Chapter 3

Stochastic integrals and martingales in Hilbert and conuclear spaces

From now on we shall be concentrating on two kinds of infinite dimensional spaces: a separable Hilbert space H and a conuclear space Φ', the strong dual of a CHNS Φ. Our aim in the present chapter is twofold: (1) To define martingales taking values in H and Φ' respectively.

While the study of such martingales (particularly H-valued martingales) is of importance in the general theory (see e.g. the books of Métivier [38] and da Prato and Zabczyk [45]), we confine our attention to discussing only those properties which are relevant to the theory of H or Φ'-valued SDE's.

(2) To introduce the definitions and study the basic properties of stochastic integrals taking values in H and Φ'. In contrast to finite dimensional stochastic calculus, we have three interested Brownian motions to consider: cylindrical Brownian motion, H-valued Brownian motion and Φ'-valued Brownian motion. We shall also define stochastic integrals with respect to a Poisson random measure.

We assume throughout that $(\Omega, \mathcal{F}, P)$ is a complete probability space with a right continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$. This chapter is organized as follows: After discussing some general properties of H-valued and Φ'-valued martingales, we introduce H-cylindrical Brownian motion (H-c.B.m), H-valued Brownian motion and Φ'-valued Wiener process. Then the stochastic integrals with respect to these processes will be defined and a representation theorem will be derived for H-valued and Φ'-valued continuous square-integrable martingales. Finally we define the stochastic integral with respect to Poisson random measure and give conditions for a Φ'-valued martingale to be represented as a stochastic integral with respect to a Poisson random measure. The two representation theorems will play important roles in later chapters in the study of stochastic differential equations on infinite dimensional spaces.

3.1 Martingales taking values in Hilbert and conuclear spaces

In this section, we study general $\mathcal{X}$-valued martingales where $\mathcal{X} = H$ or Φ'. In the latter case, we shall denote by $\{\phi_j^p\} \subset \Phi$ a CONS of Φ_p and $\{\phi_j^{-p}\}$ the CONS of Φ_{-p} conjugate to $\{\phi_j^p\}$ for $p \geq 0$. Let θ_p be the isometry from Φ_{-p} to Φ_p such that $\theta_p \phi_j^{-p} = \phi_j^p$, $\forall j \geq 1$.

First, we discuss some basic properties of $\mathcal{X}$-valued random variables.

Definition 3.1.1 *A map $X : \Omega \to \mathcal{X}$ is an $\mathcal{X}$-***valued random variable** *if it is $\mathcal{F}/\mathcal{B}(\mathcal{X})$-measurable, where $\mathcal{B}(\mathcal{X})$ is the Borel field of the topological space $\mathcal{X}$. A family $\{X_t : t \in \mathbf{R}_+\}$ of $\mathcal{X}$-valued random variables is called an* $\mathcal{X}$-**process**.

Theorem 3.1.1 *(a) $\mathcal{B}(\Phi')$ is the σ-field generated by the following class of subsets of Φ':*

$$\{f \in \Phi' : f[\phi] < a\} \quad \phi \in \Phi \text{ and } a \in \mathbf{R}. \tag{3.1.1}$$

(b) $\mathcal{B}(H)$ is the σ-field generated by the following class of subsets of H:

$$\{f \in H :< f, h >_H < a\} \quad h \in H \text{ and } a \in \mathbf{R}.$$

Proof: (a) Let $\tilde{\mathcal{B}}$ be the σ-field generated by the sets given by (3.1.1). As $\{f \in \Phi' : f[\phi] < a\}$ is an open set in the strong topology of Φ' for any $\phi \in \Phi$ and $a \in \mathbf{R}$, we have $\tilde{\mathcal{B}} \subset \mathcal{B}(\Phi')$.

On the other hand, for any bounded subset B of Φ and $\epsilon > 0$,

$$\{q_B(f) \leq \epsilon\} = \cap_{\phi \in B \cap D}\{f \in \Phi' : |f[\phi]| \leq \epsilon\} \in \tilde{\mathcal{B}}$$

where D is a countable dense subset of Φ and q_B is the seminorm on Φ' given by Definition 1.1.7 c). Therefore $\tilde{\mathcal{B}}$ contains the collection of all neighborhoods in Φ'. As Φ' can be represented as a countable union of compact subsets as follows

$$\Phi' = \cup_{p \geq 1}\{\phi \in \Phi' : \|\phi\|_{-p} \leq p\},$$

Φ' is separable. Let C be a countable dense subset of Φ'. Let G be an open subset of Φ'. Then $\forall \xi \in G$ there exists a neighborhood $U_\xi \subset G$ and hence

$$G = \cup_{\xi \in C \cap G} U_\xi \in \tilde{\mathcal{B}}.$$

Therefore $\tilde{\mathcal{B}}$ contains the collection of all open subsets of Φ' and hence $\tilde{\mathcal{B}} = \mathcal{B}(\Phi')$.

(b) can be proved in a similar fashion (note that the σ-compactness of Φ' is needed in the proof of part (a) only for the separability of Φ' and in the present case, we assumed that H is a separable Hilbert space). ∎

Corollary 3.1.1 *(a) A map $X : \Omega \to \Phi'$ is a Φ'-valued random variable iff for any $\phi \in \Phi$, $X[\phi]$ is a real-valued random variable.*
(b) A map $X : \Omega \to H$ is an H-valued random variable iff for any $h \in H$, $< X, h >_H$ is a real-valued random variable.

Proof: We only prove (a). It is clear that if X is a Φ'-valued random variable then $X[\phi]$ is a real-valued random variable, for any $\phi \in \Phi$. On the other hand, let

$$\mathcal{G} = \{C \in \mathcal{B}(\Phi') : X^{-1}(C) \in \mathcal{F}\}.$$

Then $\mathcal{G} \subset \mathcal{B}(\Phi')$ is a σ-field. As the sets of the form (3.1.1) are in $\mathcal{G}$, we have by Theorem 3.1.1 that $\mathcal{B}(\Phi') \subset \mathcal{G}$. Hence, X is a Φ'-valued random variable. ∎

The following regularization theorem is useful for constructing some Φ'-valued random variables.

Theorem 3.1.2 (Itô [19]) *Let $Y : \Phi \to L^2(\Omega, \mathcal{F}, P)$ be a continuous linear map. Then there exists a Φ'-valued random variable $\tilde{Y}$ such that $\forall \phi \in \Phi$,*

$$\tilde{Y}(\omega)[\phi] = Y(\phi)(\omega) \quad a.s.$$

Moreover there is $q > 0$ such that $P(\tilde{Y} \in \Phi_{-q}) = 1$.

Proof: Let $V(\phi) = E(Y(\phi)^2)$, $\forall \phi \in \Phi$. Since V is continuous there exist $r > 0$ and $\delta > 0$ such that if $\|\phi\|_r \leq \delta$ then $V(\phi) < 1$. Hence if $\theta = 1/\delta$ we have

$$V(\phi) \leq \theta^2 \|\phi\|_r^2, \qquad \forall \phi \in \Phi. \tag{3.1.2}$$

Let $q > r$ be such that the canonical injection from Φ_q into Φ_r is Hilbert-Schmidt. Then from (3.1.2)

$$E\left(\sum_{j=1}^{\infty} Y(\phi_j^q)^2\right) \leq \theta^2 \sum_{j=1}^{\infty} \|\phi_j^q\|_r^2 < \infty$$

i.e. if $\Omega_1 = \{\sum_{j=1}^{\infty} (Y(\phi_j^q)(\omega))^2 < \infty\}$ then $P(\Omega_1) = 1$. Define

$$\tilde{Y}(\omega) = \begin{cases} \sum_{j=1}^{\infty} Y(\phi_j^q)(\omega)\phi_j^{-q} & \text{if } \omega \in \Omega_1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\tilde{Y}$ is a Φ'-valued random variable such that $\tilde{Y} \in \Phi_{-q}$ a.s. and

$$\tilde{Y}(\omega)[\phi] = \sum_{j=1}^{\infty} Y(\phi_j^q)(\omega) < \phi, \phi_j^q >_q, \ \forall \phi \in \Phi \ a.s. \tag{3.1.3}$$

Letting $\psi_n \equiv \sum_{j=1}^{n} < \phi, \phi_j^q >_q \phi_j^q$, then $\|\psi_n - \phi\|_r \leq \|\psi_n - \phi\|_q \to 0$ as $n \to \infty$ so that from (3.1.2)

$$E\left(\sum_{j=1}^{n} Y(\phi_j^q) < \phi, \phi_j^q >_q - Y(\phi)\right)^2 \to 0 \quad \text{as } n \to \infty. \tag{3.1.4}$$

Finally, from (3.1.3) and (3.1.4) we have

$$E\left(\tilde{Y}(\omega)[\phi] - Y(\phi)(\omega)\right)^2 = 0$$

i.e.

$$\tilde{Y}[\phi] = Y(\phi) \quad \text{a.s.} \quad \forall \phi \in \Phi.$$

∎

Remark 3.1.1 *A more general regularization result can be found in Ramaswamy [46].*

In the rest of this section, we discuss H-valued and Φ'-valued martingales. Most of the results due to Mitoma [40].

Definition 3.1.2 *(a) A Φ'-valued process $M = \{M_t\}_{t \geq 0}$ is a* **Φ'-martingale** *with respect to $\{\mathcal{F}_t\}$ if for each $\phi \in \Phi$, $M_t[\phi]$ is a martingale with respect to $\{\mathcal{F}_t\}$. It is called a* **Φ'-square-integrable-martingale** *if, in addition,*

$$E(M_t[\phi]^2) < \infty, \qquad \forall \phi \in \Phi, \ t \geq 0.$$

We denote the collection of all Φ'-martingales (resp. Φ'-square-integrable-martingales) by $\mathcal{M}(\Phi')$ (resp. $\mathcal{M}^2(\Phi')$). We also denote

$$\mathcal{M}^{2,c}(\Phi') = \left\{ \begin{array}{ll} M \in \mathcal{M}^2(\Phi') : & M_t[\phi] \ has \ a \ continuous \\ & version \ for \ each \ \phi \in \Phi \end{array} \right\}.$$

(b) An H-valued process $M = \{M_t\}_{t \geq 0}$ is an **H-martingale** *with respect to $\{\mathcal{F}_t\}$ if for each $h \in H$, $< M_t, h >_H$ is a martingale with respect to $\{\mathcal{F}_t\}$. It is called an* **H-square-integrable-martingale** *if, in addition,*

$$E\|M_t\|^2 < \infty, \qquad \forall t \geq 0.$$

We denote the collection of all H-martingales (resp. H-square-integrable-martingales) by $\mathcal{M}(H)$ (resp. $\mathcal{M}^2(H)$) and write

$$\mathcal{M}^{2,c}(H) = \{M \in \mathcal{M}^2(H) : M_t \ has \ a \ continuous \ version\}.$$

Theorem 3.1.3 *Let $M \in \mathcal{M}^2(\Phi')$. Then there exists a Φ'-valued version $\tilde{M}$ of M such that the following conditions hold:*
(i) For each $T > 0$ there exists $p = p_T > 0$ such that

$$\tilde{M}|_{[0,T]} \in D([0,T], \Phi_{-p}) \ a.s.$$

(ii) $\tilde{M}$ is r.c.l.l. in the strong Φ'-topology, i.e.

$$\tilde{M} \in D([0,\infty), \Phi') \ a.s.$$

Proof: (i) Fix $T > 0$ and define $V_T^2(\phi) = E(M_T[\phi]^2)$. Then V_T satisfies the conditions of Lemma 1.3.1 and hence, there exist $\theta = \theta_T > 0$ and $r = r_T > 0$ such that

$$V_T(\phi) \leq \theta \|\phi\|_r \qquad \forall \phi \in \Phi. \tag{3.1.5}$$

Let D be a countable dense subset of [0,T]. Then by Doob's inequality

$$E\left(\sup_{t\in D} M_t[\phi]^2\right) \leq 4 \sup_{0\leq t\leq T} E(M_t[\phi]^2) = 4E(M_T[\phi]^2). \tag{3.1.6}$$

Let $p \geq r$ be such that the canonical injection from Φ_p to Φ_r is Hilbert-Schmidt. Then from (3.1.5) and (3.1.6) we have

$$\begin{aligned} E\left(\sum_{j=1}^{\infty} \sup_{t\in D} M_t[\phi_j^p]^2\right) &= \sum_{j=1}^{\infty} E\left(\sup_{t\in D} M_t[\phi_j^p]^2\right) \\ &\leq 4\theta^2 \sum_{j=1}^{\infty} \|\phi_j^p\|_r^2 < \infty. \end{aligned}$$

So, if $\Omega_1 = \left\{\omega \in \Omega : \sum_{j=1}^{\infty} \sup_{t\in D} M_t[\phi_j^p]^2(\omega) < \infty\right\}$, then $P(\Omega_1) = 1$.

Since each real-valued martingale $M_t[\phi_j^p]$ has a right continuous modification X_t^j, writing

$$\Omega_t^j \equiv \left\{\omega \in \Omega : X_t^j(\omega) = M_t[\phi_j^p](\omega)\right\},$$

we have $P(\Omega_t^j) = 1$ for $t \in D$. Then the set defined by

$$\Omega_2 \equiv \left(\cap_{t\in D} \cap_{j\geq 1} \Omega_t^j\right) \cap \Omega_1$$

has probability one and if $\omega \in \Omega_2$

$$\sum_{j=1}^{\infty} \sup_{0\leq t\leq T} X_t^j(\omega)^2 < \infty.$$

For $0 \le t \le T$, define $\tilde{M}_t(\omega) = 0$ for $\omega \notin \Omega_2$ and

$$\tilde{M}_t(\omega) = \sum_{j=1}^{\infty} X_t^j(\omega)\phi_j^{-p}, \ \omega \in \Omega_2.$$

Then for $0 \le t \le T$ we have $P(\tilde{M}_t \in \Phi_{-p}) = 1$ and $\tilde{M}_t(\omega)[\phi] = M_t(\omega)[\phi]$ for all $\phi \in \Phi$, $\omega \in \Omega_2$, i.e. $\tilde{M}_t = M_t$ a.s.

Next since for $s, t \in [0, T]$, $j \ge 1$ and $\omega \in \Omega_2$

$$|X_t^j(\omega) - X_s^j(\omega)|^2 \le 4 \sup_{0\le t\le T} X_t^j(\omega)^2,$$

by the dominated convergence theorem,

$$\begin{aligned}\lim_{s\to t+} \|\tilde{M}_t(\omega) - \tilde{M}_s(\omega)\|_{-p}^2 &= \lim_{s\to t+} \left\| \sum_{j=1}^{\infty} (X_t^j(\omega) - X_s^j(\omega))\phi_j^{-p} \right\|_{-p}^2 \\ &= \lim_{s\to t+} \sum_{j=1}^{\infty} |X_t^j(\omega) - X_s^j(\omega)|^2 \\ &= \sum_{j=1}^{\infty} \lim_{s\to t+} |X_t^j(\omega) - X_s^j(\omega)|^2 = 0,\end{aligned}$$

the last assertion follows from the right continuity of $X_t^j(\omega)$. In a similar fashion the fact that $\tilde{M}_t$ has left hand limits in the $\|\cdot\|_{-p}$-norm is shown.

Thus we have proved that for each $T > 0$ there exists $p = p_T > 0$ such that M_t has a r.c.l.l. version $\tilde{M}_t$ in the $\|\cdot\|_{-p}$-norm, i.e.

$$\tilde{M}|_{[0,T]} \in D([0,T], \Phi_{-p}), \ \text{ a.s.}$$

(ii) Let T_n increase to infinity. Then by (i) there exists p_n such that M_t has a version $\tilde{M}^n$ with

$$\tilde{M}^n|_{[0,T_n]} \in D([0,T], \Phi_{-p_n}), \ \text{ a.s.}$$

With the notation used in the proof of (i) let $\Omega_3 = \cap_{n=1}^{\infty} \Omega_2^n$. If $\omega \in \Omega_3$ define for $0 \le t < \infty$

$$\tilde{M}_t(\omega) = \tilde{M}_t^n(\omega) \qquad \text{for } T_{n-1} < t \le T_n, \ (T_0 = 0).$$

Then $P(\tilde{M}_t \in \Phi') = 1$ and $\tilde{M}_t(\omega) = M_t(\omega)$ for $\omega \in \Omega_3$.

For $t \ge 0$, let n be such that $t < T_n$. Then for $\epsilon > 0$ there exists $\delta_t > 0$ such that if $t < s < t + \delta_t$

$$\|\tilde{M}_t(\omega) - \tilde{M}_s(\omega)\|_{-p_n} < \epsilon$$

For any bounded subset B of Φ, let C be a constant such that $||\phi||_{p_n} \leq C$ $\forall \phi \in B$. Therefore

$$\sup_{\phi \in B} |(\tilde{M}_t(\omega) - \tilde{M}_s(\omega))[\phi]| < C\epsilon, \quad \forall t < s < t + \delta_t,$$

i.e. $\tilde{M}_t$ is strongly right continuous. A similar argument shows that it has left hand limits. ∎

Remark 3.1.2 *If $M \in \mathcal{M}^2(\Phi')$ such that for each $\phi \in \Phi$*

$$\sup_{0 \leq t < \infty} E\left(M_t[\phi]^2\right) < \infty,$$

there exists $p > 0$ such that M_t has a version $\tilde{M}_t \in D([0,\infty), \Phi_{-p})$ a.s. This is seen using the fact that if D is a countable dense subset of $\mathbf{R}_+$ then

$$E\left(\sup_{t \in D} M_t[\phi]^2\right) \leq 4E(M_\infty[\phi]^2).$$

The next theorem is the analogue of Theorem 3.1.3 to continuous martingales.

Theorem 3.1.4 *Let $M \in \mathcal{M}^{2,c}(\Phi')$. Then there exists a Φ'-valued version $\tilde{M}$ of M such that the following conditions hold:*
(i) For each $T > 0$ there exists $p = p_T > 0$ such that

$$\tilde{M}|_{[0,T]} \in C([0,T], \Phi_{-p}) \qquad a.s.$$

(ii) $\tilde{M}$ is continuous in the strong Φ'-topology, i.e.

$$\tilde{M} \in C([0,\infty), \Phi') \qquad a.s.$$

(iii) If for each $\phi \in \Phi$

$$\sup_{0 \leq t < \infty} E(M_t[\phi]^2) < \infty,$$

then there exists $p > 0$ such that

$$\tilde{M} \in C([0,\infty), \Phi_{-p}) \qquad a.s.$$

The following example, due to Kallianpur and Ramaswamy, gives a Φ'-valued strongly continuous Gaussian martingale M_t for which the following is **not** true: There exists p independent of t such that

$$M_t \in \Phi_{-p} \qquad \forall t \geq 0, \; a.s.$$

Example 3.1.1 *Consider the CHNS of Example 1.3.2. Using the notation of that example, we define* $f : \mathbf{R}_+ \times \Phi \to \mathbf{R}$ *as follows*

$$f(s, \phi) = \sum_{j=1}^{\infty} (1 + \lambda_j)^s < \phi, \phi_j >_0 .$$

Let $\{B_s\}_{s \geq 0}$ *be a real-valued standard Brownian motion. Since for each* $t > 0$ *and* $\phi \in \Phi$

$$\int_0^t f(s, \phi)^2 ds < \infty$$

the Wiener integral

$$X_{t,\phi} = \int_0^t f(s, \phi) dB_s$$

is a Gaussian martingale for each $\phi \in \Phi$. *Since* $f(s, \phi)$ *is linear and continuous in* Φ, *the linear random functional*

$$X_{t,\phi} : \Phi \to L^2(\Omega)$$

is Φ*-continuous. Hence by the regularization theorem there exists a* Φ'*-valued random variable* X_t *such that*

$$X_t[\phi] = X_{t,\phi} \quad a.s.\ \forall \phi \in \Phi.$$

Then $(X_t, \mathcal{F}_t^B)_{t \geq 0} \in \mathcal{M}^{2,c}(\Phi')$. *Hence by Theorem 3.1.4,* X *has a strongly continuous version also denoted by* X.

Now suppose there exists $p > 0$ *such that* $X_t \in \Phi_{-p}$ *a.s.* $\forall t \geq 0$. *Let*

$$\phi^{(n)} = \sum_{j=1}^{n} (1 + \lambda_j)^{-p-r_1} \phi_j.$$

Then $\{\phi^{(n)}\}$ *converges in* Φ_p *to an element* ϕ, *and therefore* $X_t[\phi^{(n)}] \to X_t[\phi]$. *But since* X_t *is* L^2*-continuous*

$$E(X_t[\phi^{(n)}]^2) \to E(X_t[\phi]^2) < \infty \ \forall t \geq 0 \tag{3.1.7}$$

the finiteness of the limit being a consequence of $X_t[\phi]$ *being a Gaussian random variable. On the other hand, if* $t > p + r_1$,

$$\begin{aligned} E(X_t[\phi^{(n)}]^2) &= \int_0^t f(s, \phi^{(n)})^2 ds \\ &= \int_0^t \left(\sum_{j=1}^{n} (1 + \lambda_j)^{-p-r_1+s} \right)^2 ds \\ &\geq \int_{p+r_1}^t \left(\sum_{j=1}^{n} (1 + \lambda_j)^{-p-r_1+s} \right)^2 ds. \end{aligned}$$

Then by Fatou's lemma

$$\liminf_{n\to\infty} E(X_t[\phi^{(n)}]^2) = \infty$$

which contradicts from (3.1.7). ∎

3.2 Φ'-Wiener process and cylindrical Brownian motion

In this section we introduce H-cylindrical Brownian motion, H-valued Brownian motion and Φ'-Wiener process. We give several examples of such processes and illustrate how some infinite dimensional extensions of the real valued Brownian motion (as the cylindrical Brownian motion and a sequence of independent Brownian motions) may be seen as nuclear space valued Wiener processes.

Definition 3.2.1 *Let H be a separable Hilbert space with norm $\|\cdot\|_H$. A family $\{B_t(h) : t \geq 0, h \in H\}$ of real-valued random variables is called a* **cylindrical Brownian motion (c.B.m) on H with covariance** Σ *if Σ is a continuous self-adjoint positive definite operator on H such that the following conditions hold:*
i) For each $h \in H$ such that $h \neq 0$, $< \Sigma h, h >_H^{-1/2} B_t(h)$ is a one dimensional standard Wiener process.
ii) For any $t \geq 0$, $\alpha_1, \alpha_2 \in \mathbf{R}$ and $f_1, f_2 \in H$

$$B_t(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 B_t(f_1) + \alpha_2 B_t(f_2) \qquad a.s.$$

iii) For each $h \in H$, $\{B_t(h)\}$ is an $\mathcal{F}_t^B$-martingale, where

$$\mathcal{F}_t^B = \sigma\{B_s(k) : s \leq t, k \in H\}.$$

$\{B_t(h) : t \geq 0, h \in H\}$ is called a standard H-c.B.m or simply, H-c.B.m. if it is a H-c.B.m. with covariance $\Sigma = I$.

Theorem 3.2.1 *Let $\{B_t(h) : t \geq 0, h \in H\}$ be an H-c.B.m with covariance Σ. Then there exists an H-valued process $\bar{B}_t$ such that*

$$B_t(h) =< \bar{B}_t, h >_H \qquad \forall h \in H$$

*if and only if $\Sigma \in L_{(1)}(H)$. In this case $\{\bar{B}_t\}$ is called an H-***valued Brownian motion.**

Proof: "$\Rightarrow$" For $t \geq 0$ fixed, as

$$\begin{aligned} F(h) &\equiv E\{\exp(i < \bar{B}_t, h >_H)\} \\ &= E\{\exp(iB_t(h)\} \\ &= \exp\left(-\frac{1}{2} < \Sigma h, h >_H\right), \quad h \in H, \end{aligned}$$

is a characteristic function on H and hence, by Sazonov's theorem, F is continuous with respect to S-topology. Therefore $h \to < \Sigma h, h >_H$ is S-continuous which implies that Σ is a nuclear operator.
"$\Leftarrow$" Let

$$\Sigma h = \sum_{j=1}^{\infty} \lambda_j < h, e_j >_H e_j, \quad h \in H$$

where $\lambda_j > 0$, $\sum_{j=1}^{\infty} \lambda_j < \infty$ and $\{e_j\}$ is a CONS of H. Let

$$\bar{B}_t = \sum_{j=1}^{\infty} B_t(e_j)e_j.$$

It is easy to show that $\bar{B}_t$ is well-defined and satisfies the condition of the theorem. ∎

Remark 3.2.1 1° *Let $\{B_t(h) : t \geq 0, h \in H\}$ be a standard H-c.B.m. Then it is an H-c.B.m. with identity operator as its covariance. Therefore there does not exist a process $\bar{B}_t$ in H such that*

$$B_t(h) = < \bar{B}_t, h >_H .$$

2° *If $\{B_t(h) : t \geq 0, h \in H\}$ is an H-c.B.m with covariance Σ and $S \in L(H, H)$, we define*

$$B_t^S(h) \equiv B_t(Sh), \ \forall h \in H.$$

Then $\{B_t^S(h) : t \geq 0, h \in H\}$ is an H-c.B.m with covariance $S^*\Sigma S$. As a consequence, $\{B_t^S(h) : t \geq 0, h \in H\}$ is a standard H-c.B.m if we take $S = \Sigma^{-1/2}$. Therefore we only need to consider standard H-c.B.m.

Theorem 3.2.2 *Let $\{e_n\}_{n\geq 1}$ be a CONS in H. There exists a one-to-one correspondence between an H-c.B.m. B and a sequence of independent one-dimensional Brownian motions $\{B_t^n\}$ given by*

$$B_t^n = B_t(e_n), \ n \in \mathbf{N} \tag{3.2.1}$$

and

$$B_t(h) = \sum_{n=1}^{\infty} < h, e_n >_H B_t^n, \ h \in H. \tag{3.2.2}$$

Proof: Let $\{B_t^n\}$ be a sequence of independent one-dimensional Brownian motions. Note that by Doob's inequality

$$\begin{aligned} E\sup_{0\le t\le T}\left|\sum_{n=m+1}^{m+k} < h, e_n >_H B_t^n\right|^2 &\le 4E\left|\sum_{n=m+1}^{m+k} < h, e_n >_H B_T^n\right|^2 \\ &= 4T\sum_{n=m+1}^{m+k} < h, e_n >_H^2 \to 0 \end{aligned}$$

as $m \to \infty$. Hence for any $h \in H$, $B_t(h)$ given by (3.2.2) is well-defined in the following sense: $\forall T \ge 0$

$$E\sup_{0\le t\le T}\left|B_t(h) - \sum_{n=1}^{m} < h, e_n >_H B_t^n\right|^2 \to 0, \quad \text{as } m \to \infty. \tag{3.2.3}$$

For any $0 = t_0 < t_1 < t_2 < \cdots < t_k$

$$\begin{aligned} & E\exp\left(i\sum_{j=0}^{k-1}\lambda_j(B_{t_{j+1}}(h) - B_{t_j}(h))\right) \\ = & \lim_{m\to\infty} E\exp\left(i\sum_{j=0}^{k-1}\lambda_j\sum_{n=1}^{m} < h, e_n >_H (B_{t_{j+1}}^n - B_{t_j}^n)\right) \\ = & \lim_{m\to\infty}\prod_{j=0}^{k-1}\prod_{n=1}^{m} E\exp(i\lambda_j < h, e_n >_H (B_{t_{j+1}}^n - B_{t_j}^n)) \\ = & \lim_{m\to\infty}\prod_{j=0}^{k-1}\prod_{n=1}^{m}\exp\left(-\frac{1}{2}\lambda_j^2 < h, e_n >_H^2 (t_{j+1} - t_j)\right) \\ = & \prod_{j=0}^{k-1}\exp\left(-\frac{1}{2}\lambda_j^2(t_{j+1} - t_j)\|h\|_H^2\right). \end{aligned}$$

Hence (i) of Definition 3.2.1 holds. From

$$\begin{aligned} & \alpha_1\sum_{n=1}^{m} < f_1, e_n >_H B_t^n + \alpha_2\sum_{n=1}^{m} < f_2, e_n >_H B_t^n \\ = & \sum_{n=1}^{m} < \alpha_1 f_1 + \alpha_2 f_2, e_n >_H B_t^n, \end{aligned}$$

(ii) of Definition 3.2.1 follows immediately from (3.2.3) by the uniqueness of L^2-limits.

Finally let $A \in \mathcal{F}_s^B$ which is given in (iii) of Definition 3.2.1. Then for any $t > s$ and $h \in H$

$$E\{(B_t(h) - B_s(h))1_A(\omega)\}$$

$$= \lim_{m\to\infty} E \sum_{n=1}^{m} < h, e_n >_H (B_t^n - B_s^n) 1_A(\omega) = 0.$$

This proves (iii) of Definition 3.2.1. Therefore B is a cylindrical Brownian motion on H.

