Polynomially Harmonizable Processes and Finitely Polynomially Determined Lévy Processes

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Abstract

The sequence $\{P_k(t, x)\}$ of two-variable Hermite polynomials are known to have the property that, if $\{M_t, t \geq 0\}$ denotes the standard Brownian motion, then $P_k(t, M_t)$ is a martingale for each $k \geq 1$. This property of standard Brownian motion vis-a-vis Hermite polynomials motivated the general notion of "polynomially harmonizable processes". These are processes that admit sequences of time-space harmonic polynomials, that is, two-variable polynomials which become martingales when evaluated along the trajectory of the process. For Lévy processes, this property is connected to certain properties of the associated Lévy/Kolmogorov measures. Moreover, stochastic properties of the underlying processes (like independence, stationarity of increments) turn out to be equivalent to certain algebraic/analytic properties of the corresponding sequence of polynomials. We first present a brief survey of these recently obtained general results and then describe necessary and sufficient conditions for certain classes of Lévy processes to be uniquely determined by a finite number of time-space harmonic polynomials.

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1 Introduction: General Definitions

The sequence of two-variable Hermite polynomials $\{P_k, \ k \geq 1\}$ on $[0, \infty) \times \mathbb{R}$ are defined via the classical one-variable Hermite polynomials $\{p_k, \ k \geq 1\}$ as follows:

$$P_k(t, x) = t^{k/2} p_k(\frac{x}{\sqrt{t}}),$$

where

$$p_k(x) = (-1)^k e^{x^2/2} \frac{\partial^k}{\partial x^k}(e^{-x^2/2}).$$

Some of the well-known properties of the sequence $\{P_k\}$ are:

- $P_k(t, x)$ is a polynomial in the two variables t and x, for each k.

- $P_k(t,\cdot)$ has degree k in x, with the leading term having coefficient 1.
- $\dfrac{\partial}{\partial x}P_k(t,x) = kP_{k-1}(t,x)$, for each $k \geq 1$.
- $\dfrac{\partial}{\partial t}P_k(t,x) = -\dbinom{k}{2}P_{k-2}(t,x) = -\dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}P_k(t,x)$, for each $k \geq 2$.

For the last two properties, we take $P_0(t,x) \equiv 1$. The first two properties simply tell us that we can write

$$P_k(t,x) = \sum_{j=0}^{k} p_j^{(k)}(t)x^j,$$

where the $p_j^{(k)}(t)$ are polynomials in t and $p_k^{(k)}(t) \equiv 1$.

The sequence $\{P_k\}$ of Hermite polynomials as defined above is known to have some deep connections with the standard Brownian motion. One of these is the well-known fact that if $\{M_t,\ t \geq 0\}$ denotes the standard Brownian motion, then for each k, $\{P_k(t,M_t),\ t \geq 0\}$ is a martingale (for the natural filtration of $\{M_t\}$) and standard Brownian motion is the only process with this property. Moreover, if $P(t,x)$ is any two-variable polynomial such that $\{P(t,M_t)\}$ is a martingale, then P belongs to the linear span of the sequence $\{P_k\}$.

A natural question that arises is: which stochastic processes admit such a sequence of 2-variable polynomials which when evaluated along the trajectory of the process are martingales and, if so, to what extent do these polynomials determine the process? Also, is it possible to get the sequences of polynomials so as to satisfy properties similar to those of the Hermite polynomials mentioned above? These questions were investigated in detail in Goswami and Sengupta [2] and Sengupta [6]. Following are some notations and definitions that were introduced in these works. Here we restrict ourselves only to continuous-time processes.

Let $M = \{M_t,\ t \geq 0\}$ be a stochastic process on some probability space. The *time-space harmonic polynomials* for the process M are defined to be all those two-variable polynomials $P(\cdot,\cdot)$ such that $\{P(t,M_t)\}$ is a martingale (always for the natural filtration of M). The two variables will be referred to as repectively the 'time' and the 'space' variables. The collection of all time-space harmonic polynomials for a process M will be denoted $\mathcal{P}(M)$. In other words,

$$\mathcal{P}(M) := \{P\ :\ P \text{ is a 2-variable polynomial and } \{P(t,M_t)\} \text{ is a martingale}\}$$

Any two-variable polynomial P can be written as $P(t,x) = \sum_{j=0}^{k} p_j(t)x^j$, for some k, where each $p_j(t)$ is a polynomial in t. If in the above representation, $p_k(t) \not\equiv 0$, we say that P is *of degree k in the 'space' variable x*. For a stochastic process $M = \{M_t\}$, we define $\mathcal{P}_k(M)$ to be the collection of those time-space harmonic polynomials which are of degree k in the space variable, that is,

$$\mathcal{P}_k(M) := \{P \in \mathcal{P}(M)\ :\ P \text{ is of degree } k \text{ in the space variable } x\}.$$

Clearly,

$$\mathcal{P}(M) = \bigcup_k \mathcal{P}_k(M).$$

Definition: A stochastic process M is said to be *polynomially harmonizable* (*p-harmonizable*, in short) if $\mathcal{P}_k(M) \neq \emptyset$, for all $k \geq 1$.

In this terminology, standard Brownian motion is a p-harmonizable process. Indeed, Brownian motion is p-harmonizable in a somewhat stricter sense, to be understood below.

For a process M, let us denote $\overline{\mathcal{P}}_k(M)$ to be the set of those time-space harmonic polynomials of degree k in x, for which the leading term in x is 'free' of t, that is, the coefficient of x^k is a non-zero constant. In other words,

$$\overline{\mathcal{P}}_k(M) := \{P \in \mathcal{P}_k(M) : P(t,x) = \sum_{j=0}^{k} p_j(t)x^j \text{ with } p_k(\cdot) \text{ a non-zero constant}\},$$

and we let,

$$\overline{\mathcal{P}}(M) := \bigcup_k \overline{\mathcal{P}}_k(M).$$

Clearly, $\overline{\mathcal{P}}_k(M) \subset \mathcal{P}_k(M)\ \forall\ k$ and so, $\overline{\mathcal{P}}(M) \subset \mathcal{P}(M)$. Also, if $\overline{\mathcal{P}}_k(M) \neq \emptyset$, then there is $P(t,x) = \sum_{j=0}^{k} p_j(t)x^j \in \overline{\mathcal{P}}_k(M)$ with $p_k(\cdot) \equiv 1$.

Definition: A stochastic process M is said to be *p-harmonizable in the strict sense* if $\overline{\mathcal{P}}_k(M) \neq \emptyset$, for all $k \geq 1$.

The second property of the two-variable Hermite polynomials listed earlier shows that standard Brownian motion is actually p-harmonizable in the strict sense. The other classical example of a strict sense p-harmonizable process is the Poisson process. For a Poisson process, with intensity 1 for example, a sequence of time-space harmonic polynomials is given by the so-called two-variable Charlier polynomials

$$P_k(t,x) = \sum_{j=0}^{k} \binom{k}{j} x^j \sum_{i=0}^{k-j} \left\{ \begin{array}{c} k-j \\ i \end{array} \right\} t^i,$$

where $\left\{ \begin{array}{c} l \\ i \end{array} \right\}$ denote the Stirling numbers of the second kind. The Gamma process is another example of a strict sense p-harmonizable process.

In keeping with the special properties of the sequence of Hermite polynomials mentioned earlier, we introduce here a list of properties for a sequence of two-variable polynomials. Let $\{P_k,\ k \geq 1\}$ be a sequence of two-variable polynomials with P_k being of degree k in x. We define $P_0 \equiv 1$. Let us write $P_k(t,x) = \sum_{j=0}^{k} p_j^{(k)}(t)x^j$, where the $p_j^{(k)}(t)$ are polynomials in t. We are going to refer to the following properties in the sequel.

(*i*) *Strict sense property*: For each $k \geq 1$, $p_k^{(k)}(\cdot) \equiv 1$.

(*ii*) *The Appell property*: For each $k \geq 1$, $\dfrac{\partial P_k}{\partial x} = kP_{k-1}$, that is, $jp_j^{(k)}(t) = kp_{j-1}^{(k-1)}(t)$, $1 \leq j \leq k$.

(*iii*) *The pseudo-type-zero property*: There exists a real sequence $\{h_k\}$ such that for each $k \geq 1$, $\dfrac{\partial P_k}{\partial t} = \sum_{i=1}^{k} \binom{k}{i} h_i P_{k-i}$, that is, $\dfrac{d}{dt} p_j^{(k)}(t) = \sum_{i=1}^{k-j} \binom{k}{i} h_i p_j^{(k-i)}(t)$, $1 \leq j \leq k$.

(*iv*) *Uniqueness property*: For each $k \geq 1$, $P_k(0,x) = x^k$, that is, $p_j^{(k)}(0) = 0$, $0 \leq j \leq k-1$.

The sequence of Hermite polynomials satisfies all the properties $(i)-(iv)$; property (iii) holds here with $h_2 = -1$ and $h_k = 0$ for $k \neq 2$. It is easy to verify that the two variable Charlier polynomials satisfy these properties as well. Theorems 2.3 and 2.4 in the next section will establish that these are reflections of the fact that both Brownian motion and Poisson process are homogeneous Lévy processes. Let us make some basic observations about the properties listed above. First of all, with the convention that $P_0 \equiv 1$, property (i) will always imply property (ii). Secondly, in our applications, the sequence $\{P_k\}$ will be arising as time-space harmonic polynomials of a process M. Now if, the process itself happens to be a martingale, we can always take $P_1 = x$, in which case property (ii) will actually imply a slightly stronger property than (i), namely, (i') for each $k \geq 1$, $P_k(t,x) - x^k$ has degree at most $k-2$ in x, that is, $p_k^{(k)} \equiv 1$ and $p_{k-1}^{(k)} \equiv 0$.

Properties (ii) and (iii) for a sequence of polynomials were studied analytically in an entirely different context in Sheffer [7], which is the source of our terminolgy for these properties in this context. It turns out that for a stochastic process M, the properties (ii), (iii) and some other algebraic/analytic properties the corresponding sequence of time-space harmonic polynomials are intimately connected to some stochastic properties of M.

2 Lévy Processes and p-Harmonizability

In this section, we describe some of the results on p-harmonizability of Lévy processes. Details of these can be found in [6]. Discrete-time versions of many of these results were proved earlier in [2]. For us, a *Lévy process* will mean a process $M = \{M_t, t \geq 0\}$ with independent increments and having no fixed times of discontinuity. A *homogeneous* Lévy process is one which is homogeneous as a Markov process, that is, whose increments are stationary besides being independent. In the results that follow, we will often need to impose two conditions on the process M, to be referred to as the *moment condition* and *support condition*. They are as follows:

- Condition **(Mo)** : For all t, M_t has finite moments of all orders.

- Condition **(Su)** : There is a sequence $t_n \uparrow \infty$, such that, for all $k \geq 1$, $|\text{support}(M_{t_n})| > k$ for infinitely many t_n.

