

Designs on Association Schemes

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Abstract

An association scheme partitions a finite set Ω into symmetric subsets, one of which is the diagonal subset. This paper develops the idea of a design map between two association schemes. In many designed experiments, the structure on the experimental units is an orthogonal block structure. These appear to be the structures where both the components-of-variance and patterns-of-covariance approaches (almost) agree. By replacing orthogonal block structures by association schemes, only the patterns-of-covariance model generalizes.

Keywords: association schemes; balanced design; experimental design; general balance; Latin square; orthogonal block structures

1 Introduction

Terry Speed and I worked together in the 1980s on problems in the analysis of variance. My motivation was to understand how an analysis of variance could be defined by the randomization used in setting up the experiment [3]; his was more fundamental, seeking to answer the question ‘What is an analysis of variance?’ [29]. We were both heavily influenced by John Nelder’s two papers [25, 26], in which he defines simple orthogonal block structures, makes an unsubstantiated claim about randomization, defines general balance, and shows how to analyse data from generally balanced experiments with many strata.

In joint work with Cheryl Praeger and Chris Rowley [7], we were able to generalize Nelder’s simple orthogonal block structures to a class which I now call poset block structures, and prove that Nelder’s claim about randomization holds in poset block structures. The other three authors extended this work in [27], while I showed in [4] that poset block structures are the same as the ‘complete balanced response structures’ which Kempthorne and his team at Ames, Iowa had studied extensively [21, 22, 32, 36].

More surprisingly, in [30, 31] Speed and I found that if you ignore the question of randomization then you can define an even wider class of structures in which all of Nelder’s theory carries through, with rather easy proofs. Today I use the term ‘orthogonal block structure’ for structures in this class [4]. An important input from Speed was to recognise that these orthogonal block structures are association schemes: this insight has influenced my own subsequent work enormously. A second key input from Speed was to introduce concepts from partial orders, most importantly the Möbius function,

which enables us to give explicit formulae which do not involve matrix inverses. In conversation in 1990, Oscar Kempthorne told me how important he thought the introduction of the Möbius function was to the subject. He said that the Möbius function really did the job; he wished that he and his colleagues had known about it.

Orthogonal block structures are reviewed in Section 2. They give a context for the remainder of the paper. In a very large proportion of designed experiments, the structure on the experimental units is an orthogonal block structure, but other association schemes do occur.

In [20], Houtman and Speed examined general balance in detail. In order to include as many covariance structures as possible, they did not restrict their attention to structures defined by combinatorial concepts such as ‘in the same block’. Instead, they defined a linear model to ‘have orthogonal block structure’ if all the eigenspaces of the covariance matrix are known. Everything about general balance and estimation was worked through in this framework. It is certainly true that general balance can be fruitfully defined whenever the eigenspaces of the covariance matrix are known. However, I prefer to retain the term ‘orthogonal block structure’ for the combinatorial structures defined in Section 2.

Section 5.2 of [20] discusses partially balanced incomplete-block designs. These have an association scheme defined on the set of treatments: indeed, this is the context in which association schemes were defined [9, 10]. It is fairly natural to extend the idea of partial balance to other orthogonal block structures: see [8, 18, 19] for nested block designs and [16] for nested row-column designs. However, Section 5.2 went far beyond that, because it proposed that both the set of treatments and the set of experimental units could have an arbitrary association scheme defined on them.

This idea, of two association schemes and a design map from one to the other, was given less than two pages in [20]. It is developed in the main part of this paper.

There are two rather natural ways of defining a covariance matrix on a structured set of random variables. If the structure is defined by partitions on the set, then independent random variables can be associated with each class (part) of each partition: those associated with the same partition have the same variance. This gives the components-of-variance model, which is widely used: see [28]. On the other hand, if the structure is defined by a partition on the ordered pairs from the set, one can demand that the covariance is the same for all pairs in the same part. This gives the patterns-of-covariance model, which is natural if the model is justified by randomization: see [3]. Orthogonal block structures appear to be precisely those structures where not only are both approaches possible and tractable but also the two approaches (almost) agree, as shown in Section 2. However, in generalizing from orthogonal block structures to association schemes, only the second approach is possible.

2 Orthogonal block structures

Let F be a partition of a finite set Ω . Define the subspace V_F of the real vector space \mathbb{R}^Ω to consist of all those vectors which are constant on every class of F . Then $\dim V_F$ is equal to n_F , the number of classes of F .

Two $\Omega \times \Omega$ real matrices are defined by F . The first is the *relation matrix* R_F , whose (α, β) -entry is equal to 1 if $F(\alpha) = F(\beta)$ and to 0 otherwise. Here we are writing $F(\alpha)$ for the class of F which contains α . The second is the *projection matrix* P_F . There is a natural inner product $\langle \cdot, \cdot \rangle$ on \mathbb{R}^Ω given by

$$\langle v, w \rangle = \sum_{\omega \in \Omega} v_\omega w_\omega;$$

this defines orthogonality, and P_F is just the matrix of orthogonal projection onto V_F . The (α, β) -entry of P_F is equal to $1/|F(\alpha)|$ if $F(\alpha) = F(\beta)$; otherwise it is zero.

The partition F is defined to be *uniform* if all of its classes have the same size, which must be $|\Omega|/n_F$. If F is uniform then $|\Omega|P_F = n_F R_F$.

There are two trivial uniform partitions of Ω . The *universal* partition U has a single class. Thus V_U is the 1-dimensional subspace consisting of the constant vectors. At the other extreme, the classes of the *equality* partition E are all singletons, so $V_E = \mathbb{R}^\Omega$.

Suppose that F and G are two partitions of Ω . We say that F is *finer* than G , and write $F \preceq G$, if every F -class is contained in a G -class. In this case, $V_G \leq V_F$. In particular, $E \preceq F \preceq U$ for every partition F of Ω .

More generally, the *infimum* $F \wedge G$ of F and G is defined to be the coarsest partition which is finer than both F and G . Its classes are the non-empty intersections of F -classes with G -classes. Dually, the *supremum* $F \vee G$ of F and G is the finest partition which is coarser than both F and G . Its classes are the connected components of the graph whose vertices are the elements of Ω and whose edges are the pairs $\{\alpha, \beta\}$ for which $F(\alpha) = F(\beta)$ or $G(\alpha) = G(\beta)$. It follows that $V_{F \vee G} = V_F \cap V_G$; however, there is no simple expression for $V_{F \wedge G}$.

Partitions F and G are defined to be *orthogonal* to each other if P_F commutes with P_G ; that is, if V_F is geometrically orthogonal to V_G in the sense that $V_F \cap V_{F \vee G}^\perp$ is orthogonal to $V_G \cap V_{F \vee G}^\perp$: see [33]. If $F \preceq G$ then $F \vee G = G$ so $V_G \cap V_{F \vee G}^\perp$ is the zero subspace, which is orthogonal to all subspaces, so F is orthogonal to G . In particular, F is orthogonal to U , E and itself.

Orthogonality is equivalent to a combinatorial condition that statisticians will recognise as ‘proportional meeting’. Figure 1 shows five examples where the set Ω is a rectangle. In each case F is the partition into rows, G is the partition into columns, and the numbers show the size of the row-column intersections. In (a), (c) and (d), each of F , G and $F \wedge G$ is uniform; in (e), F and G are uniform but neither $F \wedge G$ nor $F \vee G$ is; in (a)–(d), $F \vee G = U$; in (c) and (d), $F \wedge G = E$; in (a), (b), (d) and (e), F is orthogonal to G .

If F , G , $F \wedge G$ and $F \vee G$ are all uniform then there is a simple criterion for orthogonality: F is orthogonal to G if and only if, for all pairs α and β , $F(\alpha) \cap G(\beta)$ is

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Figure 1: Examples to demonstrate orthogonality

non-empty if and only if $G(\alpha) \cap F(\beta)$ is non-empty.

Sections 42 and 76 of [17] show that if F is orthogonal to G then

$$P_F P_G = P_{F \vee G}. \quad (1)$$

Definition

An *orthogonal block structure* on Ω is a set \mathcal{F} of uniform partitions of Ω such that

- (i) \mathcal{F} contains E and U ;
- (ii) if $F \in \mathcal{F}$ and $G \in \mathcal{F}$ then $F \wedge G \in \mathcal{F}$ and $F \vee G \in \mathcal{F}$;
- (iii) if $F \in \mathcal{F}$ and $G \in \mathcal{F}$ then F is orthogonal to G .

Suppose that \mathcal{F} is an orthogonal block structure. Then \mathcal{F} defines a partition of $\Omega \times \Omega$ into *associate classes* C_F labelled by elements of \mathcal{F} , as follows. Let α and β be in Ω . Since \mathcal{F} is closed under \wedge , there is a unique finest F in \mathcal{F} such that $F(\alpha) = F(\beta)$. Now the class $C(\alpha, \beta)$ containing (α, β) is C_F , and we call α and β *F-associates*. In other words, $(\alpha, \beta) \in C_F$ if and only if (i) $F(\alpha) = F(\beta)$ and (ii) if $G \in \mathcal{F}$ and $G(\alpha) = G(\beta)$ then $F \preceq G$. The $\Omega \times \Omega$ *adjacency matrix* A_F is defined to have (α, β) -entry equal to 1 if α and β are F -associates; otherwise it is zero.

Example 1

Suppose that Ω consists of b blocks, each of which is an $n \times m$ rectangular array. Let B be the partition into the blocks, F the partition into the bn rows and G the partition into the bm columns. Then $\{E, F, G, B, U\}$ is an orthogonal block structure. Moreover (α, β) is in

- C_E if $\alpha = \beta$
- C_F if $\alpha \neq \beta$ but α and β are in the same row
- C_G if $\alpha \neq \beta$ but α and β are in the same column
- C_B if α and β are in the same block but different rows and columns
- C_U if α and β are in different blocks.

Also, given an orthogonal block structure \mathcal{F} , define

$$W_F = V_F \cap \bigcap_{G \in \mathcal{F}, F \prec G} V_G^\perp \quad (2)$$

for F in \mathcal{F} . Since \mathcal{F} is closed under \vee and satisfies the orthogonality condition, it is fairly easy to show that $W_F \perp W_G$ if $F \neq G$, and that

$$V_F = \bigoplus_{G \in \mathcal{F}, F \preceq G} W_G \quad (3)$$

for F in \mathcal{F} .

$$\zeta(F, G) = \begin{cases} 1 & \text{if } F \preceq G \\ 0 & \text{otherwise.} \end{cases}$$

The elements of \mathcal{F} can be written in such an order that, as a matrix, ζ is upper triangular with all diagonal elements equal to 1. Therefore, ζ has an inverse matrix μ , and it is this which is called the Möbius function.

The definition of A_F shows that

$$R_F = \sum_{G \in \mathcal{F}} \zeta(G, F) A_G$$

for all F in \mathcal{F} . Hence

$$A_F = \sum_{G \in \mathcal{F}} \mu(G, F) R_G$$

for all F in \mathcal{F} , and $\text{span}\{A_F : F \in \mathcal{F}\} = \text{span}\{R_F : F \in \mathcal{F}\}$. Since all the partitions are uniform, $|\Omega|P_F = n_F R_F$ for all F in \mathcal{F} , and $\text{span}\{P_F : F \in \mathcal{F}\} = \text{span}\{R_F : F \in \mathcal{F}\}$. Finally, let S_F be the matrix of orthogonal projection onto W_F . Equation (3) shows that

$$P_F = \sum_{G \in \mathcal{F}} \zeta(F, G) S_G$$

for all F in \mathcal{F} , and hence

$$S_F = \sum_{G \in \mathcal{F}} \mu(F, G) P_G.$$

Therefore

$$\begin{aligned} \text{span}\{A_F : F \in \mathcal{F}\} &= \text{span}\{R_F : F \in \mathcal{F}\} = \text{span}\{P_F : F \in \mathcal{F}\} \\ &= \text{span}\{S_F : F \in \mathcal{F}\}. \end{aligned} \quad (4)$$

Now suppose that Ω is the set of experimental units in an experiment. We observe a data vector that is a realization of a random vector Y . What should we assume about the covariance matrix $\text{Cov}(Y)$?

One common assumption is that there are independent random variables associated with every class of every partition in \mathcal{F} : all those associated with F have variance σ_F^2 . This gives

$$\text{Cov}(Y) = \sum_{F \in \mathcal{F}} \sigma_F^2 R_F, \quad (5)$$

which is called the *components-of-variance* model. A second assumption is that all pairs of F -associates have the same covariance γ_F , for all F in \mathcal{F} . This gives the *patterns-of-covariance* model

$$\text{Cov}(Y) = \sum_{F \in \mathcal{F}} \gamma_F A_F. \quad (6)$$

Because of Equation (4), both of Equations (5) and (6) can be reparametrized as

$$\text{Cov}(Y) = \sum_{F \in \mathcal{F}} \xi_F S_F. \quad (7)$$

This shows that the spaces W_F are eigenspaces of $\text{Cov}(Y)$ in both cases, with eigenvalues ξ_F . Nelder called these eigenspaces *strata*, so the quantities ξ_F are called the *stratum variances*. His proposed analysis of the data begins by projecting the data onto each stratum, where it has effectively a scalar covariance matrix, so that ordinary least squares can be applied: see also [1].

However, models (5) and (6) are not identical. A covariance matrix is non-negative definite, so Equation (7) is constrained by

$$\xi_F \geq 0 \quad \text{for all } F \text{ in } \mathcal{F}. \quad (8)$$

Variances must also be non-negative, so (5) is constrained by

$$\sigma_F^2 \geq 0 \quad \text{for all } F \text{ in } \mathcal{F}. \quad (9)$$

Now,

$$\sum_F \sigma_F^2 R_F = \sum_F \sigma_F^2 \frac{|\Omega|}{n_F} P_F = \sum_F \frac{|\Omega|}{n_F} \sigma_F^2 \sum_G \zeta(F, G) S_G$$

so

$$\xi_G = \sum_F \zeta(F, G) \frac{|\Omega|}{n_F} \sigma_F^2,$$

and therefore condition (9) is stronger than condition (8).

In [20], Houtman and Speed effectively started with Equation (7) for known projectors S_F . By replacing orthogonal block structures by association schemes, we can also retain Equation (6) for known adjacency matrices A_F . That is, the patterns-of-covariance model generalizes but the components-of-variance model does not.

3 Association schemes

A subset of $\Omega \times \Omega$ can be identified with its $\Omega \times \Omega$ adjacency matrix A , whose (α, β) -entry is equal to 1 if (α, β) is in the subset and to 0 otherwise. The subset is said to be *symmetric* if its adjacency matrix is a symmetric matrix. The *diagonal* subset is $\{(\omega, \omega) : \omega \in \Omega\}$: its adjacency matrix is the identity matrix I . The adjacency matrix of $\Omega \times \Omega$ is the all-1 matrix J .

Definition

An *association scheme* on Ω is a partition of Ω into symmetric subsets, called *associate classes*, one of which is the diagonal subset, such that the product of any two of its adjacency matrices is a real linear combination of the adjacency matrices of associate classes.

The *trivial* association scheme has just one non-diagonal associate class. If B is a non-trivial uniform partition of Ω then B defines a *group-divisible* association scheme on Ω : its non-diagonal classes are

$$\{(\alpha, \beta) \in \Omega \times \Omega : B(\alpha) = B(\beta) \text{ but } \alpha \neq \beta\} \text{ and}$$

$$\{(\alpha, \beta) \in \Omega \times \Omega : B(\alpha) \neq B(\beta)\}.$$

If \mathcal{P} is an association scheme, the set $\mathcal{A}(\mathcal{P})$ of all real linear combinations of its adjacency matrices forms an algebra, called the *Bose–Mesner algebra*. A key theorem for association schemes (see [14, Chapter 17]) is that $\mathcal{A}(\mathcal{P})$ is commutative and hence has a basis $\{S_e : e \in \mathcal{E}\}$ consisting of the matrices of orthogonal projection onto its mutual eigenspaces W_e , for e in some suitable index set \mathcal{E} . If the adjacency matrices are A_i for i in I then $|I| = |\mathcal{E}| = \dim \mathcal{A}(\mathcal{P})$, but there is not usually any canonical bijection between I and \mathcal{E} . The subspace V_U is always a common eigenspace, with projector $|\Omega|^{-1}J$.

Equations (4) and (1) show that the non-zero adjacency matrices A_F of an orthogonal block structure \mathcal{F} form an association scheme, and the common eigenspaces are the non-zero strata W_F defined by Equation (2). It is convenient to extend the term ‘stratum’ to all association schemes. If none of the A_F is zero then $I = \mathcal{F} = \mathcal{E}$ and none of the W_F is zero: here there is a natural bijection between the associate classes and the strata.

4 Designs

I take the view, explained in [2], that a design is a function h from one structured set Ω , consisting of the experimental units, to another structured set Θ , consisting of the treatments. The treatment assigned to experimental unit ω is just $h(\omega)$. In this paper, the structures on Ω and Θ are both association schemes.

Information about the design map can be recorded in the $\Omega \times \Theta$ design matrix X , whose (ω, θ) -entry is equal to 1 if $h(\omega) = \theta$ and to 0 otherwise. If A is the adjacency matrix of a subset Δ of $\Omega \times \Omega$, then the (θ, ϕ) -entry in $X'AX$ is equal to

$$|\{(\alpha, \beta) \in \Delta : h(\alpha) = \theta \text{ and } h(\beta) = \phi\}|.$$

Here X' denotes the transpose of X . In particular, $X'X = X'IX$ is diagonal with (θ, θ) -entry equal to the replication of treatment θ , which is $|h^{-1}(\theta)|$, while the (θ, ϕ) -entry of $X'JX$ is equal to $|h^{-1}(\theta)| |h^{-1}(\phi)|$.

Definition

Let \mathcal{P} be an association scheme on Ω with adjacency matrices A_i , for i in I , and let Q be an association scheme on Θ with adjacency matrices B_j , for j in J . Let $h: \Omega \rightarrow \Theta$ be a design with design matrix X . Then h is *partially balanced* for \mathcal{P} with respect to Q if there are integers λ_{ij} for (i, j) in $I \times J$ such that

$$X'A_iX = \sum_j \lambda_{ij} B_j$$

for all i in I ; that is, if θ and ϕ are j -th associates in Θ then there are λ_{ij} pairs of i -th associates α and β in Ω such that $h(\alpha) = \theta$ and $h(\beta) = \phi$.

When \mathcal{P} is group divisible, this definition agrees with the usual definition of a partially balanced block design. In general, the definition is identical to the definition of (\mathcal{P}, Q) -balance in Section 5.2 of [20]. However, the usual definition of a balanced block design is more restrictive: a block design is balanced if it is partially balanced, in the above sense, with respect to the trivial association scheme on Θ . It therefore seems less confusing to reserve the unqualified term ‘balance’ for the case in which Q is trivial: that is, h is balanced for \mathcal{P} if it is partially balanced for \mathcal{P} with respect to the trivial association scheme on Θ . Such balanced designs are investigated in [6].

If \mathcal{P} is the association scheme defined by an orthogonal block structure then Equation (4) shows that an equivalent definition of partial balance is that there are integers λ_{ij}^* such that $X'R_iX = \sum_j \lambda_{ij}^* B_j$ for all i . Thus Figure 2 shows a design which is partially balanced for the association scheme of the orthogonal block structure in Example 1 (with $b = n = 2$ and $m = 3$) with respect to the group-divisible scheme defined by the partition $A, B \parallel C, D \parallel E, F$.

It is usual to use the label 0 to index the diagonal associate class. In a partially balanced design every treatment has replication λ_{00} , so the design is equi-replicate. It is conventional to write r for λ_{00} .



Figure 2: A partially balanced design on the orthogonal block structure in Ex. 1

Given a random vector Y on Ω , a natural assumption is that

$$\text{Cov}(Y) = \sum_j \gamma_j A_j; \quad (10)$$

that is, that $\text{cov}(Y_\alpha, Y_\beta)$ depends only on the associate class containing (α, β) . Equation (10) can be reparametrized as

$$\text{Cov}(Y) = \sum_e \xi_e S_e, \quad (11)$$

where S_e are the stratum projectors in \mathcal{P} and ξ_e are the stratum variances.

The other assumption for a linear model for a designed experiment is that

$$\mathbb{E}(Y) = X\tau$$

for some unknown vector τ in \mathbb{R}^Θ . Projection onto the stratum W_e gives

$$\mathbb{E}(S_e Y) = S_e X\tau \text{ and}$$

$$\text{Cov}(S_e Y) = S_e \text{Cov}(Y) S_e' = \xi_e S_e,$$

which is scalar on W_e .

Put $L_e = X' S_e X$, which is called the *information matrix* for stratum W_e . If $x \in \text{Im } L_e$ then there is a vector z in \mathbb{R}^Θ such that $L_e z = x$. Ordinary least-squares theory shows that the best linear unbiased estimator of $\langle x, \tau \rangle$ from $S_e Y$ is $z' X' S_e Y$, whose variance is $z' X' S_e' (\xi_e S_e) S_e X z = \xi_e z' L_e z$. In particular, if x is an eigenvector of L_e with eigenvalue $r\epsilon$ then this variance is equal to $\xi_e x' x / r\epsilon$.

In the textbook situation, where $\text{Cov}(Y) = \sigma^2 I$, the variance is $x' x \sigma^2 / r$. The ratio $\sigma^2 \epsilon / \xi_e$ is called the *efficiency* for x in stratum W_e , while ϵ , which depends on the design and not on the values of the stratum variances, is called the *efficiency factor* for x in stratum W_e .

Now, S_e is a linear combination of the adjacency matrices A_i , so L_e is a linear combination of the matrices $X' A_i X$. If the design is partially balanced for \mathcal{P} with respect to Q then each of the matrices $X' A_i X$ is in $\mathcal{A}(Q)$, so $L_e \in \mathcal{A}(Q)$. Therefore the strata of Q are (contained in) eigenspaces of L_e . Write ϵ_{ef} for the efficiency factor for vectors from stratum f (in Q) in stratum W_e (of \mathcal{P}). If the strata in Q have projection matrices T_f for f in \mathcal{F} then

$$L_e = r \sum_{f \in \mathcal{F}} \epsilon_{ef} T_f. \quad (12)$$

The matrices L_e are non-negative definite and sum to rI , so, for each fixed f in \mathcal{F} , the efficiency factors ϵ_{ef} are non-negative and sum to 1. If there is any e such that $\epsilon_{ef} = 1$ then any contrast $\langle x, \tau \rangle$ with x in $\text{Im } T_f$ is estimated only in stratum W_e . Otherwise,

information has to be combined from two or more strata, as described in [20]. If every efficiency factor is equal to 0 or 1 then no combining is needed and the design is said to be *orthogonal*.

Both \mathcal{P} and Q have the one-dimensional stratum labelled U . Moreover,

$$L_U = |\Omega|^{-1} X' J_\Omega X = |\Omega|^{-1} r^2 J_\Theta = r |\Theta|^{-1} J_\Theta = r T_U.$$

Therefore, $\varepsilon_{UU} = 1$, $\varepsilon_{Uf} = 0$ if $f \neq U$ and $\varepsilon_{eU} = 0$ if $e \neq U$.

If the design is balanced, V_U^\perp is the only other stratum in Q . It is convenient to give it no label, and write ε_e for the eigenvalue of L_e on V_U^\perp .

Just as for incomplete-block designs, for a more general association scheme \mathcal{P} the $\mathcal{E} \times \mathcal{F}$ table of efficiency factors gives important information about the design. Proposed designs for an experiment are compared on the basis of these tables. In Section 6 onwards, some partially balanced designs and their efficiency factors are given for those association schemes which are not orthogonal block structures but which are plausible for the set of experimental units in a designed experiment, as noted in [3]. First, Section 5 gives some theory which aids subsequent calculations.

5 Composite designs

If $h_1: \Omega \rightarrow \Theta$ and $h_2: \Theta \rightarrow \Psi$ are functions then we can form the composite function $h_2 \circ h_1: \Omega \rightarrow \Psi$, as shown in Figure 3. If h_1 and h_2 are both designs, then so is $h_2 \circ h_1$, and it is natural to call $h_2 \circ h_1$ a *composite* design, although this conflicts with the terminology in [11]. If h_i is equi-replicate with replication r_i for $i = 1, 2$ then $h_2 \circ h_1$ is equi-replicate with replication $r_1 r_2$.

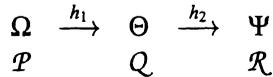


Figure 3: A composite design

Theorem 1

Let \mathcal{P} , Q and \mathcal{R} be association schemes on Ω , Θ and Ψ respectively. Let $h_1: \Omega \rightarrow \Theta$ and $h_2: \Theta \rightarrow \Psi$ be designs. If h_1 is partially balanced for \mathcal{P} with respect to Q and h_2 is partially balanced for Q with respect to \mathcal{R} then $h_2 \circ h_1$ is partially balanced for \mathcal{P} with respect to \mathcal{R} .

Proof

Let the adjacency matrices for \mathcal{P} be A_i for i in I , for Q be B_j for j in J , and for \mathcal{R} be C_k for k in \mathcal{K} . For $i = 1, 2$ let X_i be the design matrix for h_i . There are integers λ_{ij} , for

(i, j) in $I \times J$, and v_{jk} , for (j, k) in $J \times \mathcal{K}$, such that

$$X_1' A_i' X_1 = \sum_j \lambda_{ij} B_j$$

for i in I and

$$X_2' B_j' X_2 = \sum_k v_{jk} C_k$$

for j in J . Now, the design matrix for $h_2 \circ h_1$ is $X_1 X_2$, and

$$(X_1 X_2)' A_i (X_1 X_2) = X_2' X_1' A_i X_1 X_2 = X_2' \sum_j \lambda_{ij} B_j X_2 = \sum_k \sum_j \lambda_{ij} v_{jk} C_k,$$

for all i in I , and so $h_2 \circ h_1$ is partially balanced for \mathcal{P} with respect to \mathcal{R} . ■

The following theorem gives a partial converse.

Theorem 2

Let \mathcal{P} , Q and \mathcal{R} be association schemes on Ω , Θ and Ψ respectively. Let A_i , for i in I , be the adjacency matrices for \mathcal{P} . Let $h_1: \Omega \rightarrow \Theta$ and $h_2: \Theta \rightarrow \Psi$ be designs. If $\{X_1' A_i X_1 : i \in I\}$ spans $\mathcal{A}(Q)$ and $h_2 \circ h_1$ is partially balanced for \mathcal{P} with respect to \mathcal{R} then h_1 is partially balanced for \mathcal{P} with respect to Q and h_2 is partially balanced for Q with respect to \mathcal{R} .

Proof

If $\{X_1' A_i X_1 : i \in I\}$ spans $\mathcal{A}(Q)$ then $X_1' A_i X_1 \in \mathcal{A}(Q)$ for all i in I and so h_1 is partially balanced for \mathcal{P} with respect to Q . Moreover, if B_j is an adjacency matrix for Q then $B_j = X_1' M X_1$ for some M in $\mathcal{A}(\mathcal{P})$. If $h_2 \circ h_1$ is partially balanced for \mathcal{P} with respect to \mathcal{R} then $(X_1 X_2)' M (X_1 X_2) \in \mathcal{A}(\mathcal{R})$; that is, $X_2' B_j X_2 \in \mathcal{A}(\mathcal{R})$. Hence h_2 is partially balanced for Q with respect to \mathcal{R} . ■

If $\{X_1' A_i X_1 : i \in I\}$ spans $\mathcal{A}(Q)$ then the information matrices for h_1 span $\mathcal{A}(Q)$, so their mutual eigenspaces are precisely the strata in Q . Otherwise there is at least one pair of strata in Q with the same efficiency factors in every stratum of \mathcal{P} . In some sense, a design h_1 in which $\{X_1' A_i X_1 : i \in I\}$ spans $\mathcal{A}(Q)$ has *full rank* with respect to \mathcal{P} and Q .

Theorem 3

Let \mathcal{P} , Q and \mathcal{R} be association schemes on Ω , Θ and Ψ respectively, with stratum projectors S_e for e in \mathcal{E} , T_f for f in \mathcal{F} , and U_g for g in \mathcal{G} respectively. Let $h_1: \Omega \rightarrow \Theta$ be a partially balanced design for \mathcal{P} with respect to Q whose efficiency factors are ε_{ef} for (e, f) in $\mathcal{E} \times \mathcal{F}$, and let $h_2: \Theta \rightarrow \Psi$ be a partially balanced design for Q with respect to \mathcal{R} whose efficiency factors are ε_{fg} for (f, g) in $\mathcal{F} \times \mathcal{G}$. Then the efficiency factors ε_{eg} of $h_2 \circ h_1$ are given by

$$\varepsilon_{eg} = \sum_{f \in \mathcal{F}} \varepsilon_{ef} \varepsilon_{fg}$$

for (e, g) in $\mathcal{E} \times \mathcal{G}$.

Proof

Let r_1 and r_2 be the replications of h_1 and h_2 respectively. Then Equation (12) gives

$$X_1' S_e X_1 = r_1 \sum_{f \in \mathcal{F}} \varepsilon_{ef} T_f \quad \text{and} \quad X_2' T_f X_2 = r_2 \sum_{g \in \mathcal{G}} \varepsilon_{fg} U_g.$$

Hence

$$(X_1 X_2)' S_e (X_1 X_2) = r_1 r_2 \sum_{g \in \mathcal{G}} \left(\sum_{f \in \mathcal{F}} \varepsilon_{ef} \varepsilon_{fg} \right) U_g. \quad \blacksquare$$

A version of Theorem 1 is used in [13] for the multitiered experiments described in [12] to show that if the component designs h_1 and h_2 are generally balanced then so is their composite. For example, $h_2 \circ h_1$ can be a two-phase experiment. In the first phase, treatments Ψ are applied to field plots Θ according to design h_2 . In the second phase, the treatments are the produce from Θ , which are allocated to evaluation-occasions Ω according to design h_1 . [13] uses Theorem 3 to construct analysis-of-variance tables for the composite designs.

By contrast, we shall use Theorems 2 and 3 in the case that \mathcal{P} is group-divisible. Then h_1 and $h_2 \circ h_1$ are both block designs. Knowledge about block designs will be exploited to deduce properties of h_2 .

Thus we now switch notation so that h_2 is the design function h of Section 4, with the associated notation for adjacency matrices and stratum projectors. Meanwhile, h_1 becomes a design function g from Γ to Ω , where Γ has the group-divisible association scheme defined by the orthogonal block structure $\{U, B, E\}$ for some non-trivial uniform partition B of Γ . See Figure 4, which applies to the next two sections.

$$\begin{array}{ccccc} \Gamma & \xrightarrow{g} & \Omega & \xrightarrow{h} & \Theta \\ \{U, B, E\} & & \mathcal{P} & & \mathcal{Q} \end{array}$$

Figure 4: Another composite design

6 Triangular association schemes

If \mathcal{P} is a triangular scheme $T(n)$ then Ω consists of all unordered pairs from an n -set: two elements of Ω are i -th associates if their intersection has size $2 - i$, for $i = 0, 1, 2$. This can happen in an experiment where the treatments are tasks to be carried out by teams of two people playing the same role. It can also happen in half-diallel experiments, where the experimental units consist of all crosses between n parental lines, excluding self-crosses, in situations where the gender of the parent is irrelevant.

A design h on \mathcal{P} can conveniently be shown as a symmetric square with the diagonal missing, as in Figures 5–6. The symbol in row a and column b is $h(\{a, b\})$, the

treatment on the element $\{a, b\}$ of Ω . This square layout also suggests a suitable block design for g : it has n blocks of size $n - 1$, and block a contains every pair $\{a, b\}$ with $b \neq a$. Now the composite design $h \circ g$ also has n blocks of size $n - 1$; the treatments in block a are the symbols occurring in row a of the square.

The diallel context gives a way of naming the strata for $T(n)$. They are:

- W_0 the one-dimensional space V_U ;
- W_p the $(n - 1)$ -dimensional space for contrasts between parents;
- W_q $(W_0 + W_p)^\perp$.

The efficiency factors for g are

$$\begin{array}{lll} \epsilon_{U0} = 1 & \epsilon_{Up} = 0 & \epsilon_{Uq} = 0 \\ \epsilon_{B0} = 0 & \epsilon_{Bp} = \frac{n-2}{2(n-1)} & \epsilon_{Bq} = 0 \\ \epsilon_{E0} = 0 & \epsilon_{Ep} = \frac{n}{2(n-1)} & \epsilon_{Eq} = 1. \end{array}$$

No two columns are identical, so g has full rank. Therefore, design h is partially balanced for $T(n)$ with respect to an association scheme Q on Θ if and only if the block design $h \circ g$ is partially balanced with respect to Q . Theorem 3 shows that, for stratum f in Q ,

$$\epsilon_{Bf} = \frac{n-2}{2(n-1)} \epsilon_{pf} \tag{13}$$

$$\epsilon_{Ef} = \frac{n}{2(n-1)} \epsilon_{pf} + \epsilon_{qf}. \tag{14}$$

In a block design we usually want the efficiency factors ϵ_{Bf} to be as small as possible. In a design on $T(n)$, it is plausible that $\xi_p \gg \xi_q$, so we also want the efficiency factors ϵ_{pf} to be as small as possible. Thus a strategy for finding a good design h is to find a good design g' and see if it can be arranged in a symmetric square so that $h \circ g = g'$: not all block designs g' can be so arranged.

Example 2

Figure 5 gives two balanced designs for seven treatments on the association scheme $T(7)$. The design h is constructed by omitting the main diagonal of a symmetric idempotent Latin square. Its composite design $h \circ g$ is a binary balanced block design with $\epsilon_B = 1/36$ and $\epsilon_E = 35/36$. Hence h is balanced with $\epsilon_p = 1/15$ and $\epsilon_q = 14/15$, by Equations (13) and (14). Although the design h' is also balanced, its composite design $h' \circ g$ is not binary. Now the composite design has $\epsilon_B = 2/9$ and $\epsilon_E = 7/9$ so h' has $\epsilon_p = 8/15$ and $\epsilon_q = 7/15$. Thus h is better than h' .

Example 3

Figure 6 shows a design h for 12 treatments A, \dots, L in $T(9)$. The composite design is a binary incomplete-block design which is partially balanced with respect to the group-divisible association scheme defined by the partition $A, B, C \parallel D, E, F \parallel G, H, I \parallel J, K, L$.

Although this is an orthogonal block structure, we shall label the classes and strata without reference to U and E , to avoid confusion with the labels B and E for the block design g . Label the within-group class (pairs such as $\{A, B\}$) by 1 and the between-group class (pairs such as $\{D, H\}$) by 2. Label the strata so that

$$\begin{aligned} W_0 &= V_U \\ W_g &= \text{the space for contrasts between groups} \\ W_w &= (W_0 + W_g)^\perp. \end{aligned}$$

In the composite design, $\lambda_{B1} = 3, \lambda_{B2} = 4$ and $\epsilon_{Bg} = 0$. Theorem 2.2 of [15] shows that the composite design is optimal among binary incomplete-block designs in the sense of maximizing the harmonic mean of the efficiency factors in stratum W_E , counted according to multiplicity. Equations (13)–(14) suggest that h will therefore be a good design for $T(9)$.

The design h is constructed by taking $T(9)$ to consist of unordered pairs of points in the affine plane over $GF(3)$. The letters A, \dots, L are the twelve lines of the plane, in their four parallel classes. Let π be a permutation of the parallel classes of cycle type 2^2 . Any two points a and b in the plane lie on a line ℓ containing a third point c . The line ℓ lies in a parallel class \mathcal{L} . Define $h(\{a, b\})$ to be the line through c in parallel class $\pi(\mathcal{L})$. Then row a of the square contains all lines which do not pass through a .

7 Latin-square schemes

Let Ω consist of the n^2 cells of a square array on which there are $s - 2$ mutually orthogonal Latin squares of order n , for some s with $2 \leq s \leq n - 1$. Let F_1 be the partition

	B	C	D	E	F	G
B		D	E	F	G	A
C	D		F	G	A	B
D	E	F		A	B	C
E	F	G	A		C	D
F	G	A	B	C		E
G	A	B	C	D	E	

Design h

	A	G	A	E	E	G
A		B	A	B	F	F
G	B		C	B	C	G
A	A	C		D	C	D
E	B	B	D		E	D
E	F	C	C	E		F
G	F	G	D	D	F	

Design h'

Figure 5: Two balanced designs for 7 treatments in $T(7)$

	A	B	F	E	I	H	K	L
A		C	G	J	E	L	F	H
B	C		J	G	K	F	I	E
F	G	J		D	L	B	A	I
E	J	G	D		A	K	H	B
I	E	K	L	A		G	D	C
H	L	F	B	K	G		C	D
K	F	I	A	H	D	C		J
L	H	E	I	B	C	D	J	

Figure 6: Group-divisible design for 12 treatments in T(9)

of Ω into rows, F_2 the partition of Ω into columns, and, for $i = 3, \dots, s$, let F_i be the partition of Ω into subsets defined by the letters of square i . Then $\{U, E, F_1, \dots, F_s\}$ is an orthogonal block structure on Ω . Put

$$\begin{aligned}
 A_0 &= I \\
 A_b &= A_{F_1} + \dots + A_{F_s} \\
 A_c &= J - A_0 - A_b.
 \end{aligned}$$

Then A_0, A_b and A_c are the adjacency matrices of an association scheme on Ω which is said to have *Latin-square* type L(s, n). Its strata are

$$\begin{aligned}
 W_0 &= V_U \\
 W_b &= W_{F_1} + \dots + W_{F_s} \\
 W_c &= (W_0 + W_b)^\perp.
 \end{aligned}$$

We are mostly concerned with the case that $s = 2$.

If the plots in a field trial have an $n \times m$ rectangular array, it is usually appropriate to regard them as having the rectangular association scheme R(n, m), which is the orthogonal block structure whose two non-trivial partitions correspond to the rows and columns. Even if $n = m$ the rectangular scheme may still be appropriate, because the plots may not be square or the columns may be in the direction of ploughing. However, if $m = n$ and the plots are square and cultivation is by hand then L($2, n$) may be appropriate.

A design h on L(s, n) can obviously be shown in a square array: see Figures 7 and 8. If $s = 2$ the labels $h(\omega)$, for ω in the square array, give all the information. If $s \geq 3$ then the letters of the Latin squares must also be shown. The natural choice for the block design g is a square lattice design [34]. It has sn blocks of size n , whose ‘treatments’ are the elements of Ω in the classes of F_1, \dots, F_s . The composite design $h \circ g$ also has

sn blocks of size n ; the treatments in a block are those occurring in a row, or a column, or a letter of a Latin square, in the square array.

The efficiency factors for the lattice design g are

$$\begin{array}{lll} \epsilon_{U0} = 1 & \epsilon_{Ub} = 0 & \epsilon_{Uc} = 0 \\ \epsilon_{B0} = 0 & \epsilon_{Bb} = \frac{1}{s} & \epsilon_{Bc} = 0 \\ \epsilon_{E0} = 0 & \epsilon_{Eb} = \frac{s-1}{s} & \epsilon_{Ec} = 1. \end{array}$$

Hence g has full rank, so h is partially balanced for $L(s, n)$ with respect to Q if and only if the block design $h \circ g$ is partially balanced with respect to Q . Moreover,

$$\epsilon_{Bf} = \frac{1}{s} \epsilon_{bf} \quad (15)$$

$$\epsilon_{Ef} = \frac{1}{s} [(s-1)\epsilon_{bf} + s\epsilon_{cf}] = 1 - \frac{1}{s} \epsilon_{bf} \quad (16)$$

for strata f of Q .

For the association scheme $L(s, n)$ it is plausible that $\xi_b \gg \xi_c$, so we want ϵ_{bf} to be as small as possible for all f . Once again, it appears that h will be a good design if $h \circ g$ is good.

Example 4

Figure 7 shows a design h for treatments A, \dots, G on $L(2, 4)$. The composite design $h \circ g$ is partially balanced with respect to the group-divisible scheme defined by the partition

$$A, B \parallel C, D \parallel E, F \parallel G, H$$

of Θ . Labelling the strata of the latter scheme as in Example 3, we find that the efficiency factors for $h \circ g$ are

$$\begin{array}{ll} \epsilon_{Bg} = 0 & \epsilon_{Bw} = