Chapter 8

Diffusion approximation and Φ'-valued diffusion processes

The study of SDE's in Chapter 6 is motivated by various practical problems. One of the applications is to the voltage potential of spatially extended neurons. The stimuli received by a neuron are the form of electrical impulses and are modelled by Poisson random measures. When the pulses arrive frequently enough and the magnitudes are small enough, it is reasonable to expect that the compensated Poisson random measures are approximated by Gaussian white noises in space-time and hence, the discontinuous processes of voltage potentials of spatially extended neurons governed by Poisson random measures are approximated by diffusion processes.

In this chapter, we study the existence and uniqueness for the solution of a diffusion equation on the dual of a CHNS. We shall consider it as the limiting case of the SDE's driven by Poisson random measures investigated in Chapter 6.

Let $(U, \mathcal{E})$ be a measurable space and μ^n a sequence of σ-finite measures on U. Let N^n be a sequence of Poisson random measures on $\mathbf{R}_+ \times U$ with characteristic measures μ^n. Let $A^n : \mathbf{R}_+ \times \Phi' \to \Phi'$ and $G^n : \mathbf{R}_+ \times \Phi' \times U \to \Phi'$ be two sequences of measurable mappings on the corresponding spaces. We consider a sequence of SDE's

$$X_t^n = X_0^n + \int_0^t A^n(s, X_s^n)ds + \int_0^t \int_U G^n(s, X_{s-}^n, u)\tilde{N}^n(duds) \qquad (8.0.1)$$

where $\{X_0^n\}$ is a sequence of Φ'-valued random variables and $\tilde{N}^n$ is the compensated random measure of N^n.

We prove that, under suitable conditions, the sequence of unique solutions of the SDE (8.0.1) converges in distribution to the unique solution of the following diffusion equation

$$X_t = X_0 + \int_0^t A(s, X_s)ds + \int_0^t B(s, X_s)dW_s \tag{8.0.2}$$

where $A : \mathbf{R}_+ \times \Phi' \to \Phi'$ and $B : \mathbf{R}_+ \times \Phi' \to L(\Phi', \Phi')$ are two measurable mappings and W is a Φ'-valued Wiener process.

Diffusion equations of the type (8.0.2) have been studied by various authors, e.g. Kallianpur and Wolpert [27], Tuckwell [55] and Walsh [56]. Most of the above mentioned authors deal with linear or quasilinear equations. A result for the general equation was obtained by Kallianpur, Mitoma and Wolpert [24]. As a consequence of a diffusion approximation result in [31], under conditions weaker than those of [24], we established the existence and uniqueness of solution of (8.0.2). In this chapter, we present the arguments of [31].

8.1 Martingale problem of a diffusion equation

In this section we consider the tightness of the weak solutions of the SDE sequence (8.0.1). We will show that under suitable conditions, the limit points of the sequence which solves (8.0.1) can be identified as the solutions of the martingale problem corresponding to the diffusion equation (8.0.2).

Making use of the results in Chapter 6, we see that the condition (A1)(2°) is satisfied if we assume that there exists $r_0 \geq 0$ such that λ_0^n can be regarded as probability measures on Φ_{-r_0} and

$$\sup_n \int_{\Phi_{-r_0}} \|v\|_{-r_0}^2 \lambda_0^n(dv) < \infty \tag{8.1.1}$$

where λ_0^n is the distribution on Φ' of the random variables X_0^n. We make the following assumption for $\{A^n, G^n, \mu^n, \lambda_0^n\}$:
(DA1). The conditions (A1)(1°) of Chapter 6 and (8.1.1) hold.

Under assumption (DA1), it follows from Theorem 6.2.2 and Corollary 6.1.1 that the condition (A1) of Chapter 6 holds, i.e. there exists a sequence $\{\lambda^n\}$ of probability measures on $D([0, T], \Phi_{-p_1})$ which is the weak solution to the SDE's (8.0.1) and

$$\int_{D([0,T],\Phi_{-p_1})} \sup_{0\leq t\leq T} \|Z_t\|_{-p}^2 \lambda^n(dZ) \leq \tilde{K} \tag{8.1.2}$$

where $p = p(T) = \max(p_0(T), r_0)$ and $p_1 = p_1(T) \geq p(T)$ such that the canonical injection from Φ_{-p} to Φ_{-p_1} is Hilbert-Schmidt. By Lemma 6.1.2, the sequence $\{\lambda^n\}$ is tight in $D([0, T], \Phi_{-p_1})$.

To characterize the limit points of the sequence $\{\lambda^n\}$, we introduce the following

Assumption (DA2): There exist a covariance function Q on $\Phi \times \Phi$ and two measurable maps $A : \mathbf{R}_+ \times \Phi' \to \Phi'$ and $B : \mathbf{R}_+ \times \Phi' \to L(\Phi', \Phi')$ such that $\forall t \in [0,T]$, $\phi \in \Phi$, $a > 0$, $p \geq p_0$ and compact subset C_0 of Φ_{-p}, we have
(1°)

$$\lim_{n\to\infty} \sup_{v\in C_0} \|A^n(t,v) - A(t,v)\|_{-q} = 0.$$

(2°)

$$\lim_{n\to\infty} \sup_{v\in C_0} \mu^n\{u : |G^n(t,v,u)[\phi]| > a\} = 0, \tag{8.1.3}$$

$$\lim_{n\to\infty} \sup_{v_1,v_2\in C_0} \left| \int_U G^n(t,v_1,u)[\phi]G^n(t,v_2,u)[\phi]\mu^n(du) - Q(B(t,v_1)'\phi, B(t,v_2)'\phi) \right| = 0, \tag{8.1.4}$$

and

$$\lim_{M\to\infty} \sup_{v\in C_0, n\in\mathbf{N}} \int_U |G^n(t,v,u)[\phi]|^2 1_{|G^n(t,v,u)[\phi]|\geq M} \mu^n(du) = 0. \tag{8.1.5}$$

The condition (DA2)(2°) ensures that any cluster point of the sequence $\{\lambda^n\}$ is supported on continuous paths.

Theorem 8.1.1 *Let λ^* be a cluster point of the sequence $\{\lambda^n\}$ on $D([0,T], \Phi_{-p_1})$. If the sequence $(A^n, G^n, \mu^n, \lambda^n_0)$ satisfies the conditions (DA1) and (DA2)(2°), then*

$$\lambda^*(C([0,T], \Phi_{-p_1})) = 1. \tag{8.1.6}$$

Proof: Let g be a non-negative continuous function on $\mathbf{R}$ vanishing in a neighborhood of 0 and ∞ (g_m, $m \in \mathbf{N}$, of Lemma 6.1.8 are examples of such functions). For any $\phi \in \Phi$, let $\{F^n\}$ be a sequence of maps from $D([0,T], \Phi_{-p_1})$ to $\mathbf{R}$ given by

$$F^n(Z) \equiv \sum_{0<s\leq T} g(\Delta Z_s[\phi]) - \int_0^T \int_U g(G^n(s, Z_s, u)[\phi])\mu^n(du)ds.$$

Without loss of generality, we assume that λ^n converges to λ^* weakly. Making use of Skorohod's Theorem, there exists a probability space $(\Omega, \mathcal{F}, P)$ and $D([0,T], \Phi_{-p_1})$-valued random variables ξ^n and ξ with distributions λ^n and λ^* respectively such that ξ^n tends to ξ, P-a.s.

We now divide the proof into four steps.
Step 1. First we show that

$$F^n(\xi^n) \to \sum_{0<s\leq T} g(\Delta\xi_s[\phi]) \qquad \text{in probability.} \tag{8.1.7}$$

By the tightness of $\{\lambda^n\}$, for any $\epsilon > 0$, there exists a compact subset C of $D([0,T], \Phi_{-p_1})$ such that $\lambda^n(C) > 1 - \epsilon$. Let C_0 be a compact subset of Φ_{-p_1} and M a constant such that

$$C \subset \{Z \in D([0,T], \Phi_{-p_1}) : Z_t \in C_0, \ \forall t \in [0,T]\}$$

and

$$C_0 \subset \{v \in \Phi_{-p_1} : ||v||_{-p_1} \leq M\}.$$

Let $b > 0$ such that $g(x) = 0$ for any $|x| \leq b$. Then, $\forall a > 0$,

$$\begin{aligned} & P\left(\omega : \int_0^T \int_U g(G^n(s, \xi_s^n, u)[\phi])\mu^n(du)ds > a\right) \\ \leq \ & P(\omega : \xi^n \notin C) + \frac{1}{a}E\int_0^T \int_U g(G^n(s, \xi_s^n, u)[\phi])\mu^n(du)ds 1_C(\xi^n) \\ \leq \ & \epsilon + \frac{1}{a}E\int_0^T \mu^n\{u : |G^n(s, \xi_s^n, u)[\phi]| > b\}1_C(\xi^n)||g||_\infty ds \\ \leq \ & \epsilon + \frac{||g||_\infty}{a}\int_0^T \sup_{v \in C_0} \mu^n\{u : |G^n(t, v, u)[\phi]| > b\}ds. \end{aligned}$$

Since

$$\begin{aligned} & \sup_{v \in C_0} \mu^n\{u : |G^n(t, v, u)[\phi]| > b\} \\ \leq \ & \sup_{v \in C_0} \frac{||\phi||_{p_1}^2}{b^2}\int_U ||G^n(t, v, u)||_{-p_1}^2 \mu^n(du) \leq \frac{||\phi||_{p_1}^2}{b^2}K(1 + M^2), \end{aligned}$$

it follows from (DA2)(2°) and the bounded convergence theorem that

$$\limsup_{n \to \infty} P\left(\omega : \int_0^T \int_U g(G^n(s, \xi_s^n, u)[\phi])\mu^n(du)ds > a\right) \leq \epsilon.$$

i.e.

$$\int_0^T \int_U g(G^n(s, \xi_s^n, u)[\phi])\mu^n(du)ds \to 0 \qquad \text{in probability.}$$

On the other hand, we have

$$\sum_{0<s\leq T} g(\Delta\xi_s^n[\phi]) \to \sum_{0<s\leq T} g(\Delta\xi_s[\phi]) \qquad \text{P-a.s.,}$$

and hence, (8.1.7) holds.
Step 2. $\{F^n(\xi^n)\}_{n \in \mathbf{N}}$ is uniformly integrable.

For each n, let p^n and D^n be the point process and jump set respectively corresponding to the Poisson random measure N^n. Let X^n be a process on a stochastic basis $(\Omega^n, \mathcal{F}^n, P^n, \{\mathcal{F}^n_t\})$ and solve the SDE (8.0.1). It follows from the proof of Theorem 6.1.3 that

$$F^n(X^n) = \int_0^T \int_U g(G^n(s, X^n_{s-}, u)[\phi])\tilde{N}^n(duds) \tag{8.1.8}$$

and

$$\sup_n E|F^n(\xi^n)|^2 = \sup_n E^{P^n}|F^n(X^n)|^2 \leq K_g\|\phi\|^2_{p_1}K(1+\tilde{K})T,$$

where $K_g \equiv \sup\{(g(x)/x)^2 : x \in \mathbf{R}\} < \infty$. This proves the assertion of step 2.

Step 3.

$$E^{\lambda^*}\sum_{0<s\leq T} g(\Delta Z_s[\phi]) = 0.$$

It follows from (8.1.8) that $E^{P^n}(F^n(X^n)) = 0$ for any $n \in \mathbf{N}$. Hence

$$\begin{aligned} E^{\lambda^*}\sum_{0<s\leq T} g(\Delta Z_s[\phi]) &= E\sum_{0<s\leq T} g(\Delta\xi_s[\phi]) = \lim_{n\to\infty} E(F^n(\xi^n)) \\ &= \lim_{n\to\infty} E^{P^n}(F^n(X^n)) = 0. \end{aligned}$$

Step 4. (8.1.6) holds.

Let $\{g_m\}$ be given by Lemma 6.1.8. As $\{g_m(x)\}$ increases to x^2 as m tends to ∞, we have

$$E^{\lambda^*}\sum_{0<s\leq T} |\Delta Z_s[\phi]|^2 = 0, \ \forall\phi \in \Phi.$$

Taking $\phi = \phi^{p_1}_j$, $j = 1, 2, \cdots$ and adding, we have

$$E^{\lambda^*}\sum_{0<s\leq T} \|\Delta Z_s\|^2_{-p_1} = 0.$$

This proves (8.1.6) and hence finishes the proof of the theorem. ∎

To characterize λ^*, we need to consider the martingale problem posed by (8.0.2). Let $\mathcal{D}^\infty_0(\Phi')$ be given by Chapter 6. For $F \in \mathcal{D}^\infty_0(\Phi')$, consider a map $\mathcal{D}_sF : \Phi' \to \mathbf{R}$ defined by

$$\mathcal{D}_sF(v) \equiv A(s,v)[\phi]h'(v[\phi]) + \frac{1}{2}h''(v[\phi])Q(B(s,v)'\phi, B(s,v)'\phi) \tag{8.1.9}$$

where $B(s,v)' : \Phi \to \Phi$ is the dual operator of $B(s,v)$. For $Z \in C([0,T], \Phi')$, let

$$M^F(Z)_t \equiv F(Z_t) - F(Z_0) - \int_0^t \mathcal{D}_sF(Z_s)ds. \tag{8.1.10}$$

Let $\mathcal{B}_T = \mathcal{B}_T(C([0,T], \Phi'))$ be the Borel σ-field of $C([0,T], \Phi')$. For each $t \in [0,T]$, let $\mathcal{B}_t = \pi_t^{-1}\mathcal{B}_T$ where $\pi_t : C([0,T], \Phi') \to C([0,T], \Phi')$ is given by $(\pi_t x)_s = x_{t\wedge s}$, $\forall s \in [0,T]$.