The moment condition (Mo) is clearly necessary for the process to be p-harmonizable. The role of the condition (Su) is more technical in nature. However, it may be noted that any homogeneous Lévy process always satisfies this condition (unless, of course, it is deterministic). For a general Lévy process, a simpler condition that gurantees (Su) is that $M_t - M_s$ be non-degenerate for all $0 \leq s < t$, that is, the increments are all non-degenerate. We now state some of the main results from [6].

Theorem 2.1. *Any homogeneous Lévy process $M = \{M_t, t \geq 0\}$ with $M_0 \equiv 0$ and satisfying the conditions (Mo) and (Su) is p-harmonizable in the strict sense. Moreover, there exists a unique sequence $P_k \in \overline{\mathcal{P}}_k(M)$, $k \geq 1$ satisfying properties $(i) - (iv)$ and such that $\mathcal{P}(M)$ is just the linear span of $\{P_k,\ k \geq 1\}$. Further, the process M is uniquely determined by the sequence $\{P_k\}$ upto all the moments of its finite-dimensional distributions.*

Remark: (i) The fact that $\mathcal{P}(M)$ equals the linear span of $\{P_k,\ k \geq 1\}$ implies, in particular, that $\mathcal{P}(M) = \overline{\mathcal{P}}(M)$. This is actually a special case of a more general fact proved by Goswami and Sengupta in [2], namely, that for any process M satisfying (Su), if $\overline{\mathcal{P}}_k(M) \neq \emptyset \ \forall\ k$, then $\overline{\mathcal{P}}_k(M) = \mathcal{P}_k(M)\ \forall\ k$.
(ii) The property of M being determined by the sequence $\{P_k\}$ can be strengthened as follows. If we assume, for example, that for some $t > 0$ and $\epsilon > 0$, $E(\exp\{\alpha M_t\}) < \infty\ \forall\ |\alpha| < \epsilon$, then the sequence $\{P_k\}$ completely determines the distribution of the process M.

Theorem 2.2. *Let $M = \{M_t, t \geq 0\}$ be a Lévy process with $M_0 \equiv 0$ and satisfying the conditions (Mo) and (Su). Then M is p-harmonizable if and only if for each $k \geq 1$, $E(M_t^k)$ is a polynomial in t. In this case, there exists a unique sequence $P_k \in \overline{\mathcal{P}}_k(M)$, $k \geq 1$ satisfying properties $(i), (ii)$ and (iv) and such that $\mathcal{P}(M)$ is just the linear span of $\{P_k,\ k \geq 1\}$. Further, the process M is uniquely determined by the sequence $\{P_k\}$ upto all the moments of its finite-dimensional distributions.*

Remark: Note the absence of the pseudo-type-zero property (iii) in this case. In fact, property (iii) would not hold unless the process is homogeneous. [see Theorem 2.4].

We now describe a characterization of p-harmonizability of a Lévy process M in terms of the underlying Lévy measure, or, equivalently the Kolmorov measure. Associated to any Lévy process, there is a σ-finite measure m on $[0, \infty) \times (\mathbb{R} \setminus \{0\})$, called its *Lévy measure*, such that,

$$\begin{aligned} \phi_t(\alpha) &= E\{\exp(i\alpha M_t)\} \\ &= \exp\left[i\alpha\mu(t) - \tfrac{1}{2}\alpha^2\sigma^2(t) + \int \left(e^{i\alpha u} - 1 - \frac{i\alpha u}{1+u^2}\right) m([0,t] \otimes du)\right], \end{aligned}$$

where $\mu(\cdot)$ and $\sigma^2(\cdot)$ are the mean and variance functions of the 'gaussian part' of M. It can be shown that p-harmonizability of M is equivalent to requiring that all the following functions be polynomials in t:

$$h_1(t) = \mu(t) + \int \frac{u^3}{1+u^2} m([0,t] \otimes du), h_2(t) = \sigma^2(t) + \int u^2 m([0,t] \otimes du),$$

and for $k > 2$,

$$h_k(t) = \int u^k m([0,t] \otimes du).$$

The above characterization takes on a slightly simpler form when expressed in terms of what is known as the *Kolmogorov measure* associated with the process. It is the unique Borel mesure L on $[0, \infty) \times \mathbb{R}$ such that

$$\log\ E\{\exp(i\alpha M_t)\} = i\alpha\nu(t) + \int \left(\frac{e^{i\alpha u} - 1 - i\alpha u}{u^2} \right) L([0,t] \otimes du),$$

where $\nu(t) = EM_t$ is the mean function of the process M. We refer to Ito [3] for the definition and the transformation that connects the Kolmogorov measure and the Lévy measure. A necessary and sufficient condition for p-harmonizability of the process M is that : $\nu(t)$ as well as the functions $\tilde{h}_k(t) = \int u^{k-2} L([0,t] \otimes du)$, $k \geq 2$ are all polynomials in t.

We have seen that for any Lévy process M satisfying the conditions (Mo) and (Su), we can get a sequence $P_k \in \overline{\mathcal{P}}_k(M)$, $k \geq 1$, such that Appel property (ii) holds. Moreover, if M is homogeneous, then the sequence $\{P_k\}$ can be chosen so as to satisfy the pseudo-type-zero property (iii). The next two results show that, under some conditions, the converse is also true. In both the following theorems, $M = \{M_t, t \geq 0\}$ will denote a continuous-time stochastic process with r.c.l.l. paths starting at $M_0 \equiv 0$ and satisfying conditions (Mo) and (Su) and $\{\mathcal{F}_t, t \geq 0\}$ will denote the natural filtration of M.