On the other hand, let B be a cylindrical Brownian motion on H and define $\{B_t^n\}$ by (3.2.1). It follows from (i) of Definition 3.2.1 that $\{B_t^n\}$ is a sequence of one-dimensional Brownian motions. Now we prove they are independent, i.e. for any $0 \le t_{j1} < \cdots < t_{jm}$, $\lambda_{jr} \in \mathbf{R}$, $n_j \in \mathbf{N}$, $j = 1, \cdots, k$ and $r = 1, \cdots, m_j$, we have

$$E \exp\left(i \sum_{j=1}^{k} \sum_{r=1}^{m_j} \lambda_{jr} B_{t_{jr}}^{n_j} \right) = \prod_{j=1}^{k} E \exp\left(i \sum_{r=1}^{m_j} \lambda_{jr} B_{t_{jr}}^{n_j} \right).$$

We may assume that m_j and t_{jr} do not depend on j, otherwise we only need to rearrange $\{t_{jr} : j = 1, \cdots, k \text{ and } r = 1, \cdots, m_j\}$ as $\{t_1, t_2, \cdots, t_m\}$ and define

$$\lambda_{js} = 0 \qquad \text{if } t_s \notin \{t_{jr} : r = 1, \cdots, m_j\},$$

for $j = 1, \cdots, k$ and $s = 1, \cdots, m$. Let

$$\mu_{jr} = \sum_{s=1}^{r} \lambda_{js} \quad \text{and} \quad \mu_{j0} = 0, \ t_0 = 0.$$

Then

$$\begin{aligned}
& E \exp\left(i \sum_{j=1}^{k} \sum_{r=1}^{m} \lambda_{jr} B_{t_r}^{n_j} \right) \\
= \;& E \exp\left(i \sum_{j=1}^{k} \sum_{r=1}^{m} (\mu_{jr} - \mu_{j(r-1)}) B_{t_r}(e_{n_j}) \right) \\
= \;& E \exp\left(i \sum_{j=1}^{k} \sum_{r=0}^{m-1} (\mu_{jm} - \mu_{jr}) (B_{t_{r+1}}(e_{n_j}) - B_{t_r}(e_{n_j}) \right) \\
= \;& E \exp\left[i \sum_{r=0}^{m-1} \left(B_{t_{r+1}} \left(\sum_{j=1}^{k} (\mu_{jm} - \mu_{jr}) e_{n_j} \right) \right.\right. \\
& \left.\left. - B_{t_r} \left(\sum_{j=1}^{k} (\mu_{jm} - \mu_{jr}) e_{n_j} \right) \right) \right] \\
= \;& \exp\left(-\frac{1}{2} \sum_{r=0}^{m-1} (t_{r+1} - t_r) \left\| \sum_{j=1}^{k} (\mu_{jm} - \mu_{jr}) e_{n_j} \right\|_H^2 \right)
\end{aligned}$$

$$
\begin{aligned}
&= \exp\left(-\frac{1}{2}\sum_{r=0}^{m-1}(t_{r+1}-t_r)\sum_{j=1}^{k}(\mu_{jm}-\mu_{jr})^2\right)\\
&= \prod_{j=1}^{k}\exp\left(-\frac{1}{2}\sum_{r=0}^{m-1}(t_{r+1}-t_r)(\mu_{jm}-\mu_{jr})^2\right)\\
&= \prod_{j=1}^{k}E\exp\left(i\sum_{r=1}^{m}\lambda_{jr}B_{t_r}^{n_j}\right).
\end{aligned}
$$

Therefore $\{B_t^n\}$ is a sequence of independent one-dimensional Brownian motions. ∎

Next we give an example of H-c.B.m. We need the following definition.

Definition 3.2.2 *Let $\mathcal{O} \subset \mathbf{R}^d$ be a measurable set. A real-valued function W on $\Omega \times \mathcal{B}_f(\mathbf{R}_+ \times \mathcal{O})$ is called a* **white noise random measure** *if*
i) For $A \in \mathcal{B}_f(\mathbf{R}_+ \times \mathcal{O})$, $W(\cdot,A)$ is a $N(0,|A|)$ random variable;
ii) For disjoint Borel sets A_1, A_2 in $\mathcal{B}_f(\mathbf{R}_+ \times \mathcal{O})$, $W(\cdot, A_1)$ and $W(\cdot, A_2)$ are independent and

$$W(\cdot, A_1 \cup A_2) = W(\cdot, A_1) + W(\cdot, A_2) \quad a.s.$$

where

$$\mathcal{B}_f(\mathbf{R}_+ \times \mathcal{O}) = \{A \in \mathcal{B}(\mathbf{R}_+ \times \mathcal{O}) : |A| < \infty\}$$

and $|A|$ is the Lebesgue measure of A.

Next we define Brownian sheet as a random field.

Definition 3.2.3 *Let $(E, \mathcal{E})$ be a measurable space. A real-valued measurable function f on $E \times \Omega$ is called a* **random field on E**. *It is a* **Gaussian random field** *if $\{f(x,\cdot), x \in E\}$ is a Gaussian system.*

For each $(t,x) \in \mathbf{R}_+ \times \mathcal{O}$, let

$$A_{t,x} \equiv \{(s,y) \in \mathbf{R}_+ \times \mathcal{O} : 0 \le s \le t,\ y_j \le x_j,\ j = 1, \cdots, d\},$$

where $x = (x_1, \cdots, x_d)$ and $y = (y_1, \cdots, y_d)$. We assume that $|A_{t,x}| < \infty$, $\forall (t,x) \in \mathbf{R}_+ \times \mathcal{O}$.

Definition 3.2.4 *A real-valued function B on $\Omega \times \mathbf{R}_+ \times \mathcal{O}$ is a* **Brownian sheet (B.S.)** *or* **space-time Brownian motion** *if $\{B(\cdot,t,x) : (t,x) \in \mathbf{R}_+ \times \mathcal{O}\}$ it is a Gaussian system such that*
i) $E(B(\cdot,t,x)) = 0$, $\forall (t,x) \in \mathbf{R}_+ \times \mathcal{O}$
ii) $Cov(B(\cdot,t,x), B(\cdot,s,y)) = |A_{t,x} \cap A_{s,y}|$, $\forall (t,x)$, $(s,y) \in \mathbf{R}_+ \times \mathcal{O}$.

Remark 3.2.2 *If $\mathcal{O} = [0, b]^d$ for some $b > 0$ or*

$$\mathcal{O} = \{x \in \mathbf{R}^d : x_j \geq 0,\ j = 1, 2, \cdots, d\},$$

and B is a Brownian sheet on $\mathbf{R}_+ \times \mathcal{O}$, then

$$Cov(B(\cdot, t, x), B(\cdot, s, y)) = (s \wedge t) \prod_{j=1}^{d} (x_j \wedge y_j).$$

Remark 3.2.3 *There is a one-to-one correspondence between white noise random measure W and Brownian sheet B as follows:*

$$B(\cdot, t, x) = W(\cdot, A_{t,x}), \qquad \forall (t, x) \in \mathbf{R}_+ \times \mathcal{O}.$$

In this sense, we shall denote the Brownian sheet by W(t,x).

Remark 3.2.4 *It can be shown that W(t,x) is continuous in $(t, x) \in \mathbf{R}_+ \times \mathcal{O}$ and nowhere differentiable for a.a. ω. Therefore we can only define $\dot{W}_{t,x} \equiv \frac{\partial^2 W(t,x)}{\partial t \partial x}$ in the sense of distribution:*

$$\int \int_{\mathcal{O}} \phi(t, x) \frac{\partial^2 W(t, x)}{\partial t \partial x} dt dx = \int \int_{\mathcal{O}} \frac{\partial^2 \phi(t, x)}{\partial t \partial x} W(t, x) dt dx$$

for all smooth functions ϕ with compact supports in $\mathbf{R}_+ \times \mathcal{O}$. $\dot{W}_{t,x}$ is called the **white noise in space-time**.

Now we proceed to introduce stochastic integrals with respect to white noise random measures (or equivalently, with respect to a Brownian sheet). For convenience, we take $d = 1$ and $\mathcal{O} = [0, b]$. For a simple function f on $\mathbf{R}_+ \times [0, b]$ given by

$$f(s, x) = \sum_{i=1}^{n} a_i 1_{[t_{i-1}, t_i) \times [x_{i-1}, x_i)}(s, x) \tag{3.2.4}$$

where $0 = t_0 < t_1 < \cdots < t_n$ and $0 = x_0 < x_1 < \cdots < x_n = b$, we define

$$B_t(f) \equiv \sum_{i=1}^{n} a_i W([t_{i-1} \wedge t, t_i \wedge t) \times [x_{i-1}, x_i)),\ \forall t \geq 0. \tag{3.2.5}$$

The proof of the following theorem is straightforward and we leave it to the reader.

Theorem 3.2.3 *Let f be a simple function and let $B_t(f)$ be defined by (3.2.5). Then*
(a) $B_t(f)$ is a real-valued continuous Gaussian process such that for any

$0 \leq s < t$, *$B_s(f)$ is $\mathcal{F}_s^W$-measurable, $B_t(f) - B_s(f)$ is independent of $\mathcal{F}_s^W$ and*

$$E(B_t(f) - B_s(f))^2 = \int_s^t \int_0^b f(r,x)^2 dr dx$$

where

$$\mathcal{F}_t^W = \sigma\{W(A) : \forall A \subset [0,t] \times [0,b]\}.$$

(b) For any $t \geq 0$, $\alpha_1, \alpha_2 \in \mathbf{R}$ and simple functions f_1, f_2,

$$B_t(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 B_t(f_1) + \alpha_2 B_t(f_2) \quad a.s.$$

For general function f on $\mathbf{R}_+ \times [0,b]$ such that

$$\int_0^T \int_0^b f(s,x)^2 ds dx < \infty, \ \forall T > 0,$$

let $\{f_n\}$ be a sequence of simple functions such that

$$\int_0^T \int_0^b (f_n(s,x) - f(s,x))^2 ds dx \to 0 \quad \text{as } n \to \infty, \ \forall T > 0.$$

Since

$$\begin{aligned} & E \sup_{0 \leq t \leq T} |B_t(f_n) - B_t(f_m)|^2 \leq 4E|B_T(f_n - f_m)|^2 \\ = \ & 4T \int_0^T \int_0^b (f_n(s,x) - f_m(s,x))^2 ds dx \to 0, \ \forall T > 0, \end{aligned}$$

there exists a process, denoted by

$$B_t(f) = \int_0^t \int_0^b f(s,x) W(ds dx)$$

such that

$$E \sup_{0 \leq t \leq T} |B_t(f_n) - B_t(f)|^2 \to 0, \ \forall T > 0. \tag{3.2.6}$$

Theorem 3.2.4 *Let $H = L^2([0,b])$.*
(a) Let $W(dtdx)$ be a white noise random measure on $\mathbf{R}_+ \times [0,b]$. Then $\{B_t(f) : t \geq 0, \ f \in H\}$ defined by (3.2.6) is an H-c.B.m.
(b) Suppose that $\{\tilde{B}_t(f) : t \geq 0, \ f \in H\}$ is an H-c.B.m. Then there exists a white noise random measure $W(dtdx)$ on $\mathbf{R}_+ \times [0,b]$ such that $\{B_t(f)\}$ constructed in (3.2.6) has the property:

$$B_t(f) = \tilde{B}_t(f), \qquad a.s.$$

Proof: (a) Let f_n be a sequence of simple functions on [0,b] such that $\|f_n - f\|_H \to 0$ as $n \to \infty$. Then f and f_n can be regarded as functions on $\mathbf{R} \times [0, b]$, i.e.

$$f(t,x) \equiv f(x), \qquad \forall (t,x) \in \mathbf{R} \times [0,b].$$

Then $B_t(f)$ is well-defined by (3.2.6). By Theorem 3.2.3 and (3.2.6), it is easy to see that i), ii) of Definition 3.2.1 hold. The condition iii) of Definition 3.2.1 follows from $\mathcal{F}_t^B \subset \mathcal{F}_t^W$.

(b) For any $A \in \mathcal{B}_f(\mathbf{R}_+ \times [0,b])$, let

$$W(\cdot, A) = \lim_{n\to\infty} \tilde{B}_n(1_A). \tag{3.2.7}$$

For $m > n \to \infty$, we have

$$E|\tilde{B}_n(1_A) - \tilde{B}_m(1_A)|^2 = \int_n^m \int_0^b 1_A(s,x)dsdx \to 0$$

and hence $W(\cdot, A)$ is well-defined by (3.2.7). For $A_1, A_2 \in \mathcal{B}_f(\mathbf{R}_+ \times [0,b])$, let $\{f_n\}$, $\{g_n\}$ be two sequences of simple functions such that

$$\int_{\mathbf{R}_+} \int_0^b |f_n(s,x) - 1_{A_1}(s,x)|^2 dsdx \to 0$$

and

$$\int_{\mathbf{R}_+} \int_0^b |g_n(s,x) - 1_{A_2}(s,x)|^2 dsdx \to 0.$$

Let

$$f_n = \sum_{j=1}^n a_j^n 1_{[t_{j-1}^n, t_j^n) \times [x_{j-1}^n, x_j^n)}$$

and

$$g_n = \sum_{j=1}^n b_j^n 1_{[t_{j-1}^n, t_j^n) \times [x_{j-1}^n, x_j^n)}$$

Then

$$\begin{aligned}
& E\exp(i\alpha W(\cdot, A_1)) + i\beta W(\cdot, A_2)) \\
= \ & \lim_n E\exp(i\alpha B_n(f_n) + i\beta B_n(g_n)) \\
= \ & \lim_n E \prod_{j=1}^n \exp(i(\alpha a_j^n + \beta b_j^n) W([t_{j-1}^n, t_j^n) \times [x_{j-1}^n, x_j^n))) \\
= \ & \lim_n \prod_{j=1}^n \exp\left(-\frac{|\alpha a_j^n + \beta b_j^n|^2}{2}(t_j^n - t_{j-1}^n)(x_j^n - x_{j-1}^n)\right) \\
= \ & \lim_n \exp\left(-\frac{1}{2}\int_0^n \int_0^b (\alpha f_n(s,x) + \beta g_n(s,x))^2 dsdx\right)
\end{aligned}$$

$$= \exp\left(-\frac{1}{2}\int_{\mathbf{R}_+}\int_0^b (\alpha 1_{A_1} + \beta 1_{A_2})^2 ds dx\right)$$
$$= \exp\left(-\frac{1}{2}\alpha^2|A_1|\right)\exp\left(-\frac{1}{2}\beta^2|A_2|\right).$$

Hence $W(\cdot, A_1)$, $W(\cdot, A_2)$ are independent and $W(\cdot, A_j) \sim N(0, |A_j|)$, $j = 1, 2$.

Further, since

$$B_n(f_n) + B_n(g_n) = B_n(f_n + g_n)$$

and $f_n + g_n \to 1_{A_1 \cup A_2}$ we have

$$W(\cdot, A_1) + W(\cdot, A_2) = W(\cdot, A_1 \cup A_2), \quad a.s.$$

Therefore, by Definition 3.2.2, W is a white noise random measure.