Definition 8.1.1 *A probability measure λ on $(C([0,T],\Phi'),\mathcal{B}_T)$ is called a solution of the $\mathcal{D}$-martingale problem if, $\forall F \in \mathcal{D}_0^\infty(\Phi')$, $\left\{M^F(Z)_t\right\}$ is a λ-martingale with respect to the filtration $\{\mathcal{B}_t\}$.*

Theorem 8.1.2 *Under assumptions (DA1) and (DA2), (A,B,Q,λ_0^*) satisfies the following conditions (D): For any $T>0$ there exists an index $p_0=p_0(T)$ such that, $\forall p\ge p_0$, $\exists q\ge p$ and a constant $K=K(p,q,T)$ such that*
(D1) (Continuity) $\forall t\in[0,T]$, the maps $v\in\Phi_{-p}\to A(t,v)\in\Phi_{-q}$ and $v\in\Phi_{-p}\to B(t,v)\in L_{(2)}(H_Q,\Phi_{-p})$ are continuous.
(D2) (Coercivity) $\forall t\in[0,T]$ and $\phi\in\Phi$,

$$2A(t,\phi)[\theta_p\phi]\le K(1+\|\phi\|_{-p}^2).$$

(D3) (Growth) $\forall t\in[0,T]$ and $v\in\Phi_{-p}$, we have

$$\|A(t,v)\|_{-q}^2\le K(1+\|v\|_{-p}^2)$$

and

$$\|B(t,v)\|_{L_{(2)}(H_Q,\Phi_{-p})}^2\le K(1+\|v\|_{-p}^2).$$

(D4) (Initial) There exists an index r_0 such that

$$\int_{\Phi'}\|v\|_{-r_0}^2\lambda_0^*(dv)<\infty$$

where λ_0^* is the initial distribution induced by λ^*.

Proof: It follows from the conditions (DA2)(1°) and (DA1) that the map $v\in\Phi_{-p}\to A(t,v)\in\Phi_{-q}$ is continuous and

$$\|A(t,v)\|_{-q}^2\le K(1+\|v\|_{-p}^2),\ \forall v\in\Phi_{-p}.$$

Note that

$$\begin{aligned}&\sum_j Q(B(t,v)'\phi_j^p,B(t,v)'\phi_j^p)\\ \le\ &\liminf_{n\to\infty}\sum_j\int_U|G^n(t,v,u)[\phi_j^p]|^2\mu^n(du)\\ \le\ &K(1+\|v\|_{-p}^2)\end{aligned}$$

so that $B(t,v)'\in L_{(2)}(\Phi_p,H_Q')$. Hence $B(t,v)\in L_{(2)}(H_Q,\Phi_{-p})$ and

$$\|B(t,v)\|_{L_{(2)}(H_Q,\Phi_{-p})}^2\le K(1+\|v\|_{-p}^2).$$

Further

$$\begin{aligned}
& \|B(t,v_1)-B(t,v_2)\|^2_{L_{(2)}(H_Q,\Phi_{-p})} \\
= & \sum_j Q\left((B(t,v_1)-B(t,v_2))'\phi^p_j,(B(t,v_1)-B(t,v_2))'\phi^p_j\right) \\
\le & \liminf_{n\to\infty}\sum_j\int_U |(G^n(t,v_1,u)-G^n(t,v_2,u))[\phi^p_j]|^2\mu^n(du) \\
= & \liminf_{n\to\infty}\int_U \|G^n(t,v_1,u)-G^n(t,v_2,u)\|^2_{-p}\mu^n(du).
\end{aligned}$$

Hence the map from $v\in\Phi_{-p}$ to $B(t,v)\in L_{(2)}(H_Q,\Phi_{-p})$ is continuous. The condition (D4) can be verified by Fatou's lemma. ∎

Remark 8.1.1 *It follows from Lemma 3.2.2 that*

$$Q(\phi,\phi)=\left\|\sqrt{Q_p}\phi\right\|^2_p=\|\phi\|^2_{H'_Q},\qquad \forall\phi\in\Phi.$$

For $B\in L(\Phi',\Phi')$, let

$$|Q_B|_{-p,-p}=\sum_{j=1}^\infty Q(B'\phi^p_j,B'\phi^p_j).$$

Then $|Q_B|_{-p,-p}<\infty$ if and only if $B\in L_{(2)}(H_Q,\Phi_{-p})$. In this case, we have that $|Q_B|_{-p,-p}=\|B\|^2_{L_{(2)}(H_Q,\Phi_{-p})}$. In the paper of Kallianpur, Mitoma and Wolpert [24], the notation $|Q_{B(t,v)}|_{-p,-p}$ is used in the place of $\|B(t,v)\|^2_{L_{(2)}(H_Q,\Phi_{-p})}$ in assumption (D3).

Theorem 8.1.3 *Under assumptions (DA1) and (DA2), λ^* is a solution of the $\mathcal{D}$-martingale problem.*

Proof: For $F\in\mathcal{D}^\infty_0(\Phi')$, let $M^F_n(Z)_t$ be defined by (6.1.4) with $\{A,G,\mu\}$ replaced by $\{A^n,G^n,\mu^n\}$. Let ξ^n,ξ,C,C_0 and M be as in the proof of Theorem 8.1.1. Note that

$$|M^F_n(\xi^n)_t-M^F(\xi)_t|\le I^n_1(t)+I^n_1(0)+\int_0^t I^n_2(s)ds+\left|\int_0^t I^n_3(s)ds\right|$$

where

$$I^n_1(s)=|h(\xi^n_s[\phi])-h(\xi_s[\phi])|,$$

$$I^n_2(s)=|A^n(s,\xi^n_s)[\phi]h'(\xi^n_s[\phi])-A(s,\xi_s)[\phi]h'(\xi_s[\phi])|$$

and

$$\begin{aligned} I_3^n(s) &= \int_U \{h(\xi_s^n[\phi] + G^n(s,\xi_s^n,u)[\phi]) - h(\xi_s^n[\phi]) \\ &\quad -G^n(s,\xi_s^n,u)[\phi]h'(\xi_s^n[\phi])\}\,\mu^n(du) \\ &\quad -\frac{1}{2}h''(\xi_s[\phi])Q(B(s,\xi_s)'\phi, B(s,\xi_s)'\phi). \end{aligned}$$

Now we prove that, $\forall t \in [0,T]$

$$E|M_n^F(\xi^n)_t - M^F(\xi)_t| \to 0 \quad \text{as } n \to \infty. \tag{8.1.11}$$

It follows from the uniform continuity of h'' that, for any $\epsilon > 0$, there exists $\delta > 0$ such that $|h''(x) - h''(y)| < \epsilon$ whenever $|x-y| < \delta$. Letting

$$D_n \equiv \left\{ \begin{array}{cc} u: & |\int_0^1 \int_0^1 \alpha(h''(\xi_s^n[\phi] + \alpha\beta G^n(s,\xi_s^n,u)[\phi]) \\ & -h''(\xi_s^n[\phi]))d\alpha d\beta| > \epsilon \end{array} \right\}$$

we have

$$\mu^n(D_n)1_C(\xi^n) \le \sup_{v\in C_0} \mu^n\{u : |G^n(s,v,u)[\phi]| > \delta\} \to 0. \tag{8.1.12}$$

Next,

$$\begin{aligned} & |I_3^n(s)|1_C(\xi^n)1_C(\xi) \\ \le\ & \left|\int_U \left\{\int_0^1 \int_0^1 \alpha\left(h''(\xi_s^n[\phi] + \alpha\beta G^n(s,\xi_s^n,u)[\phi]) - h''(\xi_s^n[\phi])\right) d\alpha d\beta\right\} \right. \\ & \left. \qquad G^n(s,\xi_s^n,u)[\phi]^2\mu^n(du)\right|1_C(\xi^n) \\ & +\frac{1}{2}|h''(\xi_s^n[\phi]) - h''(\xi_s[\phi])| \sup_{v\in C_0} \int_U G^n(s,v,u)[\phi]^2\mu^n(du) \\ & +\frac{1}{2}\|h''\|_\infty \sup_{v\in C_0}\left|\int_U G^n(s,v,u)[\phi]^2\mu^n(du) - Q(B(s,v)'\phi, B(s,v)'\phi)\right| \\ & +\frac{1}{2}\|h''\|_\infty \left|\|B(s,\xi_s^n)'\phi\|_{H_Q'}^2 - \|B(s,\xi_s)'\phi\|_{H_Q'}^2\right|. \end{aligned} \tag{8.1.13}$$

It follows from the continuity of h'' that the second term at the right hand side of (8.1.13) tends to 0 P-a.s. The condition (DA2)(2°) implies that the third term converges to 0. By Theorem 8.1.2, the fourth term tends to 0 P-a.s. Note that the first term is dominated by

$$\begin{aligned} & \sup_{v\in C_0} \int_U \epsilon G^n(s,v,u)[\phi]^2\mu^n(du) \\ & +\|h''\|_\infty \sup_{v\in C_0} \int_{D_n} G^n(s,v,u)[\phi]^2\mu^n(du) \\ \le\ & \epsilon K(1+M^2)\|\phi\|_{p_1}^2 + \|h''\|_\infty \sup_{v\in C_0} \int_{D_n} G^n(s,v,u)[\phi]^2\mu^n(du). \end{aligned}$$

From (8.1.5) and (8.1.12) we have

$$\sup_{v\in C_0}\int_{D_n} G^n(s,v,u)[\phi]^2\mu^n(du)\to 0.$$

Hence by (8.1.13),

$$\limsup_{n\to\infty}|I_3^n(s)|1_C(\xi^n)1_C(\xi)\le \epsilon K(1+M^2)\|\phi\|_{p_1}^2.$$

As

$$\begin{aligned}
&|I_3^n(s)|1_C(\xi^n)1_C(\xi)\\
\le\ & \frac{1}{2}\|h''\|_\infty\Big\{\int_U G^n(s,\xi_s^n,u)[\phi])^2\mu^n(du)1_C(\xi^n)\\
&\qquad +Q(B(s,\xi_s)'\phi,B(s,\xi_s)'\phi)1_C(\xi)\Big\}\\
\le\ & \frac{1}{2}\|h''\|_\infty\Big\{K(1+\|\xi_s^n\|_{-p_1}^2)\|\phi\|_{p_1}^2 1_C(\xi^n)\\
&\qquad +\lim_{m\to\infty}\int_U G^m(s,\xi_s,u)[\phi])^2\mu^m(du)1_C(\xi)\Big\}\\
\le\ & K(1+M^2)\|h''\|_\infty\|\phi\|_{p_1}^2,
\end{aligned}$$

it follows from Fatou's lemma that

$$\begin{aligned}
&\limsup_{n\to\infty}P\left(\left|\int_0^t I_3^n(s)ds\right|>a\right)\\
\le\ & 2\epsilon+\limsup_{n\to\infty}\frac{1}{a}E\left|\int_0^t I_3^n(s)1_C(\xi^n)1_C(\xi)ds\right|\\
\le\ & 2\epsilon+\frac{\epsilon KT}{a}(1+M^2)\|\phi\|_{p_1}^2.
\end{aligned}$$

Hence, $|\int_0^t I_3^n(s)ds|$ converges to 0 in probability. Similarly we can prove that $\int_0^t I_2^n(s)ds$ converges to 0 in probability. Furthermore, it is easy to see that, $I_1^n(t)$ and $I_1^n(0)$ tends to 0 a.s. Therefore $M_n^F(\xi^n)_t$ tends to $M^F(\xi)_t$ in probability.

As X^n is a solution of (8.0.1), it follows from *Itô*'s formula that

$$M_n^F(X^n)_t=\int_0^T\int_U(h(X_{s-}^n[\phi]+G^n(s,X_{s-}^n,u)[\phi])-h(X_{s-}^n[\phi]))\tilde{N}^n(duds) \tag{8.1.14}$$

and hence

$$\begin{aligned}
&E|M_n^F(\xi^n)_t|^2=E^{P^n}|M_n^F(X^n)_t|^2\\
=\ & E^{P^n}\int_0^T\int_U|h(X_s^n[\phi]+G^n(s,X_s^n,u)[\phi])-h(X_s^n[\phi])|^2\mu^n(du)ds\\
\le\ & K(\tilde{K}+1)T\|h'\|_\infty^2\|\phi\|_{p_1}^2.
\end{aligned}$$

Thus $\forall t \in [0,T]$, $\{M_n^F(\xi^n)_t\}$ is uniformly integrable and hence (8.1.11) holds.

By (8.1.14) again, $\forall n \in \mathbf{N}$, $\{M_n^F(X^n)_t\}$ is a P^n-martingale and hence $\{M_n^F(\xi^n)_t\}$ is a P-martingale. Passing to the limit, we see that $M^F(\xi)_t$ is a P-martingale and hence, $M^F(Z)_t$ is a λ^*-martingale. i.e. λ^* is a solution of the $\mathcal{D}$-martingale problem. ∎

8.2 Weak solutions of diffusion equations

In this section we derive the weak solutions of (8.0.2) from the solutions of the corresponding martingale problem. The idea is similar to that used at the end of Section 6.1. We shall also be using the representation theorem in Chapter 3 for Φ'-valued continuous martingales.

Definition 8.2.1 *A probability measure λ on $C([0,T],\Phi')$ is called a* **weak solution** *on [0,T] of the SDE (8.0.2) with initial distribution λ_0 on the Borel sets of Φ' if there exists a stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$, a Φ'-Wiener process W with covariance function Q and a Φ'-valued process X such that λ and λ_0 are the distributions of X and X_0 respectively and for any $t \in [0,T]$, we have*

$$X_t = X_0 + \int_0^t A(s, X_s)ds + \int_0^t B(s, X_s)dW_s, \ a.s. \tag{8.2.1}$$

If [0,T] can be changed to $[0,\infty)$ and (8.2.1) holds for any $t \geq 0$, then we call λ on $C([0,\infty),\Phi')$ a **weak solution** *of (8.0.2).*

Lemma 8.2.1 *$\forall \phi \in \Phi$, let*

$$M_\phi(t,Z) = Z_t[\phi] - Z_0[\phi] - \int_0^t A(s, Z_s)[\phi]ds.$$

Under the conditions (DA1) and (DA2), $\{M_\phi(t,Z)\}_{t\leq T}$ is a continuous λ^*-square-integrable martingale.

Proof: Let $F_m \in \mathcal{D}_0^\infty(\Phi')$ be given by $F_m(v) = \rho_m(v[\phi])$ where ρ_m is given by Lemma 6.1.8. Let

$$\mathcal{X} \equiv \left\{ Z \in C([0,T], \Phi_{-p_1}) : \|Z_t\|_{-p_1} \leq (m-1)\|\phi\|_{p_1}^{-1} \ \forall t \in [0,T] \right\}.$$

Then, for $Z \in \mathcal{X}$, we have $|Z_s[\phi]| \leq m-1$ and hence, $M^{F_m}(Z)_t = M_\phi(t,Z)$. Therefore

$$\lambda^* \left(Z \in C([0,T], \Phi_{-p_1}) : \sup_{0 \leq t \leq T} |M^{F_m}(Z)_t - M_\phi(t,Z)| > \epsilon \right) \leq \lambda^*(\mathcal{X}^c)$$

$$
\begin{aligned}
&= \lambda^*\left(Z \in C([0,T],\Phi_{-p_1}) : \sup_{0\le t\le T} \|Z_t\|_{-p_1} > (m-1)\|\phi\|_{p_1}^{-1}\right)\\
&\le \frac{1}{(m-1)^2\|\phi\|_{p_1}^{-2}} E^{\lambda^*} \sup_{0\le t\le T} \|Z_t\|_{-p_1}^2 \le \frac{\|\phi\|_{p_1}^2}{(m-1)^2}\tilde{K} \to 0, \text{ as } m \to \infty.
\end{aligned}
$$

i.e.

$$M^{F_m}(Z)_t \to M_\phi(t,Z) \qquad \text{in } \lambda^*. \tag{8.2.2}$$

By Theorem 8.1.2, it is easy to show that there exists a constant C' independent of m such that

$$|M^{F_m}(Z)_t| \le C'\left(1 + \sup_{0\le t\le T} \|Z_t\|_{-p_1}^2\right). \tag{8.2.3}$$

As the left hand side of (8.2.3) is integrable with respect to λ^*, by (8.2.2), we have

$$E^{\lambda^*}|M^{F_m}(Z)_t - M_\phi(t,Z)| \to 0.$$

$\forall m \ge 1$, $\{M^{F_m}(Z)_t\}$ is a λ^*-martingale and therefore $\{M_\phi(t,Z)\}$ is a λ^*-martingale. Finally, it is easy to see that there exists a constant C'' such that

$$|M_\phi(t,Z)|^2 \le C''\left(1 + \sup_{0\le t\le T} \|Z_t\|_{-p_1}^2\right).$$

Hence $\{M_\phi(t,Z)\}$ is a λ^*-square-integrable-martingale. The continuity of $M_\phi(t,Z)$ in t is clear. ∎

Lemma 8.2.2 *Let $< M_\phi > (t,Z)$ be the quadratic variation process of the square integrable martingale M_ϕ. Under the conditions (DA1) and (DA2), we have*

$$< M_\phi > (t,Z) = \int_0^t Q(B(s,Z_s)'\phi, B(s,Z_s)'\phi)ds. \tag{8.2.4}$$

Proof: $\forall \phi \in \Phi$, let

$$
\begin{aligned}
N_\phi(t,Z) &= Z_t[\phi]^2 - Z_0[\phi]^2 - 2\int_0^t A(s,Z_s)[\phi]Z_s[\phi]ds\\
&\quad - \int_0^t Q(B(s,Z_s)'\phi, B(s,Z_s)'\phi)ds.
\end{aligned}
$$

Arguing as in the proof of Lemma 8.2.1, we see that $\{N_\phi(t,Z)\}_{t\le T}$ is a λ^*-martingale. It follows from *Itô*'s formula that

$$
\begin{aligned}
Z_t[\phi]^2 &= Z_0[\phi]^2 + 2\int_0^t A(s,Z_s)[\phi]Z_s[\phi]ds\\
&\quad + 2\int_0^t Z_s[\phi]dM_\phi(s,Z) + < M_\phi > (t,Z).
\end{aligned}
$$

Therefore

$$\begin{aligned} & < M_\phi > (t, Z) - \int_0^t Q(B(s, Z_s)'\phi, B(s, Z_s)'\phi)dt \\ = \quad & N_\phi(t, Z) - 2\int_0^t Z_s[\phi] dM_\phi(s, Z) \end{aligned}$$

is a martingale. This proves (8.2.4). ∎

Theorem 8.2.1 *Under assumptions (DA1) and (DA2), the SDE (8.0.2) has a weak solution.*

Proof: It follows from Lemma 8.2.1 and Lemma 8.2.2 that $M \in \mathcal{M}^{2,c}$ such that

$$< M_t[\phi] >= \int_0^t Q(B(s, Z_s)'\phi, B(s, Z_s)'\phi)ds, \ \forall \phi \in \Phi.$$

By Theorem 3.3.6 there exists a Φ'-valued Q-Wiener process W on an extension of the stochastic basis $(C([0,T], \Phi'), \mathcal{B}_T, \lambda^*, \{\mathcal{B}_t\})$ such that

$$M_t = \int_0^t B(s, Z_s)dW_s.$$

Therefore

$$Z_t = Z_0 + \int_0^t A(s, Z_s)ds + \int_0^t B(s, Z_s)dW_s,$$

and hence λ^* is a weak solution of the SDE (8.0.2) on [0,T]. ∎

Now we shall establish the existence of the weak solution of (8.0.1) under the conditions (D) instead of (DA1) and (DA2).

Theorem 8.2.2 *Under assumptions (D), the SDE (8.0.2) has a weak solution which can be approximated by a sequence of processes driven by Poisson random measures.*

Proof: By Lemma 3.2.2, there exists an index r and an operator $\sqrt{Q_r}$ on Φ_r such that

$$Q(\phi, \psi) = \left\langle \sqrt{Q_r}\phi, \sqrt{Q_r}\psi \right\rangle_r, \qquad \forall \phi, \psi \in \Phi.$$

Let $U = \{1, 2, \cdots\}$, $\mu^n(\{k\}) = n^2$, $X_0^n = X_0$, $A^n(t, v) = A(t, v)$ and

$$G^n(t, v, k)[\phi] = \frac{1}{n}\left\langle \sqrt{Q_r}B(t, v)'\phi, \phi_k^r \right\rangle_r,$$

for any $t \geq 0$, $k \in U$, $v \in \Phi'$. Now, we only need to verify the conditions (DA1) and (DA2). Note that

$$\begin{aligned}
&\int_U \|G^n(t,v,u)\|_{-p}^2 \mu^n(du) = \sum_j \int_U (G^n(t,v,u)[\phi_j^p])^2 \mu^n(du) \\
&= \sum_j \sum_k \left\langle \sqrt{Q_r} B(t,v)'\phi_j^p, \phi_k^r \right\rangle_r^2 = \sum_j \left\| \sqrt{Q_r} B(t,v)'\phi_j^p \right\|_r^2 \\
&= \sum_j Q(B(t,v)'\phi_j^p, B(t,v)'\phi_j^p) = \|B(t,v)\|_{L_{(2)}(H_Q,\Phi_{-p})}^2 \\
&\leq K(1+\|v\|_{-p}^2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_U \|G^n(t,v_1,u) - G^n(t,v_2,u)\|_{-p}^2 \mu^n(du) \\
&= \|B(t,v_1) - B(t,v_2)\|_{L_{(2)}(H_Q,\Phi_{-p})}^2,
\end{aligned}$$

and hence the map from $v \in \Phi_{-p}$ to $G^n(t,v,\cdot) \in L^2(U,\mu^n;\Phi_{-p})$ is continuous and uniform for n. The verification of the rest of the condition (DA1) for $(A^n, G^n, \mu^n, \lambda_0^n)$ directly follows from assumptions (D).

Next, let C_0 be any compact subset of Φ_{-p}. Note that

$$\begin{aligned}
n^2\|G^n(t,v,k)\|_{-p}^2 &\leq \sum_{r=1}^{\infty} n^2 \|G^n(t,v,r)\|_{-p}^2 \\
&= \int_U \|G^n(t,v,u)\|_{-p}^2 \mu^n(du) \\
&\leq K(1+\|v\|_{-p}^2).
\end{aligned}$$

Hence, for $n \geq \frac{\|\phi\|_p}{a}\sqrt{K(1+\sup_{v\in C_0}\|v\|_{-p}^2)}$, we have

$$\begin{aligned}
&\sup_{v\in C_0} \mu^n\{u : |G^n(t,v,u)[\phi]| > a\} \\
&\leq \sup_{v\in C_0} \mu^n\{u : K(1+\|v\|_{-p}^2)\|\phi\|_p^2 > (na)^2\} \\
&= 0.
\end{aligned}$$

This proves (8.1.3). (8.1.5) can be shown in a similar manner. For (8.1.4), we note that

$$\begin{aligned}
&\int_U G^n(t,v_1,u)[\phi] G^n(t,v_2,u)[\phi] \mu^n(du) \\
&= \sum_{k=1}^{\infty} \left\langle \sqrt{Q_r} B(t,v_1)'\phi, \phi_k^r \right\rangle_r \left\langle \sqrt{Q_r} B(t,v_1)'\phi, \phi_k^r \right\rangle_r \\
&= Q(B(t,v_1)'\phi, B(t,v_2)'\phi).
\end{aligned}$$

Hence $(A^n, G^n, \mu^n, \lambda_0^n)$ satisfies assumption (DA2). ∎

Finally, we construct a weak solution on $[0, \infty)$ for (8.0.2) by arguments similar to those at the end of Section 6.2. First of all, let us construct a sequence of measures λ_n on $\mathbf{C}^n = C([0, nT], \Phi_{-p_1(nT)})$ by induction. Taking $\lambda_1 = \lambda^*$ and assuming that λ_n on $\mathbf{C}^n$ has been constructed, we now construct λ_{n+1} on $\mathbf{C}^{n+1}$.

For $0 \le t \le T$, $v \in \Phi'$, let

$$\tilde{A}(t,v) = A(t+nT, v), \ \tilde{B}(t,v) = B(t+nT, v) \text{ and } \tilde{\lambda}_0 = \lambda_n \circ Z_{nT}^{-1}. \quad (8.2.5)$$

Then $(\tilde{A}, \tilde{B}, Q, \tilde{\lambda}_0)$ satisfies assumptions (D) with p_0 and K(p,q,T) replaced by $p_0((n+1)T)$ and $K(p, q, (n+1)T)$ respectively. The SDE

$$X_t = X_0 + \int_0^t \tilde{A}(s, X_s) ds + \int_0^t \tilde{B}(s, X_s) dW_s$$

has a $\Phi_{-p_1((n+1)T)}$-valued weak solution $\tilde{\lambda}_n^*$ on [0,T]. Since

$$\mathbf{C}^{1,n+1} \equiv C([0,T], \Phi_{-p_1((n+1)T)})$$

is a Polish space, the regular conditional probability measure

$$\hat{\lambda}_{z_0}^*(\cdot) = E^{\tilde{\lambda}_n^*}(Z \in \cdot | Z_0 = z_0)$$

exists. Let

$$\pi : \mathbf{C}^n \times \mathbf{C}^{1,n+1} \to \mathbf{C}^{n+1}$$

be given by

$$\pi(Z^1, Z^2)_t = \begin{cases} Z_t^1 & \text{as } 0 \le t \le nT \\ Z_{t-nT}^2 & \text{as } nT \le t \le (n+1)T. \end{cases}$$

Define a measure λ_{n+1}^* on $\mathbf{C}^n \times \mathbf{C}^{1,n+1}$ by

$$\lambda_{n+1}^*(C \times D) = \int_C \hat{\lambda}_{Z_1^{nT}}^*(D) \lambda_n(dZ^1)$$

for $C \subset \mathbf{C}^n$ and $D \subset \mathbf{C}^{1,n+1}$. Then λ_{n+1}^* induces the measure $\lambda_{n+1} = \lambda_{n+1}^* \circ \pi^{-1}$ on $\mathbf{C}^{n+1}$.

The λ_n's can be regarded as probability measures on $C([0,\infty), \Phi')$ and satisfy

$$\lambda_{n+1}|_{\mathcal{B}_{nT}} = \lambda_n$$

where $\mathcal{B}_{nT}$ is the natural σ-algebra on $C([0,\infty), \Phi')$ upto time nT. Hence, the set function

$$\tilde{\lambda}(B) = \lambda_n(B), \ \ \forall B \in \mathcal{B}_{nT}.$$

on the field $\cup_n \mathcal{B}_{nT}$ is well-defined and σ-additive. Therefore $\tilde{\lambda}$ can be extended to a probability measure on the σ-field $\vee_n \mathcal{B}_{nT} = \mathcal{B}$. Denoting this extension also by $\tilde{\lambda}$, we have

$$\tilde{\lambda}|_{\mathcal{B}_{nT}} = \lambda_n.$$

The proofs of the following two lemmas follow from the same arguments as those in the proof of Lemma 6.2.3. We leave them to the reader.

Lemma 8.2.3 *$\tilde{\lambda}$ is a solution of the $\mathcal{D}$-martingale problem.*

Lemma 8.2.4 *(1°) For any $\phi \in \Phi$, $\{M_\phi(t, Z)\}_{t \geq 0}$ given by Lemma 8.2.1 is a $\tilde{\lambda}$-square-integrable continuous martingale with*

$$< M_\phi > (t, Z) = \int_0^t Q(B(s, Z_s)'\phi, B(s, Z_s)'\phi) ds, \ \forall t \geq 0.$$

Now we obtain a weak solution of (8.0.1) for $t \in \mathbf{R}_+$.

Theorem 8.2.3 *Suppose that assumptions (D1)-(D3) hold and $\forall \phi \in \Phi$,*

$$E|X_0[\phi]|^2 < \infty.$$

Then (8.0.2) has a Φ'-valued weak solution satisfying the following condition: $\forall T > 0$, $\exists p_1 = p_1(T)$ such that

$$E \sup_{0 \leq t \leq T} \|X_t\|^2_{-p_1} \leq \tilde{K}(K, T, E\|X_0\|^2_{-p}).$$

Proof: It follows from the proof of Theorem 6.2.3 that there exists an index r such that, $E\|X_0\|^2_{-r_0} < \infty$. The rest of the the proof follows as in the proof of Theorem 8.2.1. ∎

8.3 Existence and uniqueness of the strong solution

In this section, we shall impose an additional condition to ensure that the SDE (8.0.2) has a unique strong solution. This will be achieved by establishing pathwise uniqueness and extending the Yamada-Watanabe argument to this setup. Replacing the Good process by Φ'-valued Wiener process, we shall follow the same procedure as in Section 6.3.

We first state some basic definitions.

Definition 8.3.1 *Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ be a stochastic basis and W a Φ'-valued Wiener process with covariance function Q. Suppose that X_0 is a Φ_{-p}-valued random variable such that $E\|X_0\|_{-p}^2 < \infty$. Then by an Φ_{-p}-valued* **strong solution** *on Ω to the SDE (8.0.2) for $t \in [0,T]$ we mean a process X_t defined on Ω such that*
(a) X_t is an Φ_{-p}-valued $\mathcal{F}_t$-measurable random variable;
(b) $X \in C([0,T], \Phi_{-p})$, a.s.;
(c) There exists a sequence (σ_n) of stopping times on Ω increasing to infinity, such that, $\forall n$

$$E\int_0^{T\wedge\sigma_n} \|B(s,X_s)\|_{L_{(2)}(H_Q,\Phi_{-p})}^2 ds < \infty, \tag{8.3.1}$$

and

$$E\int_0^{T\wedge\sigma_n} \|A(s,X_s)\|_{-q}^2 ds < \infty;$$

(d) The SDE (8.0.2) is satisfied for all $t \in [0,T]$ and almost all $\omega \in \Omega$.
If T is replaced by ∞, we call X a strong solution of (8.0.2).