Theorem 2.3. *If there exists a sequence $P_k \in \mathcal{P}_k(M)$, $k \geq 1$, satisfying the Appel property (ii), then for each $0 \leq s < t$, the conditional moments $E((M_t - M_s)^k | \mathcal{F}_s)$ are degenerate for all k. If moreover, for each t, the moment-generating function of M_t is finite on some open interval containing 0, then M is a Lévy process.*

Theorem 2.4. *If there exists a sequence $P_k \in \mathcal{P}_k(M)$, $k \geq 1$, satisfying both the Appel property (ii) and the pseudo-type-zero property (iii) and if for each t, the moment-generating function of M_t is finite on some open interval containing 0, then M is a homogeneous Lévy process.*

Remark: Under the hypothesis of either of the above theorems, it can further be shown that the sequence $\{P_k\}$ satisfies the properties (i) and (iv) as well and is the unique sequence to do so. Moreover, the sequence $\{P_k\}$ span all of $\mathcal{P}(M)$ and also determines the distribution of M.

Next, we briefly mention some connections between the time-space harmonic polynomials of a process and what is known as semi-stability property, as developed in Lamperti [5]. Recall that a process M with $M_0 \equiv 0$ is called *semi-stable of index $\beta > 0$* if for every $c > 0$, the processes $\{M_{ct}, t \geq 0\}$ and $\{c^\beta M_t, t \geq 0\}$ have the same distribution. It can be easily shown that if $\{P_k \in \mathcal{P}_k(M)\}$ is a sequence of time-space harmonic polynomials of a semi-stable process M, of index β, then each P_k satisfies the following homogeneity property:

$$P_k(t, x) = t^{\beta k} p_k\left(\frac{x}{t^\beta}\right),$$

where $p_k(\cdot)$ is the one-variable polynomial $P_k(1, \cdot)$. In other words, each P_k is *homogeneous* in t^β and x. It can be shown that, under mild technical conditions, the converse is also true, that is, the existence of a sequence $\{P_k \in \mathcal{P}_k(M)\}$ such that each P_k is homogeneous in t^β and x, for some $\beta > 0$, implies that the process M is semi-stable of index β. It is also worthwhile to point out here that if a process M admits a sequence $\{P_k\}$ of time-space harmonic polynomials which are homogeneous in t^β and x, then 2β must be an integer and that in case 2β is odd, the finite dimensional distributions of M are all symmetric about 0.

Finally, let us mention how an *intertwining relationship* between two markov processes, as developed in Carmona et al [1] relates the time-space harmonic polynomials of the two processes. If M and N are two markov processes with semigroups (P_t) and (Q_t) respectively, one says that the two processes (or, the two semigroups) are *intertwined* if there exists an operator Λ such that $\Lambda P_t = Q_t \Lambda \; \forall \; t$. In many cases, the operator Λ is given by the "multiplicative kernel" for a random variable Z, that is, $\Lambda f(x) = Ef(xZ)$. In such a case, it is easy to show that, if $P(t, x) = \sum_{j=0}^{k} p_j(t) x^j$ is a time-space harmonic polynomial for the process M, then $\overline{P}(t, x) = \Lambda P(t, x) = \sum_{j=0}^{k} p_j(t) E(Z^j x^j)$ is time-space harmonic for N. This has proved to be very useful in that if one knows the time-space harmonic polynomials of a process M, then one can get those for other processes which are intertwined with M. This is illustrated with examples in Section 4.

3 Finitely Polynomially Determined Lévy Processes

In this section, we address the main question of this article, which involves obtaining a characterization of Lévy processes whose laws are determined by finitely many of its time-space polynomials. In a sense, this is an extension of Lévy's characterization of standard Brownian motion, which says that, under the additional assumption of continuity of paths, standard Brownian motion is characterized by two of its time-space harmonic polynomials, namely, the first two 2-variable Hermite polynomials $P_1(t, x) = x$ and $P_2(t, x) = x^2 - t$. One knows that the continuity of paths is a crucial assumption here, without which the characterization does not hold. In the results that follow, the only path property we will assume is the standard assumption of r.c.l.l. paths for Lévy processes. Let us start with some general definitions.

Let $\mathcal{C}$ be a given class of processes.

Definition A process $M \in \mathcal{C}$ will be called *k-polynomially determined in $\mathcal{C}$* (in short, *k-p.d. in $\mathcal{C}$*), if $\mathcal{P}_j(M) \neq \emptyset$, $\forall\, j \leq k$, and, for any $N \in \mathcal{C}$, $\mathcal{P}_j(N) = \mathcal{P}_j(M)\ \forall\, j \leq k \Rightarrow N \stackrel{d}{=} M$. (Here $\stackrel{d}{=}$ means equality in distribution.) Processes which are k-p.d. in $\mathcal{C}$ for some $k \geq 1$ are called *finitely polynomially determined in $\mathcal{C}$* (in short, *f.p.d. in $\mathcal{C}$*).

Let us remark here that an f.p.d. process need not be p-harmonizable. A general question that we may address is: for what classes of processes $\mathcal{C}$, can one get a complete characterization of the f.p.d. members of $\mathcal{C}$? For two important classes of processes, such a complete characterzation has been obtained and are presented below.

