed measure on $(D([0,T],\Phi_{-p_1}),\mathcal{B}_r)$ by

$$\nu(A)=\int_A\left\{M^F_{\omega'}(Z)_t-M^F_{\omega'}(Z)_r\right\}\xi(\omega',dZ),\ \forall A\in\mathcal{B}_r.$$

Note that we have a constant K_2 depending on h, $p_1(T)$, $q_1(T)$, ϕ, K and $M(p)$ such that

$$|M^F_{\omega'}(Z)_t|\le K_2\left(1+\sup_{0\le t\le T}\|Z_t\|^2_{-p_1}\right),$$

so that

$$E^{P'}\int|M^F_{\omega'}(Z)_t|\xi(\omega',dZ)$$

$$
\begin{aligned}
&\leq E^{P'} \int K_2 \left(1 + \sup_{0\leq t\leq T} \|Z_t\|^2_{-p_1}\right) \xi(\omega', dZ) \\
&\leq \liminf_{n\to\infty} E^{P'} \int K_2 \left(1 + \sup_{0\leq t\leq T} \|Z_t\|^2_{-p_1}\right) \xi_n(\omega', dZ) \\
&= \liminf_{n\to\infty} E^{P} \int K_2 \left(1 + \sup_{0\leq t\leq T} \|Z_t\|^2_{-p_1}\right) \zeta_n(\omega, dZ) \\
&= \liminf_{n\to\infty} E^{P} \frac{1}{n} \sum_{j=1}^{n} K_2 \left(1 + \sup_{0\leq t\leq T} \|X^n_j(t)\|^2_{-p_1}\right) \\
&\leq K_2(1+K_1) < \infty.
\end{aligned}
$$

Hence, for almost all ω', ν^+ and ν^- are two finite measures. As $\mathcal{M}(\omega') = 0$, the integrals with respect to ν^+ and ν^- are the same for any $\mathcal{B}_r$-measurable continuous functions with compact support. Hence, $\nu^+ = \nu^-$, i.e. for P'-a.s. $\omega' \in \Omega'$, we have that $\xi(\omega')$ is a solution of the $\mathcal{L}_{\omega'}$-martingale problem with initial distribution λ_0. Therefore, by Theorem 6.3.1, $\xi(\omega')$ is the distribution of the solution of the following SDE

$$
X_t = X_0 + \int_0^t A(s, X_s, \xi(s,\omega'))ds + \int_0^t \int_U G(s, X_{s-}, u, \xi(s,\omega'))\tilde{N}(duds),
$$

i.e. $\xi(\omega')$ is a solution of the McKean-Vlasov equation (9.0.3). By the uniqueness of the solution of the McKean-Vlasov equation, we get $\xi(\omega') = \lambda$ for P'-a.s. ω'. Hence $\lambda \in \mathcal{P}(D([0,T], \Phi_{-p_1}))$ and $\eta = \delta_\lambda$. ∎