Definition 8.3.2 (pathwise uniqueness) *A Φ_{-p}-valued solution for the SDE (8.0.2) has the* **pathwise uniqueness** *property if the following is true: Suppose that X and X' are two Φ_{-p}-valued solutions defined on the same probability space $(\Omega, \mathcal{F}, P)$ with respect to the same Φ'-valued Wiener process and starting from the same initial point $X_0 \in \Phi_{-p}$. Then the paths of X and X' coincide for almost all $\omega \in \Omega$.*

Now, we impose the following monotonicity condition
(DM): $\forall t \in [0,T]$, $v_1, v_2 \in \Phi_{-p}$, we have that

$$\begin{aligned} 2 < A(t,v_1) - A(t,v_2), v_1 - v_2 >_{-q} \\ +\|B(t,v_1) - B(t,v_2)\|_{L_{(2)}(H_Q,\Phi_{-q})}^2 &\le K\|v_1 - v_2\|_{-q}^2 \end{aligned}$$

where q is introduced in assumptions (D).

Lemma 8.3.1 *Under assumptions (D) and (DM), SDE (8.0.2) satisfies the pathwise uniqueness property.*

Proof: Let X and X' be two Φ_{-p}-valued solutions. Without loss of generality, suppose that (c) of the Definition 8.3.1 holds for X and X' for the same sequence (σ_n) of stopping times. For $\phi \in \Phi$, we have

$$\begin{aligned} (X_t - X_t')[\phi] &= \int_0^t (A(s,X_s) - A(s,X_s'))[\phi] ds \\ &\quad + \sum_j \int_0^t < B(s,X_s)'\phi, v_j >_{H_Q} dW_t[\iota^{-1}v_j], \end{aligned}$$

where $\{v_j\} \subset \mathcal{R}(\iota)$ is a CONS of H_Q and ι is defined in Lemma 3.2.1. It follows from Itô's formula that

$$\begin{aligned} & Ee^{-K(t\wedge\sigma_n)}\{(X_t - X'_t)[\phi]\}^2 \\ = \;& 2E\int_0^{t\wedge\sigma_n} e^{-Ks}(X_s - X'_s)[\phi](A(s,X_s) - A(s,X'_s))[\phi]ds \\ & -E\int_0^{t\wedge\sigma_n} Ke^{-Ks}((X_s - X'_s)[\phi])^2 ds \\ & +E\int_0^{t\wedge\sigma_n} e^{-Ks}Q((B(s,X_s) - B(s,X'_s))'\phi, \\ & \qquad\qquad (B(s,X_s) - B(s,X'_s))'\phi)ds. \end{aligned}$$

Letting $\phi = \phi_k^q$, $k \in \mathbf{N}$ and adding, we have

$$\begin{aligned} & Ee^{-K(t\wedge\sigma_n)}\|X_t - X'_t\|_{-q}^2 \\ = \;& 2E\int_0^{t\wedge\sigma_n} e^{-Ks} < X_s - X'_s, A(s,X_s) - A(s,X'_s) >_{-q} ds \\ & -E\int_0^{t\wedge\sigma_n} Ke^{-Ks}\|X_s - X'_s\|_{-q}^2 ds \\ & +E\int_0^{t\wedge\sigma_n} e^{-Ks}\|B(s,X_s) - B(s,X'_s)\|_{L_{(2)}(H_Q,\Phi_{-q})}^2 ds \\ \le \;& 0. \end{aligned} \tag{8.3.2}$$

Hence, by the right continuity of X and X' and (8.3.2), $X = X'$ a.s. ∎

Definition 8.3.3 (Uniqueness in law) *We say that uniqueness in law holds for (8.0.2) if, for any two stochastic bases $(\Omega^k, \mathcal{F}^k, P^k, (\mathcal{F}_t^k))$, two Φ'-valued Wiener processes W^k with the same covariance function Q and two Φ_{-p}-valued solutions X^k of (8.0.2) with the same initial distribution on Φ_{-p}, $(k = 1, 2)$, we have that X^1 and X^2 induce the same probability measure on $C([0,T], \Phi_{-p})$.*

Suppose X' and X'' are two solutions of the SDE (8.0.2) on stochastic bases $(\Omega', \mathcal{F}', P', (\mathcal{F}'_t))$ and $(\Omega'', \mathcal{F}'', P'', (\mathcal{F}''_t))$ with initial random variables X'_0 and X''_0 (having the same distribution λ_0 on Φ_{-p_1}) and Φ'-valued Wiener processes W' and W'' (having the same covariance function Q) respectively. Let $\mathcal{X}$ be a Banach space containing H_Q such that W' and W'' take values in $C([0,T], \mathcal{X})$. Let P_W be the probability measure on $C([0,T], \mathcal{X})$ induced by either W' or W''. Let λ' and λ'' be the Borel probability measures on $C([0,T], \Phi_{-p_1}) \times C([0,T], \mathcal{X}) \times \Phi_{-p_1}$ induced by (X', W', X'_0) and (X'', W'', X''_0) respectively. Define a mapping

$$\pi : C([0,T], \Phi_{-p_1}) \times C([0,T], \mathcal{X}) \times \Phi_{-p_1} \to C([0,T], \mathcal{X}) \times \Phi_{-p_1}$$

by $\pi(w_1, w_2, x) = (w_2, x)$. Then, $\lambda' \circ \pi^{-1} = \lambda'' \circ \pi^{-1} = P_W \otimes \lambda_0$.

Let $\lambda'^{w_2,x}(dw_1)$ and $\lambda''^{w_2,x}(dw_1)$ be the regular conditional probability of w_1 given w_2 and x with respect to λ' and λ'' respectively. This is possible since $C([0,T], \Phi_{-p_1})$ is a Polish space. On the space

$$\Omega = C([0,T], \Phi_{-p_1}) \times C([0,T], \Phi_{-p_1}) \times C([0,T], \mathcal{X}) \times \Phi_{-p_1},$$

define a Borel probability measure λ^* by

$$\begin{aligned} \lambda^*(C) &= \int\int\int\int 1_C(w_1, w_2, w_3, x) \\ &\qquad \lambda'^{w_3,x}(dw_1)\lambda''^{w_3,x}(dw_2)P_W(dw_3)\lambda_0(dx) \end{aligned} \tag{8.3.3}$$

for $C \in \mathcal{B}(\Omega)$. Then, it is easy to show that (w_1, w_3, x) and (X', W', X_0') have the same distribution and so do (w_2, w_3, x) and (X'', W'', X_0'').

The proof of the following Lemma is as in Lemma 6.3.3.

Lemma 8.3.2 *For any $C \in \mathcal{B}_t(C([0,T], \Phi_{-p_1}))$, we define two functions f_1 and f_2*

$$f_1(w,x) = \lambda'^{w,x}(C) \quad \textit{and} \quad f_2(w,x) = \lambda''^{w,x}(C).$$

Then f_1 and f_2 are measurable with respect to the completion of the σ-field $\mathcal{B}_t(C([0,T], \mathcal{X})) \times \mathcal{B}(\Phi_{-p_1})$ under the probability measure $P_W \otimes \lambda_0$.

Lemma 8.3.3 *Let $\mathcal{B}_t'$ be the completion of*

$$\mathcal{B}_t(C([0,T], \Phi_{-p_1})) \times \mathcal{B}_t(C([0,T], \Phi_{-p_1})) \times \mathcal{B}_t(C([0,T], \mathcal{X})) \times \mathcal{B}(\Phi_{-p_1}).$$

Then w_3 is a Φ'-valued Wiener process on the stochastic basis $(\Omega, \mathcal{B}', \lambda, \mathcal{B}_t')$.

Proof: We only need to prove the independence of $w_3(t) - w_3(s)$ and $\mathcal{B}_s'$ for any $t > s$. This follows from the same argument as in Step 1 of the proof of Lemma 6.3.4. ∎

Theorem 8.3.1 *Under assumptions (D) and (DM), uniqueness in law holds and the SDE (8.0.2) has a unique strong solution on [0, T].*

Proof: Let X' and X'' be two solutions of the SDE (8.0.2). From the arguments above, we see that (w_1, w_3, x) and (w_2, w_3, x) are two solutions of (8.0.2) on the same stochastic basis $(\Omega, \mathcal{B}', \lambda, \mathcal{B}_t')$. By the pathwise uniqueness proved in Lemma 8.3.1, we have that $\tilde{\lambda}(w_2 = w_1) = 1$. Coming back to the original probability space, we have $\lambda(w_2 = w_1) = 1$. But, by (8.3.3),

$$\lambda(w_2 = w_1) = \int\int \lambda'^{w,x} \otimes \lambda''^{w,x}(w_2 = w_1)P_W(dw)\lambda_0(dx),$$

so, for $P_W \otimes \lambda_0$-a.s. (w,x), we have

$$\lambda'^{w,x} \otimes \lambda''^{w,x}(w_1 = w_2) = 1. \tag{8.3.4}$$

By Lemma 6.3.5 and (8.3.4), we have a mapping F from $C([0,T],\mathcal{X}) \times \Phi_{-p_1}$ to $C([0,T],\Phi_{-p_1})$ such that

$$\lambda'^{w,x} = \lambda''^{w,x} = \delta_{F(w,x)}. \tag{8.3.5}$$

For any $C \in \mathcal{B}_t(C([0,T],\Phi_{-p_1}))$, by (8.3.5), Lemma 8.3.2 and

$$1_{F^{-1}(C)}(w,x) = \lambda'^{w,x}(C),$$

it is easy to see that $F^{-1}(C)$ is in the completion of $\mathcal{B}_t(C([0,T],\mathcal{X}))\times\mathcal{B}(\Phi_{-p_1})$ under $P_W \otimes \lambda_0$, and hence, $F(w,x)$ is adapted. Then, for any Φ'-valued Wiener Process with covariance function Q and initial Φ_{-p_1}-valued random variable X_0, $F(W, X_0)$ is a strong solution of the SDE (8.0.2).

The uniqueness of the strong solution follows directly from the pathwise uniqueness of the SDE (8.0.2). The uniqueness in law follows from (8.3.5). ∎

The following theorem establishes the existence of a unique strong solution for (8.0.2) and can be proved by the same arguments as those in the proof of Theorem 6.3.2.

Theorem 8.3.2 *Under assumptions (D) and (DM), if $E|X_0[\phi]|^2 < \infty\ \forall \phi \in \Phi$, then the SDE (8.0.2) has a unique Φ'-valued strong solution.*

Next we make an additional assumption and derive the diffusion approximation of SDE's on the dual of a CHNS driven by Poisson random measures.

Assumption (DA3): For each n, (A^n, G^n, μ^n) satisfies the condition (M) of Chapter 6 where the index q and the constant K are independent of n.

Theorem 8.3.3 *Under assumptions (DA1)-(DA3), SDE (8.0.1) has a unique solution for each n. Let λ^n be the distribution of this solution on $D([0,T],\Phi_{-p_1})$. Then $\{\lambda^n\}$ converges weakly to the distribution λ of the unique solution of SDE (8.0.2).*

Proof: The first part of theorem follows from Theorem 6.3.1. Under the assumption (DA3), it is easy to verify the condition (DM) for (A,B,Q) and hence, by Theorem 8.3.1, (8.0.2) has a unique solution. Denote the distribution of the unique solution of (8.0.2) by λ. As the sequence $\{\lambda^n\}$ is tight with only a single cluster point λ, $\{\lambda^n\}$ converges to λ weakly. ∎

Finally, we apply our results to the linear case. Let (Φ, H, T_t) be a special compatible family and $\Lambda \in \mathcal{B}(\Phi')$. For each $n \geq 1$ let μ^n be a measure on $(\mathbf{R} \times \Lambda, \mathcal{B}(\mathbf{R}) \times \mathcal{B}(\Lambda))$ such that the positive definite bilinear form

$$Q^n(\phi, \psi) \equiv \int_{\mathbf{R}\times\Lambda} a^2 \eta[\phi]\eta[\psi]\mu^n(dad\eta)$$

is continuous on $\Phi \times \Phi$, and let N^n be a Poisson random measure with characteristic measure $\mu^n(dad\eta)$. Define

$$Y_t^n[\phi] \equiv \int_0^t \int_{\mathbf{R}\times\Lambda} a\eta[\phi]\tilde{N}^n(dad\eta ds)$$

where $\tilde{N}^n$ is the compensated random measure of N^n.

For $n \geq 1$ let $m^n \in \Phi'$ and consider the Φ'-valued process $\xi^n_\cdot$ given by

$$\begin{aligned} d\xi_t^n &= -L'\xi_t^n dt + m^n dt + dY_t^n \\ \xi_0^n &= \eta^n \end{aligned} \tag{8.3.6}$$

where η^n is $\mathcal{F}_0$-measurable.

Corollary 8.3.1 *Assume the following six conditions hold:*
1) There exists $r_2 > 0$ and $c > 0$ such that for $n \geq 1$

$$m^n[\phi]^2 + Q^n(\phi, \phi) \leq c\|\phi\|_{r_2}^2 \quad \forall \phi \in \Phi.$$

2) $\lim_{n\to\infty} Q^n(\phi, \phi) = Q(\phi, \phi)$ $\forall \phi \in \Phi$ for some positive definite bilinear continuous form Q on $\Phi \times \Phi$.
3) $\lim_{n\to\infty} m^n[\phi] = m[\phi]$, $\forall \phi \in \Phi$, for some $m \in \Phi'$.
4) There exists $r_0 > 0$ such that

$$\sup_n \max\{E\|\eta^n\|_{r_0}^2, E\|\eta\|_{r_0}^2\} < \infty.$$

5) There exists $r_3 > 0$ such that η^n converges in law to η on Φ'_{r_3}
6)

$$\lim_{n\to\infty} \int_{\mathbf{R}\times\Lambda} |a\eta[\phi]|^3 \mu^n(dad\eta) = 0 \quad \forall \phi \in \Phi.$$

Then for each $T > 0$ there exists $p_T > 0$ such that ξ^n converges weakly to ξ on $D([0,T], \Phi'_{p_T})$ where ξ is the unique solution of

$$\begin{aligned} d\xi_t &= -L'\xi_t dt + m dt + dW_t \\ \xi_0 &= \eta \end{aligned} \tag{8.3.7}$$

and W is a centered Φ'-valued Wiener process with covariance functional Q. Furthermore

$$\xi \in C([0,T], \Phi'_{p_T}).$$

Proof: It follows from Theorem 8.3.3 that we have only to verify Assumptions (DA1)-(DA3) for (A^n, G^n, μ^n) where

$$A^n(t,v) = -L'v + m^n \quad \text{and} \quad G^n(t,v,(a,\eta)) = a\eta[\phi]$$

with $A(t,v) = -L'v + m$, $B(t,v) = I$ and Q given by 2).