The first result characterizes the f.p.d. processes in the class of all homogeneous Lévy processes. As mentioned in the previous section, for any Lévy process M, one has the representaion

$$\begin{aligned}\log(E(e^{i\alpha M_t})) &= i\alpha\nu(t) + \int\left(\frac{e^{i\alpha u}-1-i\alpha u}{u^2}\right)L([0,t]\otimes du)\\ &= i\alpha\mu(t) - \tfrac{1}{2}\alpha^2\sigma^2(t) + \int\left(e^{i\alpha u}-1-\frac{i\alpha u}{1+u^2}\right)m([0,t]\otimes du),\end{aligned}$$

where L and m are called respectively the Kolmogorov measure and the Lévy measure associated to the process M. In case M is homogeneous, the measures L and m turn out to be the product measures

$$L(dt\otimes dx) = dt\otimes l(dx),\ m(dt\otimes dx) = dt\otimes\eta(dx),$$

where l and η are σ-finite measures on $\mathbb{R}$ and $\mathbb{R}\setminus\{0\}$ respectively and the above representations take on the following special forms

$$\begin{aligned}\log(E(e^{i\alpha M_t})) &= i\alpha\nu t + t\int\left(\frac{e^{i\alpha u}-1-i\alpha u}{u^2}\right)l(du)\\ &= i\alpha\mu t - \tfrac{1}{2}\alpha^2\sigma^2 t + t\int\left(e^{i\alpha u}-1-\frac{i\alpha u}{1+u^2}\right)\eta(du).\end{aligned}$$

It may be pointed out in this connection that the relation between the measures l and η is simply given by

$$l(A) = \sigma^2\delta_{\{0\}}(A) + \int_{A\setminus\{0\}} u^2\eta(du), \quad \text{for } A \in \mathcal{B}(\mathbb{R}).$$

An important property of l that will be used subsequently is that for all $k \geq 2$, the k-th cumulant of M_1 equals $\int u^{k-2}l(du)$. the following theorem now gives a characterization of f.p.d processes in the class of all homogeneous Lévy processes.

Theorem 3.1. *A process M is finitely polynomially determined in the class of all homogeneous Lévy processes if and only if the associated measure l, or equivalently the measure η, has finite support.*

Proof. It is immediate from the above relation between the measures l and η that whenever one of them has finite support, so does the other. In the proof, we will work with l.

Suppose first that l has finite support, say, $l = \sum_{i=1}^{n} \theta_i \delta_{\{r_i\}}$, where $\theta_i > 0$, $i = 1, \ldots, n$ and r_i's are distinct real numbers. Here $\delta_{\{r\}}$ denotes the 'dirac' mass at r. We show that M is k-p.d. among homogeneous Lévy processes with $k = 2n+2$. Let N be any homogeneous Lévy process with $\mathcal{P}_j(N) = \mathcal{P}_j(M)\ \forall\, j \leq 2n+2$. We will show that $\nu_N = \nu_M$ and $l_N = l_M$ which will imply that $N \stackrel{d}{=} M$. It is easy to see that $\mathcal{P}_j(N) = \mathcal{P}_j(M)\ \forall\, j \leq 2n+2$ implies the equality of the first $2n+2$ moments of N_1 and M_1, which in turn implies the equality of their first $2n+2$ cumulants. This entails, first of all, that $\nu_N = \nu_M$ and also, in view of the above mentioned property of l, that $\int u^j l_N(du) = \int u^j l_M(du)\ \forall\, j = 0, 1, \ldots, 2n$. From these, one can easily deduce that for any choice of distinct real numbers $a_1, \ldots, a_n$,

$$\int \prod_{i=1}^{n} (u - a_i)^2 l_N(du) = \int \prod_{i=1}^{n} (u - a_i)^2 l_M(du).$$

In particular, taking $a_i = r_i$, $\forall\, i$, one obtains that $\int \prod_{i=1}^{n} (u - a_i)^2 l_N(du) = 0$, implying that l_N is supported on $\{r_1, \ldots, r_n\}$, that is, $l_N = \sum_{i=1}^{n} \theta'_i \delta_{\{r_i\}}$, for non-negative θ'_i, $1 \leq i \leq n$. Using the facts $\nu_N = \nu_M$ and $\int u^j l_N(du) = \int u^j l_M(du)\ \forall\, j = 0, 1, \ldots, 2n$, it is now easy to conclude that $\theta'_i = \theta_i\ \forall\, i$, that is, $l_N = l_M$.

To prove the converse, suppose that M is a homogeneous Lévy process for which the associated measure l is *not* finitely supported. We show that M is *not* f.p.d. by exhibiting, for any k, a homogeneous Lévy process N, different from M, such that $\mathcal{P}_j(N) = \mathcal{P}_j(M)\ \forall\, j \leq k$. This is done as follows. Fix any $k \geq 1$. Since l is not finitely supported, we can get disjoint borel sets $A_i \subset \mathbb{R}$, $i = 1, \ldots, k$ such that $l(A_i) > 0$, $\forall\, i$. Consider the real vector space of signed measures on $\mathbb{R}$ defined as $\mathcal{V} = \left\{ \mu : \mu(\cdot) = \sum_{i=1}^{k} c_i l(\cdot \cap A_i),\ c_i \in \mathbb{R}, 1 \leq i \leq k \right\}$ and consider the linear map $\Lambda : \mathcal{V} \to \mathbb{R}^{k-1}$ defined by

$$\Lambda(\mu) = \left(\int \mu(du), \int u\mu(du), \cdots, \int u^{k-2}\mu(du) \right).$$