For any $f \in H$, let $\{f_n\}$ be a sequence of of simple function such that $||f_n - f||_H \to 0$. Let

$$f_n(x) = \sum_{j=1}^n a_j^n 1_{[x_{j-1}^n, x_j^n)}(x).$$

Then

$$\begin{aligned} B_t(f_n) &= \sum_{j=1}^n a_j^n W([0,t] \times [x_{j-1}^n, x_j^n)) \\ &= \sum_{j=1}^n a_j^n \tilde{B}_t(1_{[x_{j-1}^n, x_j^n)}) = \tilde{B}_t(f_n). \end{aligned} \tag{3.2.8}$$

By (3.2.8) and (3.2.6), we have

$$B_t(f) = \tilde{B}_t(f), \quad a.s.$$

∎

Now we introduce the concept of Φ'-valued Wiener process and its relationship with H-c.B.m.

Definition 3.2.5 *A strongly sample continuous Φ'-valued stochastic process $W = (W_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, P)$ is called a centered Φ'-* **Wiener process with covariance** $Q(\cdot,\cdot)$ *if W satisfies the following three conditions:*
a) $W_0 = 0$ a.s.
b) W has independent increments, i.e. the random variables

$$W_{t_1}[\phi_1], (W_{t_2} - W_{t_1})[\phi_2], \cdots, (W_{t_n} - W_{t_{n-1}})[\phi_n]$$

are independent for any $\phi_1, \cdots, \phi_n \in \Phi$, $0 \le t_1 \le \cdots \le t_n, n \ge 1$.
c) For each $t \ge 0$ *and* $\phi \in \Phi$

$$E\left(e^{iW_t[\phi]}\right) = e^{-tQ(\phi,\phi)/2}$$

where Q *is a* **covariance functional**, *i.e. a positive definite symmetric continuous bilinear form on* $\Phi \times \Phi$.

Remark 3.2.5 *Let* W *be a* Φ'*-Wiener process with covariance* Q. *Then*
i) $W \in \mathcal{M}^{2,c}(\Phi')$.
ii) $\{W_t[\phi] : \phi \in \Phi, t \ge 0\}$ *is a centered Gaussian system and*

$$E(W_t[\phi]W_s[\psi]) = (s \wedge t)Q(\phi, \psi), \quad \phi, \psi \in \Phi, \ s, t \ge 0.$$

Remark 3.2.6 *A* Φ'*-valued process* $(Z_t)_{t\ge 0}$ *is a* **non-centered Wiener process** *if there exists* $m \in \Phi'$ *such that* $Z_t - mt$ *is a centered Wiener process.*

Lemma 3.2.1 *i) For each* $\phi \in \Phi$, *let* $\iota\phi = Q(\phi, \cdot)$. *Then* ι *is an injective linear operator from* Φ *onto a linear subspace* $\mathcal{R}(\iota)$ *of* Φ'.
ii) For any $v_1, v_2 \in \mathcal{R}(\iota)$, *let*

$$< v_1, v_2 >_{H_Q} = Q(\iota^{-1}v_1, \iota^{-1}v_2).$$

Then $< \cdot, \cdot >_{H_Q}$ *is an inner product on* $\mathcal{R}(\iota)$. *Let* $\|\cdot\|_{H_Q}$ *be the norm on* $\mathcal{R}(\iota)$ *determined by the inner product* $< \cdot, \cdot >_{H_Q}$ *and let* H_Q *be the completion of* $\mathcal{R}(\iota)$ *with respect to* $\|\cdot\|_{H_Q}$. *Then* H_Q *is a separable Hilbert space and* $H_Q \subset \Phi'$.

Proof: The proof is standard and we leave it to the reader. ∎

Lemma 3.2.2 *i) There exists an index* r_2 *such that for any* $p \ge r_2$, $\exists$ *a positive-definite (i.e.* $\langle\sqrt{Q_p}\phi, \phi\rangle_p > 0$, $\forall\phi \in \Phi_p$, $\phi \ne 0$*) self-adjoint operator* $\sqrt{Q_p}$ *on* Φ_p *such that*

$$Q(\phi, \psi) = \left\langle\sqrt{Q_p}\phi, \sqrt{Q_p}\psi\right\rangle_p \quad \forall\phi, \psi \in \Phi.$$

ii) For $p \ge r_2$, *we have*

$$w[\theta_p v] =< w, v >_{-p}, \qquad \forall w, v \in \Phi_{-p}$$

and

$$\theta_p\sqrt{Q_p}' = \sqrt{Q_p}\theta_p : \Phi_{-p} \to \Phi_p.$$

iii) For any $p \geq r_2$, we have $H_Q = \mathcal{R}(\sqrt{Q_p}')$. Furthermore, for any $h \in \Phi_{-p}$

$$\left\| \sqrt{Q_p}' h \right\|_{H_Q} = \|h\|_{-p},$$

i.e., $\sqrt{Q_p}'$ is an isometry from Φ_{-p} to H_Q.

Proof: i) Let $V^2(\phi) = Q(\phi, \phi)$ for $\phi \in \Phi$. Then $V : \Phi \to [0, \infty)$ satisfies the conditions of Lemma 1.3.1. Therefore there exist $\theta > 0$ and $r_2 \geq 0$ such that

$$Q(\phi, \phi) \leq \theta \|\phi\|_{r_2}^2, \ \forall \phi \in \Phi.$$

Hence

$$|Q(\phi, \psi)| \leq \theta \|\phi\|_{r_2} \|\psi\|_{r_2} \leq \theta \|\phi\|_p \|\psi\|_p, \qquad \forall \phi, \psi \in \Phi, \ p \geq r_2.$$

Therefore Q can be extended to become a symmetric continuous bilinear form on $\Phi_p \times \Phi_p$, still denoted by Q. As $Q(\phi, \cdot) \in \Phi_{-p}$ for any $\phi \in \Phi_p$, it follows from Riesz's representation theorem that there exists $Q\phi \in \Phi_p$ such that

$$Q(\phi, \psi) = < Q\phi, \psi >_p, \qquad \forall \psi \in \Phi_p.$$

It is easy to show that Q is a positive definite self-adjoint operator on Φ_p and hence $\sqrt{Q_p}$ is well-defined and $Q(\phi, \psi) = \left\langle \sqrt{Q_p}\phi, \sqrt{Q_p}\psi \right\rangle_p$ for any ϕ and ψ in Φ_p.
ii) Note that, for any v and w in Φ_{-p},

$$\begin{aligned} w[\theta_p v] &= w\left[\sum_{j=1}^{\infty} < v, \phi_j^{-p} >_{-p} \phi_j^p\right] \\ &= \sum_{j=1}^{\infty} < v, \phi_j^{-p} >_{-p} \sum_{k=1}^{\infty} < w, \phi_k^{-p} >_{-p} \phi_k^{-p}[\phi_j^p] \\ &= \sum_{j=1}^{\infty} < v, \phi_j^{-p} >_{-p} < w, \phi_j^{-p} >_{-p} = < v, w >_{-p} \end{aligned}$$

and

$$\begin{aligned} w\left[\theta_p \sqrt{Q_p}' v\right] &= \left\langle w, \sqrt{Q_p}' v \right\rangle_{-p} = \left\langle \sqrt{Q_p}' w, v \right\rangle_{-p} \\ &= \left(\sqrt{Q_p}' w\right)[\theta_p v] = w\left[\sqrt{Q_p}\theta_p v\right]. \end{aligned}$$

iii) If $f_0 \in \Phi_{-p}$ such that $\left\langle f_0, \sqrt{Q_p}'\theta_{-p}\phi \right\rangle_{-p} = 0$ for any $\phi \in \Phi$, then $\sqrt{Q_p}' f_0 = 0$ and hence, $f_0 = 0$. i.e., $\sqrt{Q_p}'\theta_{-p}\Phi$ is dense in Φ_{-p}. As $\mathcal{R}(\iota)$ is dense in H_Q, we only need to show that $\sqrt{Q_p}'$ is an isometry from $\sqrt{Q_p}'\theta_{-p}\Phi$ onto $\mathcal{R}(\iota)$. Note that $\forall \phi, \psi \in \Phi$, we have

$$\begin{aligned} Q(\phi,\psi) &= \left\langle \sqrt{Q_p}\phi, \sqrt{Q_p}\psi \right\rangle_p \\ &= \left(\theta_{-p}\sqrt{Q_p}\phi\right)\left[\sqrt{Q_p}\psi\right] \\ &= \left(\sqrt{Q_p}'\theta_{-p}\sqrt{Q_p}\phi\right)[\psi]. \end{aligned}$$

Therefore for $\phi \in \Phi$

$$Q(\phi,\cdot) = \sqrt{Q_p}'\theta_{-p}\sqrt{Q_p}\phi = \sqrt{Q_p}'\sqrt{Q_p}'\theta_{-p}\phi$$

and

$$\|Q(\phi,\cdot)\|^2_{H_Q} = Q(\phi,\phi) = \left\|\sqrt{Q_p}'\theta_{-p}\phi\right\|^2_{-p}.$$

Hence $\sqrt{Q_p}'$ is an isometry from $\sqrt{Q_p}'\theta_{-p}\Phi$ onto $\mathcal{R}(\iota)$. ∎

Theorem 3.2.5 *Let Q be a covariance functional on $\Phi \times \Phi$ and let H_Q be constructed in Lemma 3.2.1. Then there exists a one-to-one correspondence between a Φ'-valued Wiener process W with covariance Q and an H_Q-c.B.m. B:*

$$W_t = \sum_{j=1}^{\infty} B_t(f_j)f_j \tag{3.2.9}$$

where $\{f_j\}$ is a CONS of H_Q;

$$B_t(v) = \lim_{n\to\infty} W_t[\iota^{-1}v_n],\ \forall v \in H_Q \tag{3.2.10}$$

where $\{v_n\} \subset \mathcal{R}(\iota)$ converges to v in H_Q.

Proof: First we assume that W is a Φ'-valued Wiener process and define B by (3.2.10). It follows from Doob's inequality that

$$\begin{aligned} & E\sup_{0\le t\le T}\left|W_t[\iota^{-1}v_n] - W_t[\iota^{-1}v_m]\right|^2 \\ =\ & 4TQ\left(\iota^{-1}(v_n - v_m), \iota^{-1}(v_n - v_m)\right) \\ =\ & 4T\|v_n - v_m\|^2_{H_Q} \to 0. \end{aligned} \tag{3.2.11}$$

Hence (3.2.10) is well-defined and $B.(v)$ is a real-valued continuous process. Further, let $0 = t_0 < t_1 < \cdots < t_k$ and $\lambda_1, \lambda_2, \cdots, \lambda_k \in \mathbf{R}$. Then

$$E\exp\left(i\sum_{j=1}^{k}\lambda_j\left(B_{t_j}(v) - B_{t_{j-1}}(v)\right)\right)$$

$$
\begin{aligned}
&= \lim_{n\to\infty} E\exp\left(i\sum_{j=1}^{k}\lambda_j\left(W_{t_j}[\iota^{-1}v_n]-W_{t_{j-1}}[\iota^{-1}v_n]\right)\right)\\
&= \lim_{n\to\infty}\prod_{j=1}^{k}\exp\left(-\frac{1}{2}(t_j-t_{j-1})\lambda_j^2 Q(\iota^{-1}v_n,\iota^{-1}v_n)\right)\\
&= \prod_{j=1}^{k}\exp\left(-\frac{1}{2}(t_j-t_{j-1})\lambda_j^2\|v\|_{H_Q}^2\right).
\end{aligned}
$$

Therefore $\{\|v\|_{H_Q}^{-1}B_t(v) : t\ge 0\}$ is a real-valued Brownian motion. This proves (i) of Definition 3.2.1.

For $v_1, v_2 \in V$, $\alpha_1, \alpha_2 \in \mathbf{R}$ and $t \ge 0$, note that

$$W_t[\alpha_1\iota^{-1}v_n^1+\alpha_2\iota^{-1}v_n^2] \to B_t(\alpha_1 v_1+\alpha_2 v_2)$$

and

$$\alpha_1 W_t[\iota^{-1}v_n^1]+\alpha_2 W_t[\iota^{-1}v_n^2] \to \alpha_1 B_t(v_1)+\alpha_2 B_t(v_2)$$

in the sense of (3.2.11), where $\{v_n^1\}, \{v_n^2\} \subset \mathcal{R}(\iota)$ such that $v_n^1 \to v_1$, $v_n^2 \to v_2$ in H_Q. (ii) of Definition 3.2.1 follows easily.

As $\mathcal{F}_t^B \subset \mathcal{F}_t^W$, it follows from (3.2.11) that $\forall v \in H_Q$, $A \in \mathcal{F}_t^B$, $r > t$

$$
\begin{aligned}
E(B_r(v)1_A) &= \lim_{n\to\infty} E(W_r[\iota^{-1}v_n]1_A)\\
&= \lim_{n\to\infty} E(W_t[\iota^{-1}v_n]1_A)\\
&= E(B_t(v)1_A),
\end{aligned}
$$

i.e., $B_t(v)$ is a $\mathcal{F}_t^B$-martingale. This proves (iii) of Definition 3.2.1 and hence B is an H_Q-c.B.m.

On the other hand, let B be an H_Q-c.B.m. and define W by (3.2.9). Let r_2 be given by Lemma 3.2.2 and $p \ge r_2$ such that the canonical injection from Φ_{-r_2} to Φ_{-p} is Hilbert-Schmidt. Then

$$
\begin{aligned}
&E\sup_{0\le t\le T}\left\|\sum_{j=n+1}^{n+k}B_t(f_j)f_j\right\|_{-p}^2\\
&= E\sup_{0\le t\le T}\sum_{i=1}^{\infty}\left(\sum_{j=n+1}^{n+k}B_t(f_j)\left\langle f_j,\phi_i^{-p}\right\rangle_{-p}\right)^2\\
&\le \sum_{i=1}^{\infty}4E\left(\sum_{j=n+1}^{n+k}B_T(f_j)\left\langle f_j,\phi_i^{-p}\right\rangle_{-p}\right)^2\\
&= \sum_{i=1}^{\infty}4T\sum_{j=n+1}^{n+k}\left\langle f_j,\phi_i^{-p}\right\rangle_{-p}^2
\end{aligned}
$$

$$= \quad 4T \sum_{j=n+1}^{n+k} \|f_j\|_{-p}^2 \to 0$$

as the canonical injection from H_Q to Φ_{-p} given by the composition $H_Q \to \Phi_{-r_2} \to \Phi_{-p}$ is Hilbert-Schmidt. Therefore (3.2.9) is well-defined and W_t is a continuous Φ_{-p}-valued process. As in the first part of the proof of this theorem, we can show that W_t satisfies the conditions of Definition 3.2.5, i.e., W is a Φ'-valued Wiener process. ∎

Corollary 3.2.1 *For any covariance functional Q on $\Phi \times \Phi$, there exists a Φ'-valued Wiener process W with covariance Q and there exists $p \geq 0$ depending only on Q such that*

$$W. \in C(\mathbf{R}_+, \Phi_{-p}) \quad a.s.$$

where $C(\mathbf{R}_+, \Phi_{-p})$ is the space of strongly continuous functions from $\mathbf{R}_+$ to Φ_{-p}.