Note that $\forall p \geq 0$, $v \in \Phi_{-p}$,

$$\begin{aligned}
\sum_j < -L'v, \phi_j >^2 (1+\lambda_j)^{-2(p+1)} &= \sum_j < v, \phi_j >^2 \lambda_j^2 (1+\lambda_j)^{-2(p+1)} \\
&\leq \sum_j < v, \phi_j >^2 (1+\lambda_j)^{-2p} \\
&= \|v\|_{-p}^2.
\end{aligned}$$

Hence

$$-L'v \in \Phi_{-(p+1)} \quad \text{and} \quad \|-L'v\|_{-(p+1)} \leq \|v\|_{-p}. \tag{8.3.8}$$

It follows from 1) that $\forall n \geq 1$,

$$m^n \in \Phi_{-r_2} \quad \text{and} \quad \|m^n\|_{-r_2} \leq \sqrt{c}. \tag{8.3.9}$$

Therefore, $\forall p \geq r_2$, $A^n(t,\cdot) : \Phi_{-p} \to \Phi_{-(p+1)}$ is continuous and uniform for n. Since

$$\begin{aligned}
\int_{\mathbf{R}\times\Lambda} \|G^n(t,v,(a,\eta))\|_{-p}^2 \mu^n(dad\eta) &= \sum_j \int_{\mathbf{R}\times\Lambda} a^2 \eta[\phi_j^p]^2 \mu^n(dad\eta) \\
&\leq c \sum_j \|\phi_j^p\|_{r_2}^2 < \infty, \qquad (8.3.10)
\end{aligned}$$

for $p \geq r_1 + r_2$, $G^n(t,v,\cdot) \in L^2(\mathbf{R}\times\Lambda, \mu^n(dad\eta); \Phi_{-p})$ and is clearly continuous in $v \in \Phi_{-p}$ uniformly in n (as $G^n(t,v,(a,\eta))$ does not depend on v). Hence (A^n, G^n, μ^n) satisfies (I1) uniformly in n with $p_0 = r_1 + r_2$ and $q = p+1$.

$\forall \phi \in \Phi$, $v, v_1, v_2 \in \Phi_{-p}$, we have

$$\begin{aligned}
2A^n(t,\phi)[\theta_p \phi] &= 2(-L'\phi + m^n) \left[\sum_j < \phi, \phi_j^{-p} >_{-p} \phi_j^p \right] \\
&= -2 \sum_j \lambda_j < \phi, \phi_j^{-p} >_{-p}^2 + \|m^n\|_{-p} \|\phi\|_{-p} \\
&\leq \sqrt{c} \|\phi\|_{-p}, \qquad (8.3.11)
\end{aligned}$$

$$\|A^n(t,v)\|_{-q}^2 \leq 2\|v\|_{-p}^2 + \sqrt{c}\|\phi\|_{-p} \leq (3+c)(1+\|v\|_{-p}^2), \tag{8.3.12}$$

and

$$\begin{aligned}
& 2 < A^n(t,v_1) - A^n(t,v_2), v_1 - v_2 >_{-q} \\
& + \int_{\mathbf{R}\times\Lambda} \|G^n(t,v_1,(a,\eta)) - G^n(t,v_2,(a,\eta))\|_{-q}^2 \mu^n(dad\eta) \\
= \ & 2 < -L'(v_1 - v_2), v_1 - v_2 >_{-q} \leq 0. \qquad (8.3.13)
\end{aligned}$$

Assumptions (I2), (I3) and (M) then follow from (8.3.10)-(8.3.13). Therefore (DA1) and (DA3) hold.

Since

$$< m - m^n, \phi_j >^2 (1+\lambda_j)^{-2q} \leq 4c(1+\lambda_j)^{-2(q-r_2)}$$

is summable for $q > p \geq p_0$, by the dominated convergence theorem, we have

$$\|A^n(t,v) - A(t,v)\|_{-q}^2 = \sum_j | < m - m^n, \phi_j > |^2 (1+\lambda_j)^{-2q} \to 0.$$

Further,

$$\mu^n((a,\eta) : |a\eta[\phi]| > \epsilon) \leq \frac{1}{\epsilon^3} \int_{\mathbf{R}\times\Gamma} |a\eta[\phi]|^3 \mu^n(dad\eta) \to 0,$$

$$\begin{aligned}
& \int_{\mathbf{R}\times\Lambda} G^n(t,v_1,(a,\eta))[\phi] G^n(t,v_2,(a,\eta))[\phi] \mu^n(dad\eta) \\
= \ & Q^n(\phi,\phi) \to Q(\phi,\phi)
\end{aligned}$$

and

$$\begin{aligned}
\sup_n \int_{\mathbf{R}\times\Lambda} |a\eta[\phi]|^2 1_{|a\eta[\phi]|>M} \mu^n(dad\eta) & \leq \frac{1}{M} \sup_n \int_{\mathbf{R}\times\Lambda} |a\eta[\phi]|^3 \mu^n(dad\eta) \\
& \to 0 \quad \text{as } M \to \infty.
\end{aligned}$$

This proves (DA3). ∎

8.4 Applications of diffusion approximation

In this section, we give some applications of diffusion approximation.

Example 8.4.1 *Reversal potential model*

In Chapter 4, we introduced the reversal potential model for a point neuron (i.e. the neuron can be regarded as a single point). For the convenience of the reader, we describe the reversal potential model for spatially extended neurons briefly.

Let $L = -\Delta + \alpha I$ be an operator on H, where

$$H = \left\{ h \in L^2(\mathcal{X}, dx) : \frac{\partial h}{\partial x_i}\bigg|_{x_i=0} = \frac{\partial h}{\partial x_i}\bigg|_{x_i=\pi} = 0 \right\} \qquad i = 1, \cdots, d \quad (8.4.1)$$

and $\mathcal{X} = [0, \pi]^d$ represents the neuron membrane, α is the leakage rate. Then L is a nonnegative-definite and self-adjoint operator on the separable Hilbert space H with discrete spectrum. Let $\lambda_{j_1 \cdots j_d}$, $\phi_{j_1 \cdots j_d}$, $j_1 \cdots j_d \geq 0$ be the eigenvalues and eigenvectors respectively of L, i.e.

$$\lambda_{j_1 \cdots j_d} = j_1^2 + \cdots + j_d^2 + \alpha \qquad \phi_{j_1 \cdots j_d}(x) = \phi_{j_1}(x_1) \cdots \phi_{j_d}(x_d)$$

and

$$\phi_0(x_k) = \sqrt{\frac{1}{\pi}} \qquad \phi_{j_k}(x_k) = \sqrt{\frac{2}{\pi}} \cos(j_k x_k) \ \ j_k \geq 1.$$

For $r_1 > \frac{d}{4}$, it is easy to show that

$$\sum_{j_1 \cdots j_d} (1 + j_1^2 + \cdots + j_d^2)^{-2r_1} < \infty. \qquad (8.4.2)$$

For $r \in \mathbf{R}$ and $h \in H$, let

$$||h||_r^2 \equiv \sum_{j_1 \cdots j_d} < h, \phi_{j_1 \cdots j_d} >^2 (1 + j_1^2 + \cdots + j_d^2)^{2r} \qquad (8.4.3)$$

and

$$\Phi \equiv \{ h \in H : ||h||_r < \infty, \forall r \in R \} \qquad (8.4.4)$$

where $< \cdot, \cdot >$ is the inner product on H. For each r, let H_r be the completion of Φ with respect to the norm $|| \cdot ||_r$. Let Φ' be the union of all H_r, $r \in \mathbf{R}$. Note that $H_0 = H$ and $< \cdot, \cdot >_0 = < \cdot, \cdot >$. Then Φ is a countably Hilbertian nuclear space and Φ' its dual space.

Suppose that there are excitatory (resp. inhibitory) ions with equilibrium potential $\eta_e \in \Phi'$ (resp. $\eta_i \in \Phi'$) arriving according to Poisson streams N_e (resp. N_i) with random magnitudes $A_e^k \geq 0$, $k = 1, 2, \cdots$ with common distribution F_e on $[0, \infty)$ (resp. $A_i^k \leq 0$, $k = 1, 2, \cdots$ with common distribution F_i on $(-\infty, 0]$). Let N_e and N_i be independent Poisson processes with parameters of f_e and f_i respectively. The random variables A_e^k, A_i^k, N_e and N_i are all taken to be mutually independent. Let $\{\tau_k\}$ and $\{\tau_k'\}$ be the jump instants of the processes N_e and N_i respectively.

Then the voltage potential ξ of the neuron can be regarded as a Φ'-valued process and characterized by the following reversal potential model:

$$\xi_t = \xi_0 - \int_0^t L'\xi_s ds + \sum_{k=1}^{N_e(t)} (\eta_e - \xi_{\tau_k -})A_e^k + \sum_{k=1}^{N_i(t)} (\xi_{\tau'_k -} - \eta_i)A_i^k. \qquad (8.4.5)$$

Let $U \equiv \Phi' \times \mathbf{R}$ and

$$N(\Lambda \times B \times [0,t]) \equiv \sum_{k=1}^{N_e(t)} 1_B(A_e^k)1_\Lambda(\eta_e) + \sum_{k=1}^{N_i(t)} 1_B(A_i^k)1_\Lambda(\eta_i) \qquad (8.4.6)$$

for any $t \geq 0$, $B \in \mathcal{B}(\mathbf{R})$ and $\Lambda \in \mathcal{B}(\Phi')$. Then N is a Poisson random measure on $\Phi' \times \mathbf{R} \times \mathbf{R}_+$ with characteristic measure

$$\mu(\Lambda \times B) = f_e 1_\Lambda(\eta_e)F_e(B) + f_i 1_\Lambda(\eta_i)F_i(B) \qquad (8.4.7)$$

for any $\Lambda \in \mathcal{B}(\Phi')$ and $B \in \mathcal{B}(\mathbf{R})$. (8.4.6) is then rewritten as

$$\xi_t = \xi_0 - \int_0^t L'\xi_s ds + \int_0^t \int_{\Phi'} \int_{\mathbf{R}} f(\xi_{s-}, \eta, a)N(d\eta da ds) \qquad (8.4.8)$$

where

$$f(v, \eta, a) = \begin{cases} (\eta - v)a & \text{if } a \geq 0 \\ (v - \eta)a & \text{if } a < 0, \end{cases} \qquad (8.4.9)$$

for $v \in \Phi'$, $\eta \in \Phi'$, $a \in \mathbf{R}$.

Now we consider a sequence of SDE's on Φ' of the form (8.4.8):

$$\xi_t^n = \xi_0^n - \int_0^t L'_n \xi_s^n ds + \int_0^t \int_{\Phi'} \int_{\mathbf{R}} f(\xi_{s-}^n, \eta, a)N^n(d\eta da ds) \qquad (8.4.10)$$

where $L_n = -\Delta + \alpha^n I$, $\{\alpha^n\}$ is a sequence of real numbers and $N^n(d\eta da ds)$ is a sequence of Poisson random measures on $\Phi' \times \mathbf{R} \times [0, \infty)$ given by (8.4.6) with f_e, f_i, F_e and F_i replaced by f_e^n, f_i^n, F_e^n and F_i^n respectively. The characteristic measures μ^n are given by (8.4.7) with f_e, f_i, F_e and F_i replaced by f_e^n, f_i^n, F_e^n and F_i^n respectively.

To derive a diffusion approximation for (8.4.10), we make the following

Assumptions R:
(R1) $\alpha^n + f_e^n a_e^n - f_i^n a_i^n \to \alpha$ and $f_e^n b_e^n + f_i^n b_i^n \to \beta^2$ in $\mathbf{R}$ where $a_e^n = \int_0^\infty a F_e^n(da)$, $b_e^n = \int_0^\infty a^2 F_e^n(da)$ and a_i^n and b_i^n are defined similarly.
(R2) For any $\epsilon > 0$, $f_e^n F_e^n\{a : a > \epsilon\} + f_i^n F_i^n\{a : a < -\epsilon\} \to 0$.
(R3) There exists a sequence $\{c^n\}$ such that $c^n f_e^n a_e^n \to \gamma_e$ and $c^n f_i^n a_i^n \to \gamma_i$.
(R4) $\sup_n \left(f_e^n \int_M^\infty a^2 F_e^n(da) + f_i^n \int_{-\infty}^{-M} a^2 F_i^n(da) \right) \to 0$ as $M \to \infty$.

For any ϕ and ψ in Φ, let $Q(\phi,\psi) =< \phi,\psi >$. Let $A : \Phi' \to \Phi'$ and $B : \Phi' \to \Phi'$ be given by

$$A(v) = -L'v + \gamma_e\eta_e - \gamma_i\eta_i \quad \text{and} \quad B(v)'\phi = \beta v[\phi]\phi_{0\cdots 0}. \tag{8.4.11}$$

Let $V^n_t = c^n u^n_t$. We have the following diffusion approximation result for $\{V^n\}$.

Theorem 8.4.1 *Suppose that we have r_0 such that $\sup_n E\|V^n_0\|^2_{-r_0} < \infty$ and $\{V^n_0\}$ converges to a Φ'-valued random variable V_0 in distribution. Then V^n converges in distribution to the unique solution of the diffusion equation on Φ':*

$$V_t = V_0 + \int_0^t A(V_s)ds + \int_0^t B(V_s)dW_s \tag{8.4.12}$$

where W is a Φ'-valued Wiener process with covariance Q.