Λ being a linear map form a space of dimension k into a space of dimension $k-1$, the nullity of Λ must be at least 1. Choose a non-zero μ in the null-space of Λ. Further, we can and do choose μ so that $|\mu(A_i)| < l(A_i)$, $\forall\, i$. If we now define $\tilde{l} = l + \mu$, then $\tilde{l}$ is a positive measure with $\tilde{l} \neq l$ but $\int u^j \tilde{l}(du) = \int u^j l(du)$, $\forall\, j = 0, \cdots, k-2$. It is now easy to see that if N is the homogeneous Lévy process with $\nu_N = \nu_M$ and Kolmogorov measure $L(dt \otimes dx) = dt \otimes \tilde{l}(dx)$, then $\mathcal{P}_j(N) = \mathcal{P}_j(M)\ \forall\, j \leq k$ but $N \stackrel{d}{\neq} M$. $\square$

Remarks: (i) A simple interpretation of the above therorem is that a homogeneous Lévy process is f.p.d. if and only if its jumps, if and when they occur, are of sizes in a fixed finite set.

(ii) The proof of 'if' part of the theorem shows that if the measure l is supported on precisely k many points, then the process is determined by its first $2k+2$

many time-space harmonic polynomials. A natural question is whether $2k+2$ is the minimum number of polynomials necessary. As we shall see in Section 4, that is indeed the case for the most common examples of homogeneous Lévy processes. We conjecture that it is perhaps true in general.

Our next reult will give a similar characterization of the f.p.d. property in a more general class of Lévy processes than the homogeneous ones. To be specific, we consider the class of those Lévy processes for which the Kolmogorov measure admits a 'disintegration' w.r.t. the Lebesgue measure on $[0,\infty)$. Formally, let us say that the Kolmogorov measure L of a Lévy process M admits a 'derivative measure' l if

$$L(dt,dx) = l(t,dx)dt,$$

where $l(t,A), t \in [0,\infty), A \in \mathcal{B}$ is a transition measure on $[0,\infty) \times \mathcal{B}$. Here $\mathcal{B}$ denotes the Borel σ-field on $\mathbb{R}$. We denote $\mathcal{C}$ to be the class of all those Lévy processes whose Kolmogorov measure admits such a derivative measure.

Clearly, all homogeneous Lévy processes belong to this class, since in that case $l(t,\cdot) \equiv l(\cdot)$. The class $\mathcal{C}$ is fairly large. For example, Gaussian Lévy processes as well as non-homogeneous compound Poisson processes belong to this class. Since $\mathcal{C}$ is clearly a vector space, any Lévy process that arises as the sum of independent Lévy processes of class $\mathcal{C}$ also belong to this class. As expected, our characterization of f.p.d. processes among the class $\mathcal{C}$ will be in terms of the derivative measure $l(t,\cdot)$ defined above and the general idea of the proof runs along the same lines as in the case of homogeneous Lévy processes. However, the actual argument becomes a little more technical. For example, we would show that a process M in the class $\mathcal{C}$ cannot be k-p.d. unless for almost all t, the derivative measure $l(t,\cdot)$ is supported on at most k points. This is the content of the following Lemma 3.1. The idea of the proof is analogous to that of the 'only if' part of Theorem 3.1 for homogeneous Lévy processes. That is, assuming the contrary is true, we will have to define a new process N in class $\mathcal{C}$ such that $\mathcal{P}_j(N) = \mathcal{P}_j(M)\ \forall\ j \leq k$ but $N \overset{d}{\neq} M$. However, getting hold of this process N or equivalently its derivative measure $\tilde{l}(t,\cdot)$ involves using an appropriate variant of a result of Descriptive Set Theory, known as Novikov's Selection Theorem, stated below as Lemma 3.2. We refer to Kechris [4] for details.

Lemma 3.1. *Suppose the process M is k-polynomially determined in class $\mathcal{C}$. Then for any version of l, the set $T \subset [0,\infty)$ defined by $T = \{t \geq 0 : |supp(l(t,\cdot))| > k\}$ is Borel and has lebesgue measure zero.*

We omit the proof of this lemma here. As mentioned above, the proof uses the following selection theorem (see [6] for details).

Lemma 3.2. *Suppose U is a standard Borel space and V is a σ-compact subset of a Polish space. Let $B \subset U \times V$ be a Borel set whose projection to U is the whole of U. Suppose further that, for each $x \in U$, the x-section of B is closed in V. Then there is a Borel measurable function $g : U \to V$ whose graph is contained in B.*

We now state and prove the characterization result for f.p.d.-processes in the class $\mathcal{C}$.

Theorem 3.2. *Let M be a Lévy process of the class $\mathcal{C}$.*
(a) If there exists an integer $k \geq 1$ and a measurable function $(x_1, \cdots, x_k, p_1, \cdots, p_k) : [0, \infty) \to \mathbb{R}^k \times [0, \infty)^k$ such that (i) for each $j = 0, 1, \ldots, 2k$, $\sum_{i=1}^{k} p_i(t)(x_i(t))^j$ is a polynomial in t almost everywhere, and, (ii) $l(t, \cdot) = \sum_{i=1}^{k} p_i(t)\delta_{\{x_i(t)\}}(\cdot)$ is a version of the derivative measure for M, then M is finitely polynomially determined (indeed, $(2k + 2)$-polynomially determined) in $\mathcal{C}$.
(b) Conversely, if M is finitely polynomially determined in $\mathcal{C}$, then there exists an integer $k \geq 1$ and a measurable function $(x_1, \cdots, x_k, p_1, \cdots, p_k) : [0, \infty) \to \mathbb{R}^k \times [0, \infty)^k$ such that a version of the derivative measure associated with M is given by $l(t, \cdot) = \sum_{i=1}^{k} p_i(t)\delta_{\{x_i(t)\}}(\cdot)$.