Proof: Let H_Q be constructed by Lemma 3.2.1. It follows from Theorem 3.2.2 that there exists an H_Q-c.B.m. and then by Theorem 3.2.5, we obtain the results of the corollary. ∎

Remark 3.2.7 *Let (Φ, H, T_t) be a special compatible family defined in Section 1.3. Suppose that Q is a covariance functional on $\Phi \times \Phi$, then there exists a Φ'-valued Wiener process W with covariance Q such that*

$$W. \in C(\mathbf{R}_+, \Phi_{-p}) \ a.s.$$

for any $p \geq r_1 + r_2$ where r_1 is given by (1.3.17) and r_2 is given by Lemma 3.2.2.

Remark 3.2.8 *It follows from Corollary 3.2.1 that the condition (iii) in Theorem 3.1.4 is not necessary.*

Now we introduce some examples of Φ'-valued Wiener processes.

Example 3.2.1 *Let (Φ, H, L) be a special compatible family such that $H = L^2([0,b])$ (cf. Remark 1.3.4). Let $W(t,x)$ be a Brownian sheet on $\mathbf{R}_+ \times [0,b]$. Let W_t be a Φ'-valued process defined by*

$$W_t[\phi] = \int_0^t \int_0^b \phi(x) W(dsdx) \qquad \forall \phi \in \Phi.$$

It is easy to see that $\{W_t\}$ is a Φ'-valued Wiener process with covariance functional Q given by

$$Q(\phi, \psi) = <\phi, \psi>_H \qquad \forall \ \phi, \psi \in \Phi.$$

Further $W. \in C(\mathbf{R}_+, \Phi_{-p})$ for $p \geq r_1$.

Example 3.2.2 *Let (Φ, H, L) be a special compatible family (see Remark 1.3.4). Recall that the injection from Φ_q to Φ_p is a Hilbert-Schmidt map for $q > p + r_1$. Let $< \cdot, \cdot >_0$ be the inner product in H and define*

$$Q_0(\phi, \psi) = < \phi, \psi >_0 \qquad \phi, \psi \in \Phi.$$

Then from Corollary 3.2.1 there exists a Φ'-valued Wiener process W with covariance Q_0 such that

$$W_. \in C(\mathbf{R}_+, \Phi_{-p}) \quad a.s. \text{ if } \quad p > r_1$$

and will be called a **standard Wiener process**. *More generally, if $r > 0$ and*

$$Q_r(\phi, \psi) = < \phi, \psi >_r \qquad \phi, \psi \in \Phi$$

then there exists a Φ'-valued Wiener process W with covariance Q_r such that $W_. \in C(\mathbf{R}_+, \Phi_{-p})$ for $p > r + r_1$.

As will be shown in later examples, in applications the Q is not always given by one of the inner products on the Hilbert spaces defining Φ. Nevertheless since Q is continuous on $\Phi \times \Phi$, then, as in the proof of Lemma 3.2.2, there exist $\theta > 0$ and $r_2 \geq 0$ such that

$$Q(\phi, \phi) \leq \theta ||\phi||_{r_2}^2, \qquad \forall \phi \in \Phi$$

and therefore there exists a Φ'-valued Wiener process W with covariance Q such that

$$W_. \in C(\mathbf{R}_+, \Phi_{-p}) \qquad a.s.$$

for any $p \geq r_1 + r_2$.

Example 3.2.3 *Let $\mathcal{S}(\mathbf{R})$ be the Schwartz space of Example 1.3.1 (see also Remark 1.3.5). Then $(\mathcal{S}, L^2(\mathbf{R}), -d^2/dx^2 + x^2/4)$ is a special compatible family where $\{\phi_j\}_{j\geq 1}$ are the Hermite functions given by (1.3.10), $\lambda_j = j - 1/2$, $j \geq 1$, $< \cdot, \cdot >_0$ is the inner product on $L^2(\mathbf{R})$ and $r_1 > 1/2$. Taking $\Phi = \mathcal{S}(\mathbf{R})$ and $H = L^2(\mathbf{R})$ in the last example, we have that if $Q_0(\phi, \psi) = < \phi, \psi >_0$ then the standard Wiener process W in $\mathcal{S}'(\mathbf{R})$ is such that $W \in C(\mathbf{R}_+, \mathcal{S}'_p)$ for $p > 1/2$. Clearly, there is no smallest p such that this happens.*

For $\phi \in \Phi$ define

$$W_t^{(1)}[\phi] = W_t[D^2\phi] \quad \text{where } D = \frac{d}{dx}.$$

Then the covariance functional of the Φ'-valued Wiener process $W^{(1)} = (W_t^{(1)})_{t\geq 0}$ is

$$Q^{(1)}(\phi, \psi) = Q_0(D^2\phi, D^2\psi) = < D^2\phi, D^2\psi >_0 .$$

We shall show that $W \in C(\mathbf{R}_+, \mathcal{S}'_p)$ for $p > 3/2$. In general we will prove the following: Let $Q_r(\phi, \psi) = <\phi, \psi>_r$, $\forall \phi, \psi \in \Phi$ and $r \geq 0$, and let $W = (W_t)_{t\geq 0}$ be the corresponding $\mathcal{S}'$-valued Wiener process. Define

$$W_t^{(1)}[\phi] = W_t[D^2\phi] \tag{3.2.12}$$

then $W^{(1)}$ is a $\mathcal{S}'$-valued Wiener process such that $W^{(1)} \in C(\mathbf{R}_+, \mathcal{S}'_p)$ for $p > r + 3/2$.

Clearly

$$Q^{(1)}(\phi, \psi) = < D^2\phi, D^2\psi >_r \qquad \phi, \psi \in \Phi$$

then from Example 1.3.1 for $\phi \in \Phi$

$$\begin{aligned} Q^{(1)}(\phi, \phi) &= \sum_{n=1}^{\infty} \left(n + \frac{1}{2}\right)^{2r} < D^2\phi, \phi_n >_0^2 \\ &= \sum_{n=1}^{\infty} \left(n + \frac{1}{2}\right)^{2r} < \phi, D^2\phi_n >_0^2 . \end{aligned}$$

It follows from the proof of lemma 1.3.4 that

$$\phi'_n(x) = \frac{\sqrt{n-1}}{2}\phi_{n-1}(x) - \frac{\sqrt{n}}{2}\phi_{n+1}(x)$$

and hence

$$\begin{aligned} &\phi''_n(x) \\ = &\frac{\sqrt{n-1}}{2}\left\{\frac{\sqrt{n-2}}{2}\phi_{n-2}(x) - \frac{\sqrt{n-1}}{2}\phi_n(x)\right\} \\ &-\frac{\sqrt{n}}{2}\left\{\frac{\sqrt{n}}{2}\phi_n(x) - \frac{\sqrt{n+1}}{2}\phi_{n+2}(x)\right\} \\ = &\frac{\sqrt{(n-1)(n-2)}}{4}\phi_{n-2}(x) - \frac{2n-1}{4}\phi_n(x) - \frac{\sqrt{n(n+1)}}{4}\phi_{n+2}(x). \end{aligned}$$

Therefore

$$\begin{aligned} &Q^{(1)}(\phi, \phi) \\ = &\sum_{n=1}^{\infty} \left(n + \frac{1}{2}\right)^{2r} \Big\langle \phi, \\ &\frac{\sqrt{(n-1)(n-2)}}{4}\phi_{n-2} - \frac{2n-1}{4}\phi_n - \frac{\sqrt{n(n+1)}}{4}\phi_{n+2}\Big\rangle_0^2 \\ \leq &\frac{3}{16}\sum_{n=1}^{\infty} \left(n + \frac{1}{2}\right)^{2r} \{(n-1)(n-2) < \phi, \phi_{n-2} >_0^2 \end{aligned}$$

$$
\begin{aligned}
& +(2n-1)^2 < \phi, \phi_n >_0^2 + n(n+1) < \phi, \phi_{n+2} >_0^2 \} \\
\leq\ & \frac{3}{16} \sum_{n=1}^{\infty} \left(n + \frac{1}{2} \right)^{2r+2} \{ < \phi, \phi_{n-2} >_0^2 + 4 < \phi, \phi_n >_0^2 + < \phi, \phi_{n+2} >_0^2 \} \\
\leq\ & \alpha \| \phi \|_{r+1}^2
\end{aligned}
$$

where α is a constant. Since the injection $\mathcal{S}_p \to \mathcal{S}_{r+1}$ is Hilbert-Schmidt for $p > r + 1 + \frac{1}{2} = r + \frac{3}{2}$, we have shown that the $\mathcal{S}'$-valued Wiener process given by (3.2.12) is such that, for $p > r + 3/2$,

$$
W^{(1)} \in C(\mathbf{R}_+, \mathcal{S}'_p) \quad \text{a.s.}
$$

3.3 Stochastic integral with respect to H-c.B.m and Φ'-Wiener process

In this section, we discuss stochastic integrals with respect to H-c.B.m. and with respect to Φ'-valued Wiener process. We shall also obtain stochastic representations for H-valued and Φ'-valued continuous martingales.

3.3.1 Stochastic integral

Let H and K be two separable Hilbert spaces and let B be an H-c.B.m. Let L_B^2 be the collection of all $L_{(2)}(H, K)$-valued predictable processes f such that

$$
E \int_0^T \| f(t, \omega) \|_{(2)}^2 dt < \infty, \ \forall T > 0.
$$

Definition 3.3.1 *For $f \in L_B^2$, we define*

$$
I_t(f) = \sum_j \left(\sum_i \int_0^t < f(s, \omega)' g_j, f_i >_H dB_s(f_i) \right) g_j, \quad t \geq 0
$$

where $\{f_j\}$ and $\{g_j\}$ are CONS of H and K respectively and $f(s, \omega)' \in L(K, H)$ denotes the dual operator of $f(s, \omega) \in L(K, H)$.

Theorem 3.3.1 *$I(f) \in \mathcal{M}^{2,c}(K)$ is well-defined.*

Proof: First we show that $\forall j \geq 1$,

$$
I_t(f)_j \equiv \sum_i \int_0^t < f(s, \omega)' g_j, f_i >_H dB_s(f_i)
$$

converges. In fact

$$
\begin{aligned}
& E \sup_{0\le t\le T} \left| \sum_{i=n+1}^{n+k} \int_0^t < f(s,\omega)' g_j, f_i >_H dB_s(f_i) \right|^2 \\
\le \; & 4E \left| \sum_{i=n+1}^{n+k} \int_0^T < f(s,\omega)' g_j, f_i >_H dB_s(f_i) \right|^2 \\
= \; & 4 \sum_{i=n+1}^{n+k} E \int_0^T < f(s,\omega)' g_j, f_i >_H^2 ds \to 0
\end{aligned}
$$

as $n \to \infty$. As a consequence, we see that $I(f)_j \in \mathcal{M}^{2,c}(\mathbf{R})$ is well-defined, $\forall j \ge 1$.

Next we show that $I(f)_j$ does not depend on the choice of the CONS $\{f_i\}$ of H. Let $\{\tilde{f}_i\}$ be another CONS of H and let $\tilde{I}(f)_j \in \mathcal{M}^{2,c}(\mathbf{R})$ be given by

$$
\tilde{I}_t(f)_j = \sum_i \int_0^t < f(s,\omega)' g_j, \tilde{f}_i >_H dB_s(\tilde{f}_i).
$$

Then

$$
\begin{aligned}
& E \left| I_t(f)_j - \tilde{I}_t(f)_j \right|^2 = E|I_t(f)_j|^2 + E|\tilde{I}_t(f)_j|^2 - 2E I_t(f)_j \tilde{I}_t(f)_j \\
= \; & \sum_i \int_0^t E < f(s,\omega)' g_j, f_i >_H^2 ds + \sum_i \int_0^t E < f(s,\omega)' g_j, \tilde{f}_i >_H^2 ds \\
& -2 \sum_{i,r} \int_0^t E < f(s,\omega)' g_j, f_i >_H < f(s,\omega)' g_j, \tilde{f}_r >_H < f_i, \tilde{f}_r >_H ds \\
= \; & 2 \int_0^t E \| f(s,\omega)' g_j \|_H^2 ds - 2 \int_0^t E \| f(s,\omega)' g_j \|_H^2 ds = 0.
\end{aligned}
$$

By similar arguments, we can show that

$$
I_t(f) = \sum_j I(f)_j g_j
$$

converges and does not depend on the choice of the CONS $\{g_j\}$ of K. This proves our assertion. ∎

As a consequence of the definition we have the following inequality.