Proof: Note that

$$V^n_t = V^n_0 + \int_0^t A^n(V^n_s)ds + \int_0^t \int_{\Phi'} \int_0^\infty G^n(V^n_{s-},\eta,a)\tilde{N}^n(d\eta da ds) \tag{8.4.13}$$

where

$$A^n(v) = -L'_n v + a^n_e f^n_e(c^n\eta_e - v) + a^n_i f^n_i(v - c^n\eta_i) \tag{8.4.14}$$

and

$$G^n(v,\eta,a) = \begin{cases} (c^n\eta - v)a & \text{if } a \geq 0 \\ (v - c^n\eta)a & \text{if } a < 0, \end{cases}$$

for $v \in \Phi'$, $\eta \in \Phi'$ and $a \in \mathbf{R}$.

First we show that $c^n \to 0$. In fact,

$$\begin{aligned}
& f^n_e b^n_e + f^n_i b^n_i \\
= \;& f^n_e \left[\int_0^\epsilon a^2 F^n_e(da) + \int_\epsilon^M a^2 F^n_e(da) + \int_M^\infty a^2 F^n_e(da)\right] \\
& + f^n_i \left[\int_{-\epsilon}^0 a^2 F^n_i(da) + \int_{-M}^{-\epsilon} a^2 F^n_i(da) + \int_{-\infty}^{-M} a^2 F^n_i(da)\right] \\
\leq \;& \epsilon\left(f^n_e a^n_e - f^n_i a^n_i\right) + M^2\left(f^n_e F^n_e\{a : a > \epsilon\} + f^n_i F^n_i\{a : a < -\epsilon\}\right) \\
& + \sup_n \left(f^n_e \int_M^\infty a^2 F^n_e(da) + f^n_i \int_{-\infty}^{-M} a^2 F^n_i(da)\right).
\end{aligned}$$

Taking $n \to \infty$ and then $M \to \infty$, we have

$$\beta^2 \leq \epsilon \liminf_{n\to\infty}\left(f^n_e a^n_e - f^n_i a^n_i\right). \tag{8.4.15}$$

Letting $\epsilon \to 0$, then

$$\liminf_{n\to\infty} (f_e^n a_e^n - f_i^n a_i^n) \geq \lim_{\epsilon\to 0} \frac{\beta^2}{\epsilon} = \infty. \tag{8.4.16}$$

It then follows from (R3) that $c_n \to 0$.

Now we show that (A^n, G^n, μ^n) satisfies Assumptions (DA1)-(DA3). It follows from similar arguments as those leading to (8.3.8) that $\forall v \in \Phi_{-p}$, we have

$$-L_n' v \in \Phi_{-(p+1)} \quad \text{and} \quad \|-L_n' v\|_{-(p+1)}^2 \leq 2(1+|\alpha^n|^2)\|v\|_{-p}^2. \tag{8.4.17}$$

Let $q = p+1$ and p_0 be such that $\eta_e,\ \eta_i \in \Phi_{-p_0}$. Then for $p \geq p_0$, $v \in \Phi_{-p}$, we have $A^n(v) \in \Phi_{-q}$ and

$$\|A^n(v)\|_{-q} \leq K(1+\|v\|_{-p})$$

by choosing K such that

$$K \geq 2 + 2\sup_n |\alpha^n + a_e^n f_e^n - a_i^n f_i^n| \tag{8.4.18}$$

and

$$K \geq \sup_n \|c^n a_e^n f_e^n \eta_e - c^n a_i^n f_i^n \eta_i\|_{-q}. \tag{8.4.19}$$

Similarly

$$\|A^n(v_1) - A^n(v_2)\|_{-q} \leq K\|v_1 - v_2\|_{-p}, \ \ \forall v_1,\ v_2 \in \Phi_{-p}.$$

Note that

$$\begin{aligned}
&\int_{\Phi'}\int_{\mathbf{R}} \|G^n(v_1,\eta,a) - G^n(v_2,\eta,a)\|_{-p}^2 \mu^n(d\eta da)\\
=\ & f_e^n \int_0^\infty \|a(c^n\eta_e - v_1) - a(c^n\eta_e - v_2)\|_{-p}^2 F_e^n(da)\\
&+ f_i^n \int_{-\infty}^0 \|a(v_1 - c^n\eta_i) - a(v_2 - c^n\eta_i)\|_{-p}^2 F_i^n(da)\\
=\ & (f_e^n b_e^n + f_i^n b_i^n)\,\|v_1 - v_2\|_{-p}^2\\
\leq\ & K\|v_1 - v_2\|_{-p}^2
\end{aligned}$$

by choosing K such that

$$K \geq \sup_n (f_e^n b_e^n + f_i^n b_i^n)\,. \tag{8.4.20}$$

Similarly, we have

$$\begin{aligned}
\int_{\Phi'}\int_{\mathbf{R}} \|G^n(v,\eta,a)\|_{-p}^2 \mu^n(d\eta da) &= f_e^n b_e^n \|c^n\eta_e - v\|_{-p}^2 + f_i^n b_i^n \|v - c^n \eta_i\|_{-p}^2\\
&\leq K(1+\|v\|_{-p}^2)
\end{aligned}$$

by choosing K such that

$$K \geq \sup_n (2f_e^n b_e^n + 2f_i^n b_i^n) \tag{8.4.21}$$

and

$$K \geq \sup_n \left(2|c^n| f_e^n b_e^n \|\eta_e\|_{-p}^2 + 2|c^n| f_i^n b_i^n \|\eta_i\|_{-p}^2\right). \tag{8.4.22}$$

For any $\phi \in \Phi$, we have

$$\begin{aligned} L'\phi[\theta_p \phi] &= L'\phi[\sum_j < \phi, \phi_j^{-p} >_{-p} \phi_j^p] \\ &= -\sum_j < \phi, \phi_j^{-p} >_{-p} \lambda_j < \phi, \phi_j^{-p} >_{-p} \leq 0. \end{aligned}$$

Taking K to be the largest one among the right hand sides of (8.4.18)-(8.4.22), we see that (DA1) holds. (DA3) can be verified similarly.

Finally, we verify (DA2). It is clear that $A^n(v) \to A(v)$. Further,

$$\begin{aligned} & \mu^n\{(\eta, a) : |G^n(v, \eta, a)[\phi]| > \epsilon\} \\ \leq\ & f_e^n F_e^n \left\{a : a > \frac{\epsilon}{|c^n \eta_e[\phi] - v[\phi]|}\right\} + f_i^n F_i^n \left\{a : a < -\frac{\epsilon}{|v[\phi] - c^n \eta_i[\phi]|}\right\} \\ \to\ & 0, \end{aligned}$$

$$\begin{aligned} & \int_{\Phi'} \int_{\mathbf{R}} G^n(v_1, \eta, a)[\phi] G^n(v_2, \eta, a)[\phi] \mu^n(d\eta da) \\ =\ & f_e^n b_e^n (c^n \eta_e[\phi] - v_1[\phi])(c^n \eta_e[\phi] - v_2[\phi]) \\ & + f_i^n b_i^n (v_1[\phi] - c^n \eta_i[\phi])(v_2[\phi] - c^n \eta_i[\phi]) \\ \to\ & \beta^2 v_1[\phi] v_2[\phi] \end{aligned}$$

and

$$\begin{aligned} Q(B(v_1)'\phi, B(v_2)'\phi) &= < \beta v_1[\phi]\phi_0, \beta v_2[\phi]\phi_0 >_0 \\ &= \beta^2 v_1[\phi] v_2[\phi]. \end{aligned}$$

(8.1.5) follows from (R4) easily. This proves (RD2) and hence, by Theorem 8.3.3, we complete the proof. ∎

Next, we show that the limiting process is in fact in H_0 and can thus be regarded as the unique solution of a stochastic partial differential equation.

Theorem 8.4.2 *Suppose that η_e, $\eta_i \in H_0$ and V_0 is an H_0-valued random variable such that $E\|V_0\|_0^2 < \infty$, then $V \in C([0,T], H_0)$. Let $V(t, \cdot) = V_t$, then*

$$V(t,x) = V(0,x) - \int_0^t (L'V(s,x) + \gamma_e \eta_e(x) - \gamma_i \eta_i) ds + \int_0^t \beta V(s,x) dB_s \tag{8.4.23}$$

where B is a one-dimensional Brownian motion which is independent of the initial random field $\{V(0,x) : x \in \mathcal{X}\}$.

Proof: It follows from (8.4.11) and (8.4.12) that, for $\phi \in \Phi$ such that $L\phi = \lambda\phi$,

$$\begin{aligned} V_t[\phi] &= V_0[\phi] + \int_0^t A(V_s)[\phi]ds + \int_0^t < B(V_s)'\phi, dW_s >_0 \qquad (8.4.24) \\ &= V_0[\phi] - \int_0^t (\lambda V_s[\phi] - \gamma_e\eta_e[\phi] - \gamma_i\eta_i[\phi])ds + \int_0^t \beta V_s[\phi]dW_s[\phi_0]. \end{aligned}$$

Making use of Itô's formula, we have

$$\begin{aligned} V_t[\phi]^2 &= V_0[\phi]^2 - \int_0^t 2V_s[\phi](\lambda V_s[\phi] - \gamma_e\eta_e[\phi] - \gamma_i\eta_i[\phi])ds \\ &\quad + \int_0^t 2\beta V_s[\phi]^2 dW_s[\phi_0] + \int_0^t \beta^2(V_s[\phi])^2 ds. \qquad (8.4.25) \end{aligned}$$

From the Burkholder-Davis-Gundy inequality (see Dellacherie and Meyer [7], p285, (90.1)) we have

$$\begin{aligned} f(r) &\equiv E\sup_{0\le t\le r} V_t[\phi]^2 \qquad (8.4.26) \\ &\le EV_0[\phi]^2 + \int_0^r (2|\alpha| + 1 + \beta^2)EV_s[\phi]^2 ds \\ &\quad + (\gamma_e\eta_e[\phi] - \gamma_i\eta_i[\phi])^2 r + 8\beta E\sqrt{\int_0^r V_s[\phi]^4 ds} \\ &\le EV_0[\phi]^2 + (2|\alpha| + 1 + \beta^2)\int_0^r f(s)ds \\ &\quad + (\gamma_e\eta_e[\phi] - \gamma_i\eta_i[\phi])^2 r + 8\beta E\left(\sup_{0\le t\le r}|V_s[\phi]|\sqrt{\int_0^r V_s[\phi]^2 ds}\right) \\ &\le EV_0[\phi]^2 + (2|\alpha| + 1 + \beta^2)\int_0^r f(s)ds + (\gamma_e\eta_e[\phi] - \gamma_i\eta_i[\phi])^2 r \\ &\quad + \frac{1}{2}f(r) + 32\beta^2\int_0^r E(V_s[\phi])^2 ds. \end{aligned}$$

i.e.

$$\begin{aligned} f(r) &\le 2EV_0[\phi]^2 + 2(\gamma_e\eta_e[\phi] - \gamma_i\eta_i[\phi])^2 r \\ &\quad + 2(2|\alpha| + 1 + 33\beta^2)\int_0^r f(s)ds. \qquad (8.4.27) \end{aligned}$$

Gronwall's inequality then yields

$$\begin{aligned} &E\sup_{0\le t\le T} V_t[\phi]^2 \qquad (8.4.28) \\ &\le (2EV_0[\phi]^2 + 2(\gamma_e\eta_e[\phi] - \gamma_i\eta_i[\phi])^2 T)\exp(2(2|\alpha| + 1 + 33\beta^2)T). \end{aligned}$$

Letting $\phi = \phi_{j_1 \cdots j_d}$ and adding, we have

$$E \sum_{j=0}^{\infty} \sup_{0 \le t \le T} V_t[\phi_j]^2 \tag{8.4.29}$$
$$\le (2E\|V_0\|_0^2 + 4(\gamma_e^2\|\eta_e\|_0^2 + \gamma_i^2\|\eta_i\|_0^2 T)\exp(2(2|\alpha| + 1 + 33\beta^2)T).$$

The continuity of $V_t[\phi_j]$ is obvious. It follows from (8.4.29) that $V \in C([0,T], H_0)$. (8.4.23) easily follows upon setting $B_t = W_t[\phi_0]$. ∎

Example 8.4.2 *White noise current injection at a point*

Wan and Tuckwell [58] considered this problem and first used the expression "white noise current injection at a point".

Let $H = L^2([0,\pi], dx)$ and $-L$ be as in the example for the stochastic cable equation (cf. Section 4.2). We shall now introduce a SDE in which the driving Gaussian white noise process is not generated by the Brownian sheet. It will be shown that the resulting equation has a unique H-valued solution. The SDE describes the evolution of the voltage potential of a neuron when it receives random impulses only at a single point, say $x_0 \in [0,\pi]$. As explained in Chapter 4, first consider impulses arriving at x_0 with arrival rate measure of the form

$$\mu^n(A \times B) = \mu_1^n(A) 1_B(x_0), \;\; A \in \mathcal{B}(\mathbf{R}_+), \; B \in \mathcal{B}([0,\pi])$$

where

$$\mu_1^n(A) = \sum_{k=1}^{p} f_e^{k,n} 1_A(a_e^{k,n}) + \sum_{\ell=1}^{q} f_i^{\ell,n} 1_A(-a_i^{\ell,n})$$

and $a_e^{k,n} > 0$ are the magnitudes of the excitatory pulses and $-a_i^{\ell,n} > 0$ are the magnitudes of the inhibitory pulses. $f_e^{k,n}$, $f_i^{\ell,n}$ are the characteristic measures of the Poisson processes $N_e^{k,n}$, $N_i^{\ell,n}$. Let

$$\sigma_n^2 = \sum_{k=1}^{p} f_e^{k,n}(a_e^{k,n})^2 + \sum_{\ell=1}^{q} f_i^{\ell,n}(a_i^{\ell,n})^2,$$

$$\gamma_n = \sum_{k=1}^{p} f_e^{k,n} a_e^{k,n} - \sum_{\ell=1}^{q} f_i^{\ell,n} a_i^{\ell,n}$$

and define

$$Y_t^n[\phi] = \sum_{k=1}^{p} a_e^{k,n} \int_0^t \int_0^\pi \phi(x) \tilde{N}_e^{k,n}(dxds) - \sum_{\ell=1}^{q} a_i^{\ell,n} \int_0^t \int_0^\pi \phi(x) \tilde{N}_i^{k,n}(dxds).$$

Here, $\tilde{N}_e^{k,n}$ and $\tilde{N}_i^{k,n}$ are independent, compensated Poisson random measures with characteristic measures given by $f_e^{k,n}\nu(dx)$ and $f_i^{k,n}\nu(dx)$ with $\nu(B) = 1_B(x_0)$. We have

$$EY_t^n[\phi] = 0 \quad \text{and} \quad EY_t^n[\phi]Y_s^n[\psi] = (t \wedge s)Q^n(\phi, \psi)$$

where

$$Q^n(\phi, \psi) = \sigma_n^2 \phi(x_0)\psi(x_0).$$

For each n, the evolution of the voltage potential ξ^n is described by the following SDE driven by the Poisson martingale Y^n:

$$d\xi_t^n = \{-L'\xi_t^n + \gamma^n \phi(x_0)\}dt + dY_t^n, \quad t > 0. \tag{8.4.30}$$

We take the initial value ξ_0^n to be zero for all n. In order to derive the limiting behavior of ξ^n, impose the following conditions on the parameters:

(i)

$$\lim_{n\to\infty} \max_{k,\ell}\{a_e^{k,n}, a_i^{\ell,n}\} = 0;$$

(ii)

$$\lim_{n\to\infty} \sigma_n^2 = \sigma^2, \quad 0 < \sigma^2 < \infty;$$

(iii)

$$\lim_{n\to\infty} \gamma_n = \gamma, \quad |\gamma| < \infty.$$

Then

$$\lim_{n\to\infty} Q^n(\phi, \psi) = Q(\phi, \psi) \equiv \sigma^2 \phi(x_0)\psi(x_0)$$

and the convergence to normality applies (cf. Corollary 8.3.1). The processes ξ^n converge weakly to ξ which is the unique solution of

$$d\xi_t = \{-L'\xi_t + \gamma\phi(x_0)\}dt + dW_t, \quad \xi_0 = 0, \tag{8.4.31}$$

where W_t is a Φ'-valued Wiener process with $EW_t[\phi] = 0$ and

$$EW_t[\phi]W_s[\psi] = \sigma^2(t \wedge s)\phi(x_0)\psi(x_0).$$

To simplify the discussion take $\sigma^2 = 1$ and $\gamma = 0$. W_t is Φ'-valued in a degenerate sense for we may take $W_t = Z_t\delta_{x_0}$ where Z_t is a real valued standard Wiener process and δ_{x_0} is the Dirac measure at x_0. The solution of (8.4.31) can be seen to be given by $\xi_t \in \Phi'$ with

$$\xi_t[\phi] = \sum_{j=0}^{\infty} \xi_t^j < \phi, \phi_j >, \quad \xi_t^j = \xi_t[\phi_j].$$

We now consider the convergence of the series. From (8.4.31), we have

$$d\xi_t^j = -\lambda_j \xi_t^j dt + dW_t^j$$

and

$$\xi_t^j = -\int_0^t \lambda_j \xi_s^j ds + W_t^j$$

or

$$\xi_t^j = \phi_j(x_0) \int_0^t e^{-\lambda_j(t-s)} dZ_s. \tag{8.4.32}$$

For different j, the ξ_t^j are Ornstein-Uhlenbeck processes but they are not independent.

Theorem 8.4.3 *Let ξ be the unique solution of equation (8.4.31). Then $\xi \in C([0,T],H)$ a.s.*

Proof: We divide the proof into three steps.
Step 1: Let B be a real-valued Brownian motion. Then

$$K \equiv E \sup_{0 \le t < \infty} \frac{B_t^2}{(t+1)[\log\log(t+2e)]^2} < \infty. \tag{8.4.33}$$

To show this, let

$$\Theta = \left\{ \theta \in C([0,\infty)) : \theta_0 = 0 \text{ and } \lim_{t\to\infty} \frac{\theta_t^2}{(t+1)[\log\log(t+2e)]^2} = 0 \right\}.$$

Then Θ is a separable Banach space with norm $\|\cdot\|_\Theta$ given by

$$\|\theta\|_\Theta^2 = \sup_{0 \le t < \infty} \frac{\theta_t^2}{(t+1)[\log\log(t+2e)]^2}.$$

It follows from Strassen's law of the iterated logarithm (see Hida [14]) that $B_\cdot \in \Theta$ a.s. and hence, $\{B_t\}$ induces a centered Gaussian measure on $(\Theta, \mathcal{B}(\Theta))$. It follows from Fernique's theorem (see Kuo [35] or Deuschel and Stroock [8]) that there exists $\alpha > 0$ such that $E\exp\left(-\alpha\|B_\cdot\|_\Theta^2\right) < \infty$. As a consequence, (8.4.33) holds.
Step 2. There exists a constant $K_1 > 0$ such that

$$E \sup_{0 \le t \le T} (\xi_t^j)^2 \le K_1 \frac{(\log \lambda_j)^2}{\lambda_j} \quad \forall j \ge 0. \tag{8.4.34}$$

There exists a real-valued Brownian motion $\hat{B}$ such that

$$\int_0^t e^{-\lambda_j t} dZ_s = \hat{B}_{\frac{e^{2\lambda_j t}-1}{2\lambda_j}} \stackrel{\mathcal{L}}{=} \frac{1}{\sqrt{2\lambda_j}} \hat{B}_{e^{2\lambda_j t}-1}.$$

Therefore

$$\begin{aligned}
E \sup_{0\le t\le T} (\xi_t^j)^2 &= \frac{\phi_j(x_0)^2}{2\lambda_j} E \sup_{0\le t\le T} e^{-2\lambda_j t}\left(\hat{B}_{e^{2\lambda_j t}-1}\right)^2 \\
&\le \frac{1}{\pi\lambda_j} E \sup_{0\le t\le e^{2\lambda_j T}-1} \frac{\hat{B}_t^2}{t+1} \\
&\le \frac{1}{\pi\lambda_j}[\log\log(e^{2\lambda_j T} - 1 + 2e)]^2 \\
&\quad E \sup_{0\le t\le e^{2\lambda_j T}-1} \frac{\hat{B}_t^2}{(t+1)[\log\log(t+2e)]^2} \\
&\le K_1 \frac{(\log\lambda_j)^2}{\lambda_j}.
\end{aligned}$$

Step 3: $\xi \in C([0,T], H)$ a.s.

It is clear that $\xi_\cdot^j \in C([0,T], \mathbf{R})$ a.s. $\forall j \ge 0$. As $\sum_j \frac{(\log\lambda_j)^2}{\lambda_j} < \infty$, by the dominated convergence theorem and (8.4.34), it easily follows that $\xi \in C([0,T], H)$ a.s. ∎

Example 8.4.3 *White noise current injection at a point ($d > 1$)*

Let $H = L^2([0,\pi]^d, dx)$ and $L = -\Delta + I$ be a differential operator on H with Neumann boundary condition. We consider the following equation which is similar to (8.4.31) on Φ':

$$d\xi_t = -L'\xi_t dt + dW_t \tag{8.4.35}$$

where $x_0 \in [0,\pi]^d$, W_t is a Φ'-valued Wiener process with covariance $Q(\phi,\psi) \equiv \sigma^2\phi(x_0)\psi(x_0)$ and Φ is the nuclear space constructed by (8.4.3) and (8.4.4).

For simplicity of notation we denote $(j_1,\cdots,j_d)$ by $\vec{j}$. Let $\xi_t^{\vec{j}} = \xi_t[\phi_{\vec{j}}]$ and $W_t = \delta_{x_0} Z_t$ where Z_t is a real-valued Wiener process. Then

$$\xi_t^{\vec{j}} = \phi_{\vec{j}}(x_0)\int_0^t e^{-\lambda_{\vec{j}}(t-s)} dZ_s.$$

Lemma 8.4.1 *i) $\forall t \in [0,T]$, ξ_t is not an H-valued random variable.*
ii) There exists $p > 0$ such that $\xi_\cdot \in C([0,T], \Phi_{-p})$.

Proof: i) For simplicity, assume $x_0 = 0$. If ξ_t is an H-valued random variable, then $E\|\xi_t\|_H^2 < \infty$ since it has a Gaussian distribution. But

$$E\|\xi_t\|_H^2 = \left(\frac{2}{\pi}\right)^d \sum_{\vec{j}} \frac{1 - e^{-2(1+|\vec{j}|^2)t}}{2(1+|\vec{j}|^2)} = \infty.$$

Therefore, ξ_t is not an H-valued random variable.
ii) We have only to verify Conditions (D) and (DM) for

$$A(v) = -L'v, \ B(v) = I \text{ and } Q(\phi, \psi) = \phi(x_0)\psi(x_0). \tag{8.4.36}$$

Similar to (8.4.17), $\forall p \geq 0$, $\exists q = p+1$ such that A is a continuous map from Φ_{-p} to Φ_{-q} and

$$\|A(v)\|_{-q} \leq \|v\|_{-p}, \qquad \forall v \in \Phi_{-p}. \tag{8.4.37}$$

As

$$|\phi_{\vec{j}}(x)| \leq \left(\frac{2}{\pi}\right)^{d/2}, \ \forall x \in [0,\pi]^d$$

we have

$$|\phi(x_0)|^2 \leq \left(\frac{2}{\pi}\right)^d \sum_{\vec{j}} < \phi, \phi_{\vec{j}} >^2 (1+\lambda_{\vec{j}})^{2r_2} \sum_{\vec{j}} (1+\lambda_{\vec{j}})^{-2r_2} \equiv \theta \|\phi\|_{r_2}^2$$

for $r_2 > \frac{d}{4}$. Then for $p > \frac{d}{2}$, the canonical injection from H_Q to Φ_{-p} is Hilbert-Schmidt, i.e. B defined in (8.4.36) is a continuous map from Φ_{-p} to $L_{(2)}(H_Q, \Phi_{-p})$. This proves (D1). The conditions (D2)-(D4) and (DM) can be verified easily. ∎

If we replace the assumption that all the impulses arrive at the point x_0 by the more realistic assumption that they arrive in the vicinity of x_0, then the covariance functional $Q(\phi, \psi)$ of W_t has the form

$$Q(\phi, \psi) = < f_\epsilon, \phi >< f_\epsilon, \psi >$$

where

$$f_\epsilon = \frac{1}{(2\epsilon)^d} 1_{\{y \in [0,\pi]^d : x_0^i - \epsilon \leq y^i \leq x_0^i + \epsilon \ i=1,\cdots,d\}} \in H.$$

Then we have $W_t = Z_t f_\epsilon$ and

$$\xi_t^{\epsilon,\vec{j}} = -\int_0^t \lambda_{\vec{j}} \xi_s^{\epsilon,\vec{j}} ds + f^{\epsilon,\vec{j}} Z_t$$

where $f^{\epsilon,\vec{j}} = f_\epsilon[\phi_{\vec{j}}]$.

Lemma 8.4.2 $\forall \epsilon > 0$, $\xi^\epsilon_\cdot \in C([0,T], H)$.

Proof: We omit the index ϵ for convenience of writing. From Itô's formula, we have

$$\begin{aligned}
(\xi_t^{\vec{j}})^2 &= -2\lambda_{\vec{j}} \int_0^t (\xi_s^{\vec{j}})^2 ds + 2f^{\vec{j}} \int_0^t \xi_s^{\vec{j}} dZ_s + \int_0^t (f^{\vec{j}})^2 ds \\
&\leq (f^{\vec{j}})^2 T + 2f^{\vec{j}} \int_0^t \xi_s^{\vec{j}} dZ_s,
\end{aligned}$$

$\forall 0 \leq t \leq T$. Let $G_{\vec{j}}(r) = E\sup_{0\leq t\leq r}(\xi_t^{\vec{j}})^2$. Then

$$\begin{aligned}
G_{\vec{j}}(r) &\leq (f^{\vec{j}})^2 T + 2|f^{\vec{j}}| E \sup_{0\leq t\leq r}\left|\int_0^t \xi_s^{\vec{j}} dZ_s\right| \\
&\leq (f^{\vec{j}})^2 T + 4|f^{\vec{j}}|\sqrt{E\sup_{0\leq t\leq r}\int_0^t (\xi_s^{\vec{j}})^2 ds} \\
&\leq (f^{\vec{j}})^2 T + 4|f^{\vec{j}}|\sqrt{\int_0^r G_{\vec{j}}(s) ds} \\
&\leq (f^{\vec{j}})^2 T + 2(f^{\vec{j}})^2 + 2\int_0^r G_{\vec{j}}(s) ds \\
&\leq 2(T+1)(f^{\vec{j}})^2 + 2\int_0^r G_{\vec{j}}(s) ds.
\end{aligned}$$

By Gronwall's inequality,

$$G_{\vec{j}}(r) \leq 2(T+1)(f^{\vec{j}})^2 e^2. \tag{8.4.38}$$

Hence

$$\begin{aligned}
E\sum_{\vec{j}}\sup_{0\leq t\leq T}(\xi_t^{\vec{j}})^2 &\leq e^2\sum_{\vec{j}} 2(T+1)(f^{\vec{j}})^2 \\
&= 2e^2(T+1)\|f_\epsilon\|_H^2.
\end{aligned}$$

The above inequality, together with the fact that $\xi_t^{\epsilon,\vec{j}}$ is continuous in t for each $\vec{j}$, implies that $\xi_\cdot^\epsilon \in C([0,T],H)$. ∎

Theorem 8.4.4

$$\lim_{\epsilon\to 0} E\sup_{0\leq t\leq T}\|\xi_t^\epsilon - \xi_t\|_{-p}^2 = 0.$$

Proof: As

$$\eta_t^{\epsilon,\vec{j}} \equiv \xi_t^{\epsilon,\vec{j}} - \xi_t^{\vec{j}} = -\lambda_{\vec{j}}\int_0^t \eta_s^{\epsilon,\vec{j}} ds + (f_\epsilon^{\vec{j}} - \phi_{\vec{j}}(x_0))Z_t.$$

It follows from the same arguments as in (8.4.38) that

$$E\sup_{0\leq t\leq T}(\eta_t^{\epsilon,\vec{j}})^2 \leq 2e^2(T+1)(f_\epsilon^{\vec{j}} - \phi_{\vec{j}}(x_0))^2.$$

Then

$$\begin{aligned}
E\sup_{0\leq t\leq T}\|\eta_t^\epsilon\|_{-p}^2 &\leq \sum_{\vec{j}}(1+\lambda_{\vec{j}})^{-2p} E\sup_{0\leq t\leq T}(\eta_t^{\epsilon,\vec{j}})^2 \\
&\leq \sum_{\vec{j}}(1+\lambda_{\vec{j}})^{-2p} 2e^2(T+1)(f_\epsilon^{\vec{j}} - \phi_{\vec{j}}(x_0))^2 \to 0
\end{aligned}$$

by the dominated convergence theorem. ∎

8.5 Examples of nuclear-space-valued SDE's

To justify the theory of stochastic differential equations in nuclear spaces developed in the previous sections of this chapter it is expedient to give concrete examples to show that the occurrence of such stochastic equations is not a pathology but probably as natural as the appearance of generalized functions (or distributions) in functional analysis or the theory of partial differential equations.