As mentioned above, the idea of the proof is similar to the homogeneous case except that it is a little more technical. One of the key observations used in the proof is that for a process M in the class $\mathcal{C}$, $\mathcal{P}_j(M) \neq \emptyset$, $1 \leq j \leq k$ if and only if the first cumulant $c_1(t)$ of the process M is a polynomial in t and for all $2 \leq j \leq k$, the functions $t \mapsto \int u^{j-2} l(t, du)$ are polynomials in t almost everywhere, where l is a version of the derivative measure associated to M. Using this, here is a brief sketch of the proof of the theorem.

Proof. (a) In view of the above observation, the conditions (i) and (ii) clearly imply that $\mathcal{P}_j(M) \neq \emptyset$, $1 \leq j \leq 2k + 2$. If now N is another process of class $\mathcal{C}$ with $\mathcal{P}_j(N) = \mathcal{P}_j(M)\ \forall\, 1 \leq j \leq 2k + 2$, then it will follow that N has the same mean function as M and also for all $0 \leq j \leq 2k$, $\int u^j l_N(t, du) = \int u^j l_M(t, du)$ for almost all $t \in [0, \infty)$, where l_N and l_M denote (versions of) the derivative measures associated with N and M respectively. Consequently, one will have $\int \prod_{i=1}^{k} (u - x_i(t))^2 l_N(t, du) = \int \prod_{i=1}^{k} (u - x_i(t))^2 l_M(t, du)$. By the same argument as in the proof of the 'if' part of Theorem 3.1, we get $l_N(t, \cdot) = l_M(t, \cdot)$ for almost every t, and hence $N \stackrel{d}{=} M$.
(b) Suppose that M is k-polynomially determined in $\mathcal{C}$. Using Lemma 3.1, one can get a version $l(t, \cdot)$ of the derivative measure associated to M such that $|supp(l(t, \cdot))| \leq k$ for *all* $t \in [0, \infty)$. For each $1 \leq j \leq k$, let $T_j = \{t \in [0, \infty) : |supp(l(t, \cdot))| = j\}$. It can be shown that each T_j and hence $\cup_j T_j$ is a Borel set. For $t \in T_j$, order the elements of $supp(l(t, \cdot))$ as $x_1(t) < \cdots < x_j(t)$ and denote the $l(t, \cdot)$-measures of these points by $p_1(t), \cdots, p_j(t)$ respectively. Also, for $j < i \leq k$, set $x_i(t) = x_j(t) + 1$ and $p_i(t) = 0$. Finally, for $t \notin \cup_j T_j$, set $x_i(t) \equiv y_i$ and $p_i(t) \equiv 0$ for all $1 \leq i \leq k$, where $y_1, \cdots, y_k$ is any arbitrarily chosen set of k points. With these notations, it is clear that $l(t, \cdot)$ has the form asserted. One can now show that the mapping $t \mapsto (x_1(t), \cdots, x_k(t), p_1(t), \cdots, p_k(t))$ as defined above is measurable and that completes the proof. □

Remark: In the next section, we will see some examples of possible forms of the functions $x_i(t)$ and $p_i(t)$. Let us remark here that it is possible to formulate the definition of the class $\mathcal{C}$ in terms of the Lévy measures and then to give a characterization involving the 'derivative measure' arising out of the Lévy measure. However, it is not clear how to go beyond the class $\mathcal{C}$ and to even formulate a condition that will, for example, characterize the f.p.d. processes among all Lévy processes.

4 Some Examples

The most commonly known examples of polynomially harmonizable processes are the standard Brownian motion and the standard Poisson process. One can easily see that for a Brownian motion with μ and σ^2 as its drift and diffusion coefficients respectively, a canonical sequence of time-space harmonic polynomials is given by $P_k(t,x) = (\sigma t)^{k/2} p_k(\frac{x-\mu t}{\sqrt{\sigma t}})$, where the p_k are the usual one-variable Hermite polynomials as defined in Section 1.

Similarly, for the Poisson process with intensity λ, a sequence of of time-space harmonic polynomials is given by $P_k(t,x) = \sum_{j=0}^{k} \binom{k}{j} x^j \sum_{i=0}^{k-j} \left\{ {k-j \atop i} \right\} (\lambda t)^i$, where $\left\{ {1 \atop i} \right\}$ denote the Stirling numbers of the second kind.

If M is a non-homogeneous compound Poisson process with intensity function $\lambda(\cdot)$ and jump-size distribution F, then it is not difficult to see, using the results described in Section 2, that M is polynomially harmonizable if and only if $\lambda(\cdot)$ is a polynomial function and F has finite moments of all orders. It is possible, though cumbersome, to get an explicit sequence of time-space harmonic polynomials.

A not so well-known example of a p-harmonizable process is the process $M = \mathrm{BES}^2(1)$, the square of the 1-dimensional Bessel process. It is well-known that this is a semi-stable markov process whose generator is given by $\frac{d}{dx} + 2x\frac{d^2}{dx^2}$. Using this, one can show that M is polynomially harmonizable and that a sequence of its time-space harmonic polynomials is given by $P_k(t,x) = \sum_{j=0}^{k} (-2)^j \cdot \frac{k!}{(2j)!(k-j)!} t^{k-j} x^j$.

Using the technique mentioned at the end of Section 2, we can now get other examples of p-harmonizable processes that arise as markov processes whose semigroups are intertwined with that of the process $\mathrm{BES}^2(1)$. Some examples of random variables which lead to interesting semigroups intertwined with that of $\mathrm{BES}^2(1)$, in the sense described in Section 2, are
(i) $Z = Z_{\frac{1}{2},b}$, having the beta distribution with parameters $\frac{1}{2}$ and b,
and,
(ii) $Z = 2Z_{b+\frac{1}{2}}$, where $Z_{b+\frac{1}{2}}$ has gamma distribution with parameter $b + \frac{1}{2}$.
The first one leads to the process $\mathrm{BES}^2(2b)$, the square of the Bessel process of

dimension $2b$, while the latter leads to a certain process detailed in Yor [8] with "increasing saw-teeth" paths.
Another interesting example of a process intertwined with $\mathrm{BES}^2(1)$ in the same way is what is called Azema's martingale (see Yor [9]) defined as $M_t = sgn(B_t) \cdot \sqrt{t - g_t}$, $t \geq 0$, where B is the standard Brownian motion and g_t denotes the last zero of B before time t. The multiplicative kernel here is given by the random variable m_1, ther terminal value of " Brownian meander". In [9], Yor uses Chaotic Representation Property to give an alternative proof of p-harmonizability of Azema's martingale as well as each member of the class of "Emery's martingales'. As an illustration of our method, we use the time-space harmonic polynomials of $\mathrm{BES}^2(1)$ as obtained above and the intertwinning to describe time-space harmonic polynomials for two of the cases mentioned above.

In the case of $\mathrm{BES}^2(2b)$, a sequence of time-space harmonic polynomials are given by $P_k(t,x) = \sum_{j=0}^{k} \frac{(-2)^j (\frac{1}{2})_j \, t^{k-j}}{(2j)!(b+\frac{1}{2})_j \, (k-j)!} x^j$, where $(y)_k$ stands for the product $\prod_{i=0}^{k-1} (y+i)$.

For the Azema's martingale, one uses the fact the m_1 has a Rayleigh distribution to obtain a sequence of time-space harmonic polynomials given by $P_k(t,x) = EH_k(t, m_1 x) = \sum_{j=0}^{k} 2^{\frac{j}{2}} \Gamma(\frac{j}{2}+1) h_j^{(k)}(t) x^j$, where $\Gamma(\cdot)$ denotes the gamma function and $H_k(t,x) = \sum_{j=0}^{k} h_j^{(k)}(t) x^j$ are the 2-variable Hermite polynomials.

We now discuss some examples of f.p.d. processes. First of all, it is not difficult to see that the only 2-p.d. Lévy processes are those that are deterministic, that is, M_t is identically equal to a polynomial $p(t)$. Our first example of a non-trivial f.p.d. process is the standard Brownian motion, which is a homogeneous Lévy process with $l(du) = \delta_{\{0\}}(du)$. Thus, by our Theorem 3.1, standard Brownian motion is uniquely determined among homogeneous Lévy processes by its first four time-space harmonic polynomials, for example, by the first four 2-variable Hrermite polynomials. This result should be contrasted with the well-known characterization due to Lévy, which says that the first two Hermite polynomials suffice if one assumes continuity of paths in addition. In contrast, our result says that among all homogeneous Lévy processes, standard Brownian motion is the only one for which the first four hermite polynomials are time-space harmonic. A natural question is whether we can do with less than four. The answer is an emphatic 'no'. An example of another homogeneous Lévy process for which the first three Hermite polynomials are time-space harmonic is the mean zero process determined by the Kolmogorov measure $L(dt, du) = dt \otimes \tilde{l}(du)$, where $\tilde{l}(du) = \frac{1}{2}\left[\delta_{\{-1\}}(du) + \delta_{\{1\}}(du)\right]$.

It is not difficult to see that any gaussian Lévy process, with mean and variance functions being polynomials, is also 4-p.d.

For the homogeneous Poisson process with intensity λ, one has $l(du) = \lambda\delta_{\{1\}}$, so that once again it is 4-p.d. among all homogeneous Lévy processes. Here also, four is the minimum number needed, since one can easily construct an example of a different homogeneous Lévy process for which the first three Charlier polynomials are time-space harmonic.

For the non-homogeneous compound Poisson process, it can easily be seen that it is f.p.d. if and only if the jump-size distribution is finitely supported and the intensity function is a polynomial function and that in this case, it is actually $(2k+2)$-p.d. where k is the cardinality of the support of the jump-size distribution.

We conclude with some examples of f.p.d. processes in the class $\mathcal{C}$. We have a characterization of such processes in Theorem 3.2. Here are some examples of possible forms of the functions $x_i(t)$ and $p_i(t)$, that appear in that Thoerem. We consider only the case $k = 2$. The simplest possible case is that $x_1(t), x_2(t)$ and $p_1(t) \geq 0, p_2(t) \geq 0$ are themselves polynomials. Another possibility is that $x_1(t) = a(t) + \sqrt{b(t)}, x_2(t) = a(t) - \sqrt{b(t)}, p_1(t) = c(t) + d(t)\sqrt{b(t)}, p_2(t) = c(t) - d(t)\sqrt{b(t)}$, where a, b, c, d are polynomials so chosen that $c + d\sqrt{b}, c - d\sqrt{b}$ are both non-negative on $[0, \infty)$. One can similarly construct other examples. From Theorem 3.2, it follows that all these would lead to processes that are f.p.d (in fact, 6-p.d.) in the class $\mathcal{C}$.

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