Theorem 3.3.2 *For $2 \le p < \infty$, there exist constants C_p depending only on p such that for a predictable $L_2(H,K)$-valued process f with*

$$
E \left[\left\{ \int_0^T \| f(s,\omega) \|_2^2 ds \right\}^{p/2} \right] < \infty
$$

one has

$$E\left[\sup_{t\leq T}\left\|\int_0^t f(s,\omega)dW_s\right\|_K^p\right] \leq C_p E\left[\left\{\int_0^T \|f(s,\omega)\|_2^2 ds\right\}^{p/2}\right].$$

Proof: It follows from Burkholder's inequality for finite dimensional martingale $(I_t(f)_1, \cdots, I_t(f)_d)$ that

$$E\left[\sup_{t\leq T}\left\{\sum_{j=1}^d I_t(f)_j^2\right\}^{p/2}\right] \leq C_p E\left[\left\{\sum_{j=1}^d \int_0^T \|f(s,\omega)f_j\|_K^2 ds\right\}^{p/2}\right].$$

The required inequality follows from this, using Fatou's lemma. ∎

Let W be a Φ'-valued Wiener process with covariance Q. The space of integrands, L_Q^2 consists of those predictable functions $f : \mathbf{R}_+ \times \Omega \to L(\Phi', \Phi')$ for which

$$E\int_0^T Q(f(s,\omega)'\phi, f(s,\omega)'\phi)ds < \infty, \qquad \forall T > 0,\ \phi \in \Phi.$$

Theorem 3.3.3 *Let $f \in L_Q^2$. Then for $T > 0$, there exists $p \equiv p_T \geq 0$ such that f can be regarded as a predictable map from $[0,T]\times\Omega$ to $L_{(2)}(H_Q, \Phi_{-p})$ and*

$$E\int_0^T \|f(s,\omega)\|^2_{L_{(2)}(H_Q,\Phi_{-p})} ds < \infty.$$

Proof: Define a map V_T from Φ to $[0,\infty)$ by

$$V_T(\phi)^2 \equiv E\int_0^T Q(f(s,\omega)'\phi, f(s,\omega)'\phi)ds.$$

It is easy to see that V_T satisfies the conditions of Lemma 1.3.1 and hence, there exist $\theta > 0$ and $r \geq 0$ such that

$$V_T(\phi) \leq \theta\|\phi\|_r,\ \forall \phi \in \Phi.$$

Let $p > r$ be such that the canonical injection from Φ_p to Φ_r is Hilbert-Schmidt. Note that for $\phi \in \Phi \subset \Phi_p \subset H_Q'$,

$$\begin{aligned}\|\phi\|^2_{H_Q'} &= \sup\left\{Q(\psi,\cdot)[\phi]^2/\|Q(\psi,\cdot)\|^2_{H_Q} : \psi \in \Phi\right\} \\ &= \sup\left\{Q(\psi,\phi)^2/Q(\psi,\psi) : \psi \in \Phi\right\} = Q(\phi,\phi).\end{aligned}$$

Hence

$$\begin{aligned}E\int_0^T \sum_j \|f(s,\omega)'\phi_j^p\|^2_{H_Q'} ds &= E\int_0^T \sum_j Q(f(s,\omega)'\phi_j^p, f(s,\omega)'\phi_j^p)ds \\ &= \sum_j \theta\|\phi_j^p\|_r^2 < \infty.\end{aligned}$$

Therefore $f(t,\omega)' \in L_{(2)}(\Phi_p, H'_Q)$, dtdP-a.e. and hence,

$$f(t,\omega) = f(t,\omega)'' \in L_{(2)}(H_Q, \Phi_{-p}) \qquad \text{dtdP-a.e.}$$

such that

$$E\int_0^T \|f(s,\omega)\|^2_{L_{(2)}(H_Q,\Phi_{-p})} ds = E\int_0^T \|f(s,\omega)'\|^2_{L_{(2)}(\Phi_p,H'_Q)} ds < \infty.$$

Let $\{v_j\}$ be a CONS of H_Q. As

$$L_{(2)}(H_Q, \Phi_{-p}) \cap L(\Phi', \Phi') = \left\{ \ell \in L(\Phi', \Phi') : \sum_j \|\ell v_j\|^2_{-p} < \infty \right\},$$

is a measurable subset of $L(\Phi', \Phi')$, f can be regarded as a predictable map from $[0,T] \times \Omega$ to $L_{(2)}(H_Q, \Phi_{-p})$. ∎

Based on Theorems 3.3.1 and 3.3.3, we now introduce the stochastic integral with respect to a Φ'-Wiener process W.

Definition 3.3.2 *Let B be the H_Q-c.B.m. given by W in Theorem 3.2.5 and $f \in L^2_Q$. For any $T > 0$, let $p = p_T$ be given by Theorem 3.3.3. For $t \le T$, we define*

$$M_t \equiv \int_0^t f(s,\omega) dW_s \equiv \int_0^t f(s,\omega) dB_s,$$

i.e.

$$M_t[\phi] = \sum_j \int_0^t (f(s,\omega)v_j)[\phi] dB_s(v_j) \tag{3.3.1}$$

where $\{v_j\}$ is a CONS of H_Q. As

$$E\int_0^T \|f(s,\omega)\|^2_{L_{(2)}(H_Q,\Phi_{-p})} ds < \infty,$$

M_t, given by (3.3.1), is a well-defined Φ_{-p}-valued martingale for $t \in [0,T]$.

The following theorem follows directly from Theorems 3.3.1, 3.3.3 and Definition 3.3.2.

Theorem 3.3.4 *M in Definition 3.3.2 is a well-defined element in $\mathcal{M}^{2,c}(\Phi')$. Further, if $p = p_T$ is given by Theorem 3.3.3, then*

$$M|_{[0,T]} \in C([0,T], \Phi_{-p}).$$

3.3.2 Representation theorems

Now we consider the stochastic integral representation for H-valued continuous square-integrable martingales. First we fix $T > 0$ and let M be an H-valued continuous square-integrable martingale. Let $f : [0,T] \times \Omega \to L_{(2)}(H,H)$ be predictable and

$$< M_t(h^1), M_t(h^2) >= \int_0^t < f(s,\omega)h^1, f(s,\omega)h^2 >_H ds \qquad (3.3.2)$$

where $h^j \in H$, $M_t(h^j) =< M_t, h^j >_H$, $j = 1,2$, and the left hand side of (3.3.2) is the quadratic covariation process of the martingales $M_t(h^1)$ and $M_t(h^2)$.

Definition 3.3.3 *We say a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \tilde{\mathcal{F}}_t)$ is an* **extension** *of a stochastic basis $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ if there exists a map $\pi : \tilde{\Omega} \to \Omega$ which is $\tilde{\mathcal{F}}/\mathcal{F}$-measurable such that i) $\tilde{\mathcal{F}}_t \supset \pi^{-1}(\mathcal{F}_t)$; ii) $P = \tilde{P}\pi^{-1}$ and iii) for every bounded random variable X on Ω,*

$$\tilde{E}(\tilde{X}(\tilde{\omega})|\tilde{\mathcal{F}}_t) = E(X|\mathcal{F}_t)(\pi\tilde{\omega}) \quad \tilde{P}\text{-a.s.},$$

where $\tilde{X}(\tilde{\omega}) = X(\pi\tilde{\omega})$, for $\tilde{\Omega} \in \tilde{\Omega}$. We shall denote $\tilde{X}$ by X if its meaning is clear from the context.

$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \tilde{\mathcal{F}}_t)$ is called a **standard extension** *of a stochastic basis $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ if we have another stochastic basis $(\Omega', \mathcal{F}', P', \mathcal{F}'_t)$ such that*

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \tilde{\mathcal{F}}_t) = (\Omega, \mathcal{F}, P, \mathcal{F}_t) \times (\Omega', \mathcal{F}', P', \mathcal{F}'_t)$$

and $\pi\tilde{\omega} = \omega$ for $\tilde{\omega} = (\omega, \omega') \in \tilde{\Omega}$.

Theorem 3.3.5 *Let $M \in \mathcal{M}^{2,c}(H)$ such that (3.3.2) holds and*

$$E\int_0^T \|f(s,\omega)\|_{(2)}^2 ds < \infty.$$

Then, on a standard extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \tilde{\mathcal{F}}_t)$ of $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$, there exists an H-c.B.m B_t such that

$$M_t = \int_0^t f(s,\omega) dB_s. \qquad (3.3.3)$$

Proof: We divide the proof into three steps. For simplicity of notations, we suppress ω and write $f(s)$, $g_n(s)$, $R(s)$ for $f(s,\omega)$, $g_n(s,\omega)$, $R(s,\omega)$.
Step 1. We construct an H-c.B.m. B_t under the assumption that $\forall (s,\omega) \in [0,T] \times \Omega$, $f(s,\omega)$ is a non-negative definite self-adjoint Hilbert-Schmidt operator.

Let $g_n(s) = f(s)(f(s)^2+n^{-1}I)^{-1}$. It is easy to see that $\|f(s)g_n(s)\|_{L(H,H)} \leq 1$ and $\|g_n(s)\|_{L(H,H)} \leq \frac{\sqrt{n}}{2}$. Let $R(s)$ be the (orthogonal) projection operator from H to the range of $f(s)^2$. Then $f(s)g_n(s) = g_n(s)f(s) \to R(s)$ in $L(H,H)$ as $n \to \infty$. Let $(\Omega', \mathcal{F}', P', \mathcal{F}'_t)$ be a stochastic basis and B'_t be an H-c.B.m on this basis. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \tilde{\mathcal{F}}_t) = (\Omega, \mathcal{F}, P, \mathcal{F}_t) \times (\Omega', \mathcal{F}', P', \mathcal{F}'_t)$ be a standard extension of the stochastic basis $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$. On this extension, let

$$\begin{aligned} B_t^{(n,J)}(h) &= \sum_{j=1}^{J} \int_0^t < g_n(s)h, f_j >_H dM_s(f_j) \\ &\quad + \sum_{j=1}^{J} \int_0^t < (I - R(s))h, f_j >_H dB'_s(f_j) \end{aligned}$$

for any $t \in [0,T]$, $h \in H$, $n, J \in \mathbf{N}$, where $\{f_j\}$ is a CONS of H. Then for $h^1, h^2 \in H$, the quadratic covariation process is given by

$$\begin{aligned} &< B^{(n,J)}(h^1), B^{(m,K)}(h^2) >_t \\ &= \int_0^t < f(s)\pi_J g_n(s)h^1, f(s)\pi_K g_m(s)h^2 >_H ds \\ &\quad + \int_0^t < \pi_{J\wedge K}(I - R(s))h^1, (I - R(s))h^2 >_H ds \end{aligned}$$

where π_J is the projection operator from H to the linear span of $\{f_j : 1 \leq j \leq J\}$ on H. By the dominated convergence theorem, as $J \to \infty$

$$\begin{aligned} &E \sup_{0\leq t\leq T} \left|B^{(n,J+k)}(h) - B^{(n,J)}(h)\right|^2 \\ &\leq 4E\left\langle B^{(n,J+k)}(h) - B^{(n,J)}(h)\right\rangle_T \\ &= 4\int_0^T \|f(s)(\pi_{J+k} - \pi_J)g_n(s)h\|_H^2 ds \\ &\quad +4\int_0^T \|(\pi_{J+k} - \pi_J)(I - R(s))h\|_H^2 ds \to 0. \end{aligned} \tag{3.3.4}$$

Therefore $B^{(n,J)}(h)$ converges to a real-valued continuous square-integrable martingale, say $B^{(n)}(h)$. Then

$$\begin{aligned} < B^{(n)}(h^1), B^{(m)}(h^2) >_t &= \int_0^t < f(s)g_n(s)h^1, f(s)g_m(s)h^2 >_H ds \\ &\quad + \int_0^t < (I - R(s))h^1, h^2 >_H ds \end{aligned}$$

and

$$< B^{(n,J)}(h^1), B^{(n)}(h^2) >_t$$

$$= \int_0^t < f(s)\pi_J g_n(s)h^1, f(s)g_n(s)h^2 >_H ds$$
$$+ \int_0^t < \pi_J(I - R(s))h^1, (I - R(s))h^2 >_H ds.$$

Proceeding as in (3.3.4) we can prove that $B^{(n)}(h)$ converges to a real-valued continuous square-integrable martingale, say $B(h)$, and

$$\begin{aligned} < B(h^1), B(h^2) >_t &= \int_0^t < R(s)h^1, h^2 >_H ds \\ &\quad + \int_0^t < (I - R(s))h^1, h^2 >_H ds \\ &= t < h^1, h^2 >_H . \end{aligned}$$

It is easy to verify the conditions of Definition 3.2.1 and hence, B is an H-c.B.m.
Step 2. We now obtain the representation (3.3.3). Let

$$\tau_m = \inf\{t \in [0, T] : \|f(t)\|_{(2)} > m\}.$$

Note that

$$\sum_{j=1}^{J} \int_0^{t\wedge\tau_m} < f(s)h, f_j >_H dB_s^{(n,K)}(f_j) \tag{3.3.5}$$
$$= \sum_{k=1}^{K} \int_0^{t\wedge\tau_m} < g_n(s)\pi_J f(s)h, f_k >_H dM_s(f_k)$$
$$+ \sum_{k=1}^{K} \int_0^{t\wedge\tau_m} < (I - R(s))\pi_J f(s)h, f_k >_H dB'_s(f_k).$$

As for $s < \tau_m$,

$$\|f(s)\pi_K g_n(s)(\pi_{J+k} - \pi_J)f(s)h\|_H^2 \le m^4 \frac{n}{4} \|h\|_H^2$$

and

$$\|\pi_K(I - R(s))(\pi_{J+k} - \pi_J)f(s)h\|_H^2 \le m^2 \|h\|_H^2,$$

it follows from the dominated convergence theorem that, as $J \to \infty$,

$$E\left| \sum_{j=J+1}^{J+k} \int_0^{t\wedge\tau_m} < f(s)h, f_j >_H dB_s^{(n,K)}(f_j) \right|^2$$
$$= \sum_{i,j=J+1}^{J+k} E \int_0^{t\wedge\tau_m} < f(s)h, f_i >_H < f(s)h, f_j >_H$$

$$
\begin{aligned}
& < f(s)\pi_K g_n(s) f_i, f(s)\pi_K g_n(s) f_j >_H ds \\
& + \sum_{i,j=J+1}^{J+k} E \int_0^{t\wedge\tau_m} < f(s)h, f_i >_H < f(s)h, f_j >_H \\
& \quad < \pi_K(I - R(s)) f_i, (I - R(s)) f_j >_H ds \\
= \; & E \int_0^{t\wedge\tau_m} \| f(s)\pi_K g_n(s)(\pi_{J+k} - \pi_J) f(s)h \|_H^2 ds \\
& + E \int_0^{t\wedge\tau_m} \| \pi_K (I - R(s))(\pi_{J+k} - \pi_J) f(s) h \|_H^2 ds \to 0,
\end{aligned}
$$

i.e. the left hand side of (3.3.5) converges to

$$
\sum_{j=1}^{\infty} \int_0^{t\wedge\tau_m} < f(s)h, f_j >_H dB_s^{(n,K)}(f_j).
$$

We can similarly derive the limit (as $J \to \infty$) of the right hand side of (3.3.5). Then

$$
\begin{aligned}
& \sum_{j=1}^{\infty} \int_0^{t\wedge\tau_m} < f(s)h, f_j >_H dB_s^{(n,K)}(f_j) \\
= \; & \sum_{k=1}^{K} \int_0^{t\wedge\tau_m} < g_n(s) f(s) h, f_k >_H dM_s(f_k) \\
& + \sum_{k=1}^{K} \int_0^{t\wedge\tau_m} < (I - R(s)) f(s) h, f_k >_H dB_s'(f_k) \\
= \; & \sum_{k=1}^{K} \int_0^{t\wedge\tau_m} < g_n(s) f(s) h, f_k >_H dM_s(f_k).
\end{aligned}
$$

Note that as $K \to \infty$, we have

$$
\begin{aligned}
& E \left| \sum_{j=1}^{\infty} \int_0^{t\wedge\tau_m} < f(s)h, f_j >_H d(B_s^{(n,K)}(f_j) - B_s^{(n)}(f_j)) \right|^2 \\
= \; & \lim_{J\to\infty} \sum_{i,j=1}^{J} E \int_0^{t\wedge\tau_m} < f(s)h, f_i >_H < f(s)h, f_j >_H \\
& \quad \langle f(s)(I - \pi_K) g_n(s) f_i, f(s)(I - \pi_K) g_n(s) f_j \rangle_H \, ds \\
& + \lim_{J\to\infty} \sum_{i,j=1}^{J} E \int_0^{t\wedge\tau_m} < f(s)h, f_i >_H < f(s)h, f_j >_H \\
& \quad \langle (I - \pi_K)(I - R(s)) f_i, (I - R(s)) f_j \rangle_H \, ds \\
= \; & \lim_{J\to\infty} E \int_0^{t\wedge\tau_m} \{ \| f(s)(I - \pi_K) g_n(s) \pi_J f(s) h \|_H^2
\end{aligned}
$$

$$
\begin{aligned}
&+\|(I-\pi_K)(I-R(s))\pi_J f(s)h\|_H^2\}ds \\
= \; & E\int_0^{t\wedge\tau_m} \|f(s)(I-\pi_K)g_n(s)f(s)h\|_H^2 ds \to 0
\end{aligned}
$$

and

$$
\begin{aligned}
& E\left|\sum_{k=K+1}^{K+\ell}\int_0^{t\wedge\tau_m} < g_n(s)f(s)h, f_k >_H dM_s(f_k)\right|^2 \\
= & \sum_{j,k=K+1}^{K+\ell} E\int_0^{t\wedge\tau_m} < g_n(s)f(s)h, f_j >_H < g_n(s)f(s)h, f_k >_H \\
& \quad < f(s)f_j, f(s)f_k >_H ds \\
= & \; E\int_0^{t\wedge\tau_m} \|f(s)(\pi_{K+\ell}-\pi_K)g_n(s)f(s)h\|_H^2 ds \to 0
\end{aligned}
$$

and therefore

$$
\begin{aligned}
& \sum_{j=1}^{\infty}\int_0^{t\wedge\tau_m} < f(s)h, f_j >_H dB_s^{(n)}(f_j) \\
= & \sum_{k=1}^{\infty}\int_0^{t\wedge\tau_m} < g_n(s)f(s)h, f_k >_H dM_s(f_k).
\end{aligned}
$$

Similarly, as $n \to \infty$,

$$
\sum_{j=1}^{\infty}\int_0^{t\wedge\tau_m} < f(s)h, f_j >_H dB_s^{(n)}(f_j) \to \sum_{j=1}^{\infty}\int_0^{t\wedge\tau_m} < f(s)h, f_j >_H dB_s(f_j),
$$

and

$$
\begin{aligned}
& E\left|M_{t\wedge\tau_m}(h) - \sum_{k=1}^{\infty}\int_0^{t\wedge\tau_m} < g_n(s)f(s)h, f_k >_H dM_s(f_k)\right|^2 \\
= & \; E\left|\sum_{k=1}^{\infty}\int_0^{t\wedge\tau_m} < (I-g_n(s)f(s))h, f_k >_H dM_s(f_k)\right|^2 \\
= & \lim_{K\to\infty}\int_0^{t\wedge\tau_m} \|f(s)\pi_K(I-g_n(s)f(s))h\|_H^2 ds \\
= & \; E\int_0^{t\wedge\tau_m} \|f(s)(I-g_n(s)f(s))h\|_H^2 ds \\
\to & \; E\int_0^{t\wedge\tau_m} \|f(s)(I-R(s))h\|_H^2 ds = 0.
\end{aligned}
$$

Therefore

$$
M_{t\wedge\tau_m}(h) = \sum_j \int_0^{t\wedge\tau_m} < f(s,\omega)h, f_j >_H dB_s(f_j).
$$

Letting $m \to \infty$, we see that (3.3.3) holds.
Step 3. For general f, let $p(s)$ be an $L(H,H)$-valued predictable process such that $p(s)'p(s) = I$ and $(f(s)'f(s))^{1/2} = f(s)p(s)$. As

$$< M_t(h^1), M_t(h^2) >= \int_0^t \left\langle (f(s)'f(s))^{\frac{1}{2}} h^1, (f(s)'f(s))^{\frac{1}{2}} h^2 \right\rangle_H ds,$$

by previous steps, there exists an H-c.B.m B_t such that

$$M_t = \int_0^t (f(s)'f(s))^{\frac{1}{2}} dB_s.$$

Let

$$\tilde{B}_t(h) = \sum_j \int_0^t < p(s)'h, f_j >_H dB_s(f_j) \qquad \forall h \in H.$$

Note that

$$< \tilde{B}_t(h) >= \sum_j \int_0^t < p(s)'h, f_j >_H^2 ds = t\|h\|_H^2,$$

it is easy to show that $\tilde{B}_s$ is an H-c.B.m. Using similar arguments as in step 2, we see that (3.3.3) holds with B replaced by $\tilde{B}$. ∎

Finally we consider the stochastic integral representation for Φ'-valued continuous square-integrable martingales.

Theorem 3.3.6 *Let Q be a covariance function on $\Phi \times \Phi$. Suppose that $M \in \mathcal{M}^{2,c}(\Phi')$ and there exists $f \in L_Q^2$ such that*

$$< M_t[\phi] >= \int_0^t Q(f(s)'\phi, f(s)'\phi) ds.$$

Then on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \tilde{\mathcal{F}}_t)$ of $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$, there exists a Φ'-Wiener process W with covariance Q such that

$$M_t = \int_0^t f(s) dW_s. \tag{3.3.6}$$

Proof: For each $n \in \mathbf{N}$, let p_n be given by Theorem 3.3.3 (with $T = n$). Then $M|_{[0,n]}$ is a Φ_{-p_n}-valued continuous square-integrable martingale such that $\forall h \in \Phi_{-p_n}$, $t \in [0,n]$

$$< M(h) >_t = \int_0^t \left\| \theta_{-p_n} \sqrt{Q_{p_n}} f(s)' \theta_{p_n} h \right\|_{-p_n}^2 ds.$$

Taking $H = \Phi_{-p_n}$, it follows from Theorem 3.3.5 that there exists a standard extension $(\Omega, \mathcal{F}, P, \mathcal{F}_t) \times (\Omega^n, \mathcal{F}^n, P^n, \mathcal{F}_t^n)$ of the stochastic basis $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ and a Φ_{-p_n}-c.B.m. B^n such that

$$M_t = \int_0^t \theta_{-p_n} \sqrt{Q_{p_n}} f(s)' \theta_{p_n} dB_s^n.$$

For any $v \in \Phi_{-p_n}$, let $\hat{B}_s^n(\sqrt{Q_{p_n}}' v) = B_s^n(v)$. Then $\hat{B}_s^n$ is an H_Q-c.B.m. for $s \in [0, n]$ and, for any $\phi \in \Phi$

$$\begin{aligned} M_t[\phi] &= \sum_j \int_0^t \left\langle \theta_{-p_n} \sqrt{Q_{p_n}} f(s)' \phi, \phi_j^{-p_n} \right\rangle_{-p_n} dB_s^n(\phi_j^{-p_n}) \\ &= \sum_j \int_0^t \left\langle f(s)' \phi, \sqrt{Q_{p_n}} \phi_j^{p_n} \right\rangle_{p_n} d\hat{B}_s^n \left(\sqrt{Q_{p_n}}' \phi_j^{-p_n} \right) \\ &= \sum_j \int_0^t \left(\sqrt{Q_{p_n}}' \phi_j^{-p_n} \right) [f(s)' \phi] d\hat{B}_s^n \left(\sqrt{Q_{p_n}}' \phi_j^{-p_n} \right). \quad (3.3.7) \end{aligned}$$

Let

$$(\Omega', \mathcal{F}', P', \mathcal{F}_t') = \prod_{n=1}^{\infty} (\Omega^n, \mathcal{F}^n, P^n, \mathcal{F}_t^n).$$

On the extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \tilde{\mathcal{F}}_t) = (\Omega, \mathcal{F}, P, \mathcal{F}_t) \times (\Omega', \mathcal{F}', P', \mathcal{F}_t')$, we define B inductively:

$$B_t = \begin{cases} \hat{B}_t^1, & \text{for } t \in [0, 1]; \\ \hat{B}_n + \hat{B}_t^{n+1} - \hat{B}_n^{n+1}, & \text{for } t \in [n, n+1), \ n \geq 1. \end{cases}$$

It is easy to see that B is an H_Q-c.B.m. Let W be the Φ'-Wiener process with covariance Q corresponding to B by Theorem 3.2.5. By Definition 3.3.2 and (3.3.7), we see that (3.3.6) holds. ∎

3.4 Stochastic integral with respect to Poisson random measure

In this section, we study the stochastic integral of Φ'-valued processes with respect to Poisson random measures. We shall derive a representation theorem for a class of purely-discontinuous Φ'-valued martingales.

First we recall some basic facts without proof about real-valued semimartingales. We refer the reader to the books of Ikeda and Watanabe [18] and Jacod and Shiryaev [22] for more details. Denote by $\mathcal{M}^2(\mathbf{R})$ ($\mathcal{M}^{2,c}(\mathbf{R})$) the collection of all (continuous) real-valued square-integrable martingales. Let $\mathcal{A}$ be the collection of all adapted processes whose sample paths are of finite variations on any finite intervals.

Definition 3.4.1 *$M \in \mathcal{M}^2(\mathbf{R})$ is* **purely-discontinuous** *if $M_0 = 0$ and for any $N \in \mathcal{M}^{2,c}(\mathbf{R})$, MN is a martingale. We denote the collection of all purely-discontinuous real-valued square-integrable martingales by $\mathcal{M}^{2,d}(\mathbf{R})$.*

Theorem 3.4.1 *For any $M \in \mathcal{M}^2(\mathbf{R})$, there exists a unique decomposition $M = M^c + M^d$ such that $M^c \in \mathcal{M}^{2,c}(\mathbf{R})$ and $M^d \in \mathcal{M}^{2,d}(\mathbf{R})$. They are called respectively the* **continuous** *and the* **purely-discontinuous part** *of M.*

For any $M, N \in \mathcal{M}^2(\mathbf{R})$, we define the **quadratic variation process**

$$[M,N]_t = \lim_{\lambda\to 0}\sum_{j=0}^{n-1}(M_{t_{j+1}} - M_{t_j})(N_{t_{j+1}} - N_{t_j})$$

where $0 = t_0 < t_1 < \cdots < t_n = t$ and $\lambda = \max\{t_{j+1} - t_j : 0 \le j < n\}$.

Theorem 3.4.2 *For any $M, N \in \mathcal{M}^2(\mathbf{R})$, we have $[M,N] \in \mathcal{A}$ and $MN - [M,N]$ is a real-valued martingale. Further*

$$[M,N]_t =< M^c, N^c >_t + \sum_{s\le t}\Delta M_s \Delta N_s,$$

where $\Delta M_s = M_s - M_{s-}$. As a consequence, $M \in \mathcal{M}^2(\mathbf{R})$ is purely-discontinuous iff $\forall t > 0$

$$[M,M]_t = \sum_{s\le t}(\Delta M_s)^2, \qquad a.s.$$

Definition 3.4.2 *Let $(U,\mathcal{E})$ be a measurable space. A map $N : \Omega\times(\mathcal{B}(\mathbf{R}_+)\times\mathcal{E}) \to \mathbf{R}$ is called a* **random measure** *if $N(\omega,\cdot)$ is a measure on $\mathbf{R}_+ \times U$ for each ω and $N(\cdot, B)$ is a random variable for each $B \in \mathcal{B}(\mathbf{R}_+)\times\mathcal{E}$. A random measure N is called* **adapted** *if $N(\cdot,B)$ is $\mathcal{F}_t$-measurable for $B \subset [0,t]\times U$. A random measure N is* **σ-finite** *if there exists a sequence U_n increasing to U such that $E|N(\cdot,[0,t]\times U_n)| < \infty$ for each $n \in \mathbf{N}$ and $t > 0$.*

A random measure N is called a **martingale random measure** *if for any $A \in \Gamma_N \equiv \{A \in \mathcal{E} : E|N([0,t]\times A)| < \infty\ \forall t > 0\}$, the stochastic process $N([0,t]\times A)$ is a martingale.*

A σ-finite adapted random measure N is said to be in the **class (QL)** *if there exists a σ-finite random measure $\hat{N}$ such that $\tilde{N} \equiv N - \hat{N}$ is a martingale random measure and for any $A \in \Gamma_N$, $\hat{N}([0,t]\times A) \in \mathcal{A}$ is continuous in t. The random measure $\hat{N}$ is called the* **compensator** *of N.*

Theorem 3.4.3 *Let $(U,\mathcal{E})$ be a measurable space and let N be an integer-valued adapted random measure on $\mathbf{R}_+ \times U$. Then, there exists a sequence of stopping times $\{\tau_n\}$ and a U-valued optional process p such that*

$$N(\omega, A) = \sum_{s\ge 0} 1_D(\omega,s)1_A(s,p_s(\omega)),\ \forall A \in \mathcal{B}(\mathbf{R}_+)\times\mathcal{E},$$

where

$$D = \cup_n\{(\omega, \tau_n(\omega)) : \omega \in \Omega\} \subset \Omega \times \mathbf{R}_+.$$

The set D and the process p are called the **jump set** *and the* **point process** *corresponding to the integer-valued random measure N.*

Definition 3.4.3 *A random measure N is called* **independently scattered** *if for any disjoint $B_1, \cdots, B_n \in \mathcal{B}(\mathbf{R}_+) \times \mathcal{E}$, the random variables $N(\cdot, B_1), \cdots, N(\cdot, B_n)$ are independent.*

An independently scattered integer-valued adapted random measure is called a **Poisson random measure** *if for any $B \in \mathcal{B}(\mathbf{R}_+) \times \mathcal{E}$ such that $(dtd\mu)(B) < \infty$, $N(\cdot, B)$ is a Poisson random variable with parameter $(dtd\mu)(B)$. μ is called the* **characteristic measure** *of N.*

It is clear that any Poisson random measure N is in class (QL) with $\hat{N}([0,t] \times A) = t\mu(A)$ for any $A \in \mathcal{E}$.

Definition 3.4.4 *A real-valued function $f(t, u, \omega)$ defined on $\mathbf{R}_+ \times U \times \Omega$ is* **predictable** *if it is $\mho/\mathcal{B}(\mathbf{R})$ measurable where $\mho$ is the smallest σ-field on $\mathbf{R}_+ \times U \times \Omega$ with respect to which all g having the following properties are measurable:*
i) for each $t > 0$, $(u, \omega) \to g(t, u, \omega)$ is $\mathcal{E} \times \mathcal{F}_t$-measurable;
ii) for each (u, ω), $t \to g(t, u, \omega)$ is left continuous.

Let N be a Poisson random measure with characteristic measure μ. We introduce the following classes:

$$F_N^j = \left\{ f(t, u, \omega) : \begin{array}{l} f \text{ is predictable and } \forall t > 0 \\ E\int_0^t \int_U |f(s, u, \omega)|^j \mu(du)ds < \infty \end{array} \right\}, \quad j = 1, 2.$$

For $f \in F_N^1 \cap F_N^2$, let

$$\begin{aligned} M_t &= \int_0^t \int_U f(s, u, \omega)\tilde{N}(dsdu) \\ &\equiv \sum_{s \le t} 1_D(\omega, s) f(s, p_s(\omega), \omega) - \int_0^t \int_U f(s, u, \omega)\mu(du)ds \end{aligned} \tag{3.4.1}$$

where D and $p(s)$ are the jump set and point process corresponding to N. It is easy to prove that (3.4.1) is well-defined and $M \in \mathcal{M}^2(\mathbf{R})$ such that

$$< M >_t = \int_0^t \int_U f(s, u, \omega)^2 \mu(du)ds \tag{3.4.2}$$

and

$$\Delta M_t = 1_D(\omega, t) f(t, p_t(\omega), \omega). \tag{3.4.3}$$

For $f \in F_N^2$, let

$$f_n(t,u,\omega) = 1_{[-n,n]}(f(t,u,\omega))1_{U_n}(u)f(t,u,\omega)$$

where U_n is given by Definition 3.4.2. Then $f_n \in F_N^1 \cap F_N^2$. Define M^n by (3.4.1) with f replaced by f_n. It is easy to prove that M^n converges, say to M, in $\mathcal{M}^2(\mathbf{R})$. We call M the **stochastic integral** of f with respect to the compensated Poisson random measure $\tilde{N}$. It is easy to verify (3.4.2) and (3.4.3) for M.

Theorem 3.4.4 (Itô's formula) *Let N be a Poisson random measure with characteristic measure μ. Suppose that*

$$X_t^j = X_0^j + A_t^j + M^j + \int_0^t \int_U f^j(s,u,\omega)\tilde{N}(dsdu)$$

where $A^j \in \mathcal{A}$, $M^j \in \mathcal{M}^{2,c}(\mathbf{R})$ and $f^j \in F_N^2$, $j = 1,2,\cdots,d$. Let $F \in C^2(\mathbf{R}^d)$. Then

$$\begin{aligned} F(X_t) &= F(X_0) + \sum_{j=1}^d \int_0^t \partial_j F(X_s)dA_s^j + \sum_{j=1}^d \int_0^t \partial_j F(X_s)dM_s^j \\ &+ \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial_{ij}^2 F(X_s)d < M^i, M^j >_s \\ &+ \int_0^t \int_U \{F(X_{s-} + f(s,u,\omega)) - F(X_{s-})\}\tilde{N}(dsdu) \\ &+ \int_0^t \int_U \Big\{F(X_s + f(s,u,\omega)) - F(X_s) \\ &- \sum_{j=1}^d f^j(s,u,\omega)F_j'(X_s)\Big\}ds\mu(du). \end{aligned}$$

Theorem 3.4.5 *Let N be a Poisson random measure on $\mathbf{R}_+ \times U$ and $f \in \Gamma_N^2$. Then*

$$M_t = \int_0^t \int_U f(s,u,\omega)\tilde{N}(dsdu) \tag{3.4.4}$$

iff $M \in \mathcal{M}^{2,d}(\mathbf{R})$ and (3.4.3) holds.

Proof: "$\Rightarrow$" We only need to prove that $M \in \mathcal{M}^{2,d}(\mathbf{R})$. Let $\gamma \in \mathcal{M}^{2,c}(\mathbf{R})$. It follows from Itô's formula that

$$M_t\gamma_t = M_0\gamma_0 + \int_0^t M_s d\gamma_s + \int_0^t \int_U f(s,u,\omega)\gamma_s\tilde{N}(dsdu)$$

is a martingale. Therefore $< M, \gamma >= 0$ and hence, $M \in \mathcal{M}^{2,d}(\mathbf{R})$.
"$\Leftarrow$" Denoting the right hand side of (3.4.4) by $\tilde{M}_t$. Then $M - \tilde{M} \in \mathcal{M}^{2,d}(\mathbf{R})$.

On the other hand, $\Delta(M-\tilde{M})=0$, i.e. $M-\tilde{M}\in\mathcal{M}^{2,c}(\mathbf{R})$. Hence $M=\tilde{M}$. ∎

Theorem 3.4.6 *Let $(V,\mathcal{B}_V)$ be a measurable space and M be an adapted integer valued random measure in class (QL) with the compensator $\hat{M}(dtdv) = q(t,dv,\omega)dt$. Suppose that $(U,\mathcal{E})$ is a standard measurable space and there exists a predictable $V^*=V\cup\{\partial\}$-valued process*

$$f(t,u,\omega):[0,\infty)\times U\times\Omega\to V^*$$

such that

$$\mu\{u: f(t,u,\omega)\in A\}=q(t,A,\omega),\qquad \forall A\in\mathcal{B}_V$$

where ∂ is an extra point attached to V. Then, on an extension $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P},\tilde{\mathcal{F}}_t)$ of $(\Omega,\mathcal{F},P,\mathcal{F}_t)$, there exists a Poisson random measure N with characteristic measure μ such that

$$\begin{aligned} M((0,t]\times A) &= \int_0^t\int_U 1_A(f(s,u,\omega))N(dsdu) \\ &\equiv \sum_{s\le t} 1_D(\omega,s)1_A(f(s,p_s(\omega),\omega)), \end{aligned}$$

for every $A\in\mathcal{B}_V$.

After the above preparations, we now define the stochastic integral of Φ'-valued functions with respect to Poisson random measures. Let N be a Poisson random measure on $\mathbf{R}_+\times U$ with characteristic measure μ and f be a predictable map from $[0,\infty)\times U\times\Omega$ to Φ' such that

$$E\int_0^t\int_U |f(s,u,\omega)[\phi]|^2\mu(du)ds<\infty,\ \forall t>0,\ \forall\phi\in\Phi.$$

Define

$$M_t^\phi=\int_0^t\int_U f(s,u,\omega)[\phi]\tilde{N}(dsdu),\quad \forall\phi\in\Phi. \tag{3.4.5}$$

It is clear that there exists $M\in\mathcal{M}^{2,d}(\Phi')$, denoted by

$$M_t=\int_0^t\int_U f(s,u,\omega)\tilde{N}(dsdu),\ \forall\phi\in\Phi,$$

such that $M_t^\phi = M_t[\phi]$ for all $t\ge 0$ and $\phi\in\Phi$, where $\mathcal{M}^{2,d}(\Phi')$ is the collection of $M\in\mathcal{M}^2$ such that $M[\phi]\in\mathcal{M}^{2,d}(\mathbf{R})$ for any $\phi\in\Phi$.

As a consequence of Theorem 3.4.6, we have the following representation theorem for Φ'-valued purely-discontinuous square-integrable martingales.

Theorem 3.4.7 *For $M \in \mathcal{M}^{2,d}(\Phi')$, we define an integer-valued random measure N_M on $\mathbf{R}_+ \times (\Phi'/\{0\})$ by*

$$N_M([0,t] \times A) = \sum_{s \leq t} 1_A(\Delta M_s), \quad \forall A \in \mathcal{B}(\Phi'/\{0\}).$$

If N_M is in class (QL) with the compensator $\hat{N}_M(dtdv) = q(t, dv, \omega)dt$ and there exists a standard measurable space $(U, \mathcal{E})$ and a predictable map

$$f(t, u, \omega) : [0, \infty) \times U \times \Omega \to (\Phi'/\{0\}) \cup \{\partial\}$$

such that

$$\mu\{u : f(t, u, \omega) \in A\} = q(t, A, \omega), \ \forall A \in \mathcal{B}(\Phi'/\{0\}),$$

then on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \tilde{\mathcal{F}}_t)$ of $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$, there exists a Poisson random measure N with characteristic measure μ such that

$$M_t = \int_0^t \int_U f(s, u, \omega)\tilde{N}(dsdu). \tag{3.4.6}$$

Proof: It follows from Theorem 3.4.6 that there exists a Poisson random measure N with characteristic measure μ such that

$$N_M((0,t] \times A) = \int_0^t \int_U 1_A(f(s, u, \omega))N(dsdu)$$

for every $A \in \mathcal{B}(\Phi'/\{0\})$. Therefore

$$\Delta M_t(\omega) = 1_D(t, \omega) f(t, p_t(\omega), \omega), \quad \forall t > 0, \omega \in \Omega,$$

where D and $p(s)$ are the jump set and point process corresponding to N. Hence for any $\phi \in \Phi$, $t > 0$ and $\omega \in \Omega$, we have

$$\Delta M_t(\omega)[\phi] = \Delta \int_0^t \int_U f(s, u, \omega)[\phi]\tilde{N}(dsdu),$$

i.e

$$M_t(\omega)[\phi] - \int_0^t \int_U f(s, u, \omega)[\phi]\tilde{N}(dsdu) \in \mathcal{M}^{2,c}(\mathbf{R}) \cap \mathcal{M}^{2,d}(\mathbf{R}) = \{0\}$$

where 0 denotes the identically 0 martingale. Therefore (3.4.6) holds. ∎

This is probably the right place to discuss some special examples of purely-discontinuous Φ'-martingales.

Example 3.4.1 *Let $\mathcal{X}$ be a domain in $\mathbf{R}^d$. Let A be a closed densely defined nonnegative-definite self-adjoint operator on $H = L^2(\mathcal{X}, \rho(x)dx)$ where ρ is an appropriately chosen measurable function on $\mathcal{X}$. Suppose that the condition (1.3.17) holds and Φ is the CHNS constructed in Example 1.3.2.*

Let N be a Poisson random measure on $\mathbf{R}_+ \times \mathbf{R}_+ \times \mathcal{X}$ with characteristic measure μ on $\mathbf{R}_+ \times \mathcal{X}$ such that

$$\int_{\mathbf{R}_+\times\mathcal{X}} a^2\phi(x)^2\mu(dadx) < \infty \qquad \forall \phi \in \Phi.$$

For any $\phi \in \Phi$, let

$$M_t^\phi = \int_0^t \int_{\mathbf{R}_+\times\mathcal{X}} a\phi(x)\tilde{N}(dsdadx).$$

Theorem 3.4.8 *For any $\phi, \psi \in \Phi$ and $t, s \geq 0$, we have*

$$EM_t^\phi M_s^\psi = (t \wedge s)Q(\phi, \psi)$$

where

$$Q(\phi, \psi) = \int_{\mathbf{R}_+\times\mathcal{X}} a^2\phi(x)\psi(x)\mu(dadx).$$

Proof:

$$\begin{aligned}
&EM_t[\phi]M_s[\psi]\\
=\ & E\int_0^t \int_{\mathbf{R}_+\times\mathcal{X}} a\phi(x)\tilde{N}(drdadx)\int_0^s \int_{\mathbf{R}_+\times\mathcal{X}} a\psi(x)\tilde{N}(drdadx)\\
=\ & \int_0^{t\wedge s} \int_{\mathbf{R}_+\times\mathcal{X}} a^2\phi(x)\psi(x)dr\mu(dadx)\\
=\ & (t\wedge s)Q(\phi,\psi).
\end{aligned}$$

∎

Theorem 3.4.9 *There exists $M \in \mathcal{M}^{2,d}(\Phi')$ such that $M_t^\phi = M_t[\phi]$, $\forall t \geq 0$, $\phi \in \Phi$, iff Q is continuous on $\Phi \times \Phi$.*

Proof: "⇒" Let $t \geq 0$ be fixed and let $V : \Phi \to [0, \infty)$ be given by

$$V(\phi) = \sqrt{EM_t[\phi]^2}, \qquad \forall \phi \in \Phi.$$

It is easy to verify the conditions of Lemma 1.3.1 and hence, $\exists\theta > 0$ and $r \geq 0$ such that

$$\sqrt{EM_t[\phi]^2} \leq \theta\|\phi\|_r \qquad \forall \phi \in \Phi. \tag{3.4.7}$$

The continuity of Q then follows from Theorem 3.4.8.
"$\Leftarrow$" Similar to (3.4.7), $\exists\theta > 0$ and $r \geq 0$ such that

$$Q(\phi,\phi) \leq \theta^2 \|\phi\|_r^2 \qquad \forall \phi \in \Phi. \tag{3.4.8}$$

Let $p > r$ be such that the canonical injection from Φ_p to Φ_r is Hilbert-Schmidt. Let

$$M_t = \sum_j M_t^{\phi_j^p} \phi_j^{-p}.$$

It follows from (3.4.8) and Theorem 3.4.8 that M_t is a Φ_{-p}-valued process such that $M_t^\phi = M_t[\phi]$, $\forall t \geq 0$, $\phi \in \Phi$. $M \in \mathcal{M}^{2,d}(\Phi')$ then follows directly from the definition. ∎

Remark 3.4.1 *For most of the cases of interest to us, we have $\Phi \hookrightarrow C_b(\mathcal{X})$ (e.g. $\mathcal{S}(\mathbf{R})$ in Example 1.3.1 and Remark 1.3.5, the CHNS Φ constructed in Section 7.2). In this case, Q is continuous on $\Phi \times \Phi$. In fact, let $V : \Phi \to [0,\infty)$ be given by $V(\phi)^2 = Q(\phi,\phi)$, $\forall \phi \in \Phi$. The condition (1) of Lemma 1.3.1 follows from Fatou's lemma and the conditions (2) and (3)' follows from the linearity of Q. Therefore, $\exists\theta > 0$ and $r \geq 0$ such that $V(\phi) \leq \theta \|\phi\|_r$, $\forall \phi \in \Phi$. The continuity of Q then follows easily.*

Remark 3.4.2 *Comparing with (3.4.5), in this example, we have $U = \mathbf{R}_+ \times \mathcal{X}$, $u = (a,x)$ and $f(s,u,\omega)[\phi] = a\phi(x)$ [non-random integrand]. If $\Phi \hookrightarrow C_b(\mathcal{X})$, then $f(s,u,\omega) \in \Phi'$ for all $(s,u,\omega) \in \mathbf{R}_+ \times U \times \Omega$.*

Example 3.4.2 *Let Φ be a CHNS and let Λ be a measurable subset of Φ'. Let N be a Poisson random measure on $\mathbf{R}_+ \times \mathbf{R} \times \Lambda$ with characteristic measure μ on $\mathbf{R} \times \Lambda$ such that*

$$\int_{\mathbf{R}\times\Lambda} a^2 \eta[\phi]^2 \mu(da d\eta) < \infty \qquad \forall \phi \in \Phi.$$

For any $\phi \in \Phi$, let

$$M_t^\phi = \int_0^t \int_{\mathbf{R}\times\Lambda} a\eta[\phi] \tilde{N}(ds da d\eta).$$

Similar to the previous example, we have

Theorem 3.4.10 *(1) For any $\phi,\psi \in \Phi$ and $t,s \geq 0$, we have*

$$EM_t^\phi M_s^\psi = (t \wedge s) Q(\phi,\psi)$$

where

$$Q(\phi,\psi) = \int_{\mathbf{R}\times\Lambda} a^2 \eta[\phi]\eta[\psi]\mu(da d\eta).$$

(2) There exists $M \in \mathcal{M}^{2,d}(\Phi')$ such that $M_t^\phi = M_t[\phi]$, $\forall t \geq 0$, $\phi \in \Phi$, iff Q is continuous on $\Phi \times \Phi$.