Each of the examples in this section relates to some area of application.

Example 8.5.1 *Stochastic fluctuation of a two-dimensional neuron.*

When the neuron is regarded as a thin cylindrical segment, it is usual to model its stochastic behavior by a stochastic cable equation as in Section 4.2. While this is often considered to be a prototype of a spatially extended neuron (see remarks at the end of [26]) it is interesting to consider neuron membranes that are parts of a manifold. For simplicity, take $\mathcal{X}$ to be a square $\{(x,y): \ 0 \le x \le \pi, \ \ 0 \le y \le \pi\}$. The SPDE describing the fluctuation of the voltage potential across this membrane (with insulating edges) is assumed to be of the form

$$\frac{\partial u}{\partial t} = \Delta u - u + \dot{W}_{txy}, \quad t > 0,\ 0 < x < \pi,\ 0 < y < \pi \tag{8.5.1}$$

with Neumann boundary conditions

$$\frac{\partial u}{\partial x}(t,0,y) = \frac{\partial u}{\partial x}(t,\pi,y) = \frac{\partial u}{\partial y}(t,x,0) = \frac{\partial u}{\partial y}(t,x,\pi) = 0.$$

Since the initial value has no effect on the nature of the solution we shall take it to be zero. The generator L has eigenvalues $\lambda_{jk} = 1 + j^2 + k^2$, $(j,k = 0,1,\cdots)$ with eigenfunctions $\phi_{jk}(x,y) \equiv \phi_j(x)\phi_k(y)$ where $\phi_j(x) = \frac{1}{\sqrt{\pi}}$ for $j = 0$ and $\sqrt{\frac{2}{\pi}}\cos jx$ for $j \ge 1$. The Green function

$$G(t;x,y,x',y') = \sum_{jk} e^{-\lambda_{jk}t}\phi_{jk}(x,y)\phi_{jk}(x',y'), \quad t > 0.$$

If a random field solution of (8.5.1) exists it is easy to see that it is given by

$$\begin{aligned} u(t,x,y) &= \int_0^t \int_0^\pi \int_0^\pi G(t-s;x,y,x',y')W(dx'dy'ds) \\ &= \sum_{jk} A_{jk}(t)\phi_j(x)\phi_k(y) \end{aligned} \tag{8.5.2}$$

where

$$A_{jk} = \int_0^t e^{-\lambda_{jk}(t-s)} W_{jk}(ds)$$

and

$$W_{jk}(ds) = \int_0^\pi \int_0^\pi \phi_j(x')\phi_k(y')W(dx'dy'ds).$$

The W_{jk} are independent standard Brownian motions and hence A_{jk} are independent, centered, Gaussian (Ornstein-Uhlenbeck) processes. Hence the formal series (8.5.2) is almost surely convergent iff

$$\sum_{jk} EA_{jk}(t)^2\phi_j(x)^2\phi_k(y)^2$$

converges, i.e. iff

$$\sum_{jk} \frac{1-e^{-2\lambda_{jk}t}}{2\lambda_{jk}}\phi_j(x)^2\phi_k(y)^2 < \infty.$$

In particular, for $x = y = 0$, we must have

$$\sum_{jk} \frac{1-e^{-2\lambda_{jk}t}}{2\lambda_{jk}} < \infty.$$

But since $\lambda_{jk} = 1 + j^2 + k^2$, for $t > 0$,

$$\sum_{jk} \frac{1-e^{-2\lambda_{jk}t}}{2\lambda_{jk}} \geq \frac{1}{2}(1-e^{-2t})\sum_{jk}\frac{1}{1+j^2+k^2} = \infty.$$

Hence the formal series cannot represent the solution and the SPDE does not have a random field solution. The above example has been discussed by J. Walsh [57].

Let Φ be the nuclear space given by (8.4.4). The SPDE (8.5.1) can be considered as a SDE for $u_t \equiv u(t,\cdot,\cdot)$ in the conuclear space Φ'. In fact, it can shown that $u_t \in C([0,\infty), \Phi_{-p})$ a.s. for $p > \frac{1}{2}$.

Example 8.5.2 *Interacting diffusions*

We briefly describe here the fluctuation limit of interacting particles. It is assumed that the motion of the latter is given by the n-particle diffusion system

$$\begin{aligned} Y_k^{(n)}(t) &= \gamma_k + \frac{1}{n}\sum_{j=1}^n \int_0^t a\left(Y_k^{(n)}(s), Y_j^{(n)}(s)\right) dW_s^k &\quad (8.5.3)\\ &+\frac{1}{n}\sum_{j=1}^n \int_0^t b\left(Y_k^{(n)}(s), Y_j^{(n)}(s)\right) ds, \quad k = 1,\cdots,n &\quad (8.5.4)\end{aligned}$$

where (γ_k, W^k) are independent copies of (γ, W) where $W = (W_t)$ $(t \geq 0)$ is a real-valued Brownian motion and γ is a random variable independent of

W and satisfying the condition $E\left(e^{c_0\Upsilon^2}\right) < \infty$ for some $c_0 > 0$. The coefficient functions $a(x,y)$, $b(x,y) \in C_b^\infty$, that is, bounded and with bounded derivatives of all orders. Consider the measure-valued (so called occupation) process

$$U^{(n)}(t) = \frac{1}{n}\sum_{j=1}^{n} \delta_{Y_j^{(n)}(t)}, \quad t \geq 0$$

where δ_x is the Dirac measure at x. It has been shown by McKean [39] that for each t, $U^{(n)}(t) \to U(t)$ in probability, where $U(dx,t)$ is the probability distribution of Z_t, the latter being the solution of the real-valued SDE

$$dZ_t = \alpha(Z_t,t)dW_t + \beta(Z_t,t)dt,$$

$$\alpha(x,t) = \int_{-\infty}^{\infty} a(x,y)U(dy,t),$$

$$\beta(x,t) = \int_{-\infty}^{\infty} b(x,y)U(dy,t).$$

It has also been shown by McKean [39] that $U(dx,t)$ has a density $u(x,t)$ and that $\alpha(x,t)$, $\beta(x,t)$ and $u(x,t)$ are C^∞-functions in x and t.

The processes of interest are the measure-valued processes

$$S_n(t) \equiv n^{\frac{1}{2}}\{U^{(n)}(t) - U(t)\}. \tag{8.5.5}$$

In order to study the limit of the sequence $\{S_n(t)\}$ we need to introduce the following nuclear space and its dual. Let

$$\psi(x) = \int_{-\infty}^{\infty} \rho(x-z)dz$$

where ρ is the mollifier

$$\rho(x) = \begin{cases} c\exp\frac{1}{1-|x|^2} & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

and c is a constant such that $\int_{-\infty}^{\infty}\rho(x)dx = 1$. Introduce the test function space Φ which is a modification of the Schwartz space $\mathcal{S}$ of rapidly decreasing real-valued functions: A function $\phi \in \Phi$ if and only if $\psi\phi \in \mathcal{S}$. The topology of Φ is defined by the sequence of Hilbertian norms $\|\phi\|_n \equiv \|_{n,\mathcal{S}}$ where

$$\|f\|_{n,\mathcal{S}} = \sum_{k=0}^{n}\int_{-\infty}^{\infty}(1+x^2)^{2n}|D^k f(x)|^2 dx \quad (n \geq 0).$$

Then Φ and its dual Φ' are nuclear spaces.

Hitsuda and Mitoma [15] have shown that $S_n(t)$ converges weakly to a nuclear space valued stochastic process (i.e., a generalized process) $\{\xi_t, t \geq 0\}$ which is the unique solution of the SDE

$$\begin{aligned} d\xi_t &= [A'(t)\xi_t + B'(t)\xi_t]dt + dM_t \\ \xi_0 &= \eta \end{aligned} \tag{8.5.6}$$

where $A(t) : \Phi \to \Phi$ is given by

$$(A(t)\phi)(x) = \frac{1}{2}\alpha(x,t)^2\phi''(x) + \beta(x,t)\phi'(x) \tag{8.5.7}$$

and $B(t) : \Phi \to \Phi$ is given by

$$\begin{aligned} (B(t)\phi)(x) &= \int_{-\infty}^{\infty} b(y,x)\phi'(y)u(y,t)dy \\ &\quad + \int_{-\infty}^{\infty} \alpha(y,t)a(x,y)\phi''(y)u(y,t)dy \end{aligned} \tag{8.5.8}$$

$M = (M_t)$, $M_0 = 0$ is a zero mean, Φ'-valued, continuous Gaussian martingale with covariance functional (ϕ_1, $\phi_2 \in \Phi$)

$$EM_t[\phi_1]M_t[\phi_2] = \int_0^{t\wedge s}\int_{-\infty}^{\infty} \phi_1'(x)\phi_2'(x)\alpha(x,r)^2u(dx,r)dr. \tag{8.5.9}$$

The uniqueness of solution of (8.5.6) was shown by Mitoma [42] and, later, independently by Kallianpur and Perez-Abreu [26]. These authors also showed that A(t) generates a two-parameter evolution semigroup (or evolution system) on Φ.

Example 8.5.3 *Asymptotic behavior of a system of free Brownian particles*

An early example of a SDE governing a nuclear space valued process is due to K. Itô [20].

$B_k(t)$, $k = 1, \cdots, n$ are independent Brownian motions with common initial distribution given by a density μ. For any Borel set A, let

$$N_n(t,A) \equiv \#\{k \leq n : B_k(t) \in A\}$$

and

$$X_n(t,A) \equiv n^{-\frac{1}{2}}\{N_n(t,A) - EN_n(t,A)\}.$$

Then $X_n(t,\cdot)$ is a signed measure valued process. For ϕ belonging to a test function space Φ to be suitably chosen, define

$$X_n(t,\phi) \equiv \int_{-\infty}^{\infty} \phi(x)X_n(t,dx).$$

Itô showed that $X_n(t,\cdot)$ regarded as Φ'-valued processes converges weakly to a Φ'-valued process $\xi(t)$ which satisfies a SDE which we shall here derive as a special case of Example 8.5.2. Using the notation of that example, take $b(x,y) \equiv 0$ and $a(x,y) \equiv 1$ in (8.5.3). Then for $n \geq 1$,

$$\begin{aligned} Y_k^{(n)}(t) &= \gamma_k + W_t^k, \quad t \geq 0, \\ &\equiv Y_k(t) \qquad \text{say.} \end{aligned}$$

Let γ_k have the common Gaussian distribution with density μ. Then (γ, W) is replaced by Y and the Y_k are independent copies of a Brownian motion with initial density μ. The condition of the previous example, namely, $E\left(e^{c_0\gamma^2}\right) < \infty$ for some $c_0 > 0$ is obviously satisfied. We have

$$U^{(n)}(A,t) = \frac{1}{n}\sum_{j=1}^{n} \delta_{Y_j(t)}(A) = \frac{1}{n}N_n(t,A).$$

Also $U(A,t)$ has the density $u(x,t) = \mu * g_t(x)$ where $*$ denotes convolution and $g_t(x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$. Let Φ be the nuclear space of Example 8.5.2. From (8.5.5), we see that

$$\begin{aligned} &\int_{-\infty}^{\infty} \phi(x)S_n(dx,t) \\ &= X_n(t,\phi) + n^{\frac{1}{2}}\left\{\frac{1}{n}\sum_{j=1}^{n} E\phi(Y_j(t)) - \int_{-\infty}^{\infty} \phi(x)U(dx,t)\right\}. \end{aligned}$$

The quantity in curly brackets on the right hand side vanishes since

$$E\phi(Y_j(t)) = \int_{-\infty}^{\infty} \phi(x)U(dx,t).$$

From Example 8.5.2 it follows that $X_n(t)$ converges to $\xi(t)$ which is a solution of (8.5.6). It remains to identify $A'(t)$, $B'(t)$ and the martingale M_t.

Noting that $a(x,y) \equiv 1$ and $b(x,y) \equiv 0$, we have

$$(A(t)\phi)(x) = \frac{1}{2}\alpha(x,t)^2\phi''(x),$$

so that for $F \in \Phi'$,

$$(A'(t)F)[\phi] = F[A(t)\phi] = \frac{1}{2}F[\phi''] = \frac{1}{2}D^2F,$$

where D is differentiation in Φ'.

The Gaussian martingale M_t is centered and has the covariance

$$\begin{aligned} EM_t[\phi]M_s[\psi] &= \int_0^{t\wedge s}\int_{-\infty}^{\infty}\phi'(x)\psi'(x)u(dx,r)dr \\ &= \int_0^{t\wedge s} E\phi'(Y_n)\psi'(Y_n)dr. \end{aligned} \tag{8.5.10}$$

From (8.5.8),

$$(B(t)\phi)(x) = \int_{-\infty}^{\infty}\phi'(y)u(y,t)dy = a(t), \text{ say}$$

where $a(t)$ is a scalar independent of x (though depending on ϕ). Hence $B(t)\phi = a(t)1$ (the function $1 \in \Phi$)

$$B'(t)\xi_t[\phi] = \xi_t[B(t)\phi] = a(t)\xi_t[1].$$

From the general formula

$$E\xi_t[\phi]^2 = E\{\phi(Y_t) - E\phi(Y_t)\}^2$$

we have $E\xi_t[1]^2 = 0$ and so $\xi_t[1] = 0$ a.s. $\forall t$. It follows that, $\forall\phi \in \Phi$, $B'(t)\xi_t[\phi] = 0$, and therefore, $B'(t)\xi_t = 0$ a.s. $\forall t$.

In fact, since, from Equation (8.5.6), ξ_t is a.s. continuous Φ'-valued process, we conclude that almost surely, $B'(t)\xi_t = 0$ for all t.

Combining all of the above calculations, we find that Itô's process ξ_t satisfies the following version of (8.5.6):

$$d\xi_t = \frac{1}{2}D^2\xi_t dt + dM_t. \tag{8.5.11}$$

(8.5.11) is precisely the equation derived by Itô. It should be noted that the nuclear space Φ' is different from the space chosen by Itô. Finally, we also obtain the uniqueness of the solution of (8.5.11), a fact inherited from Example 8.5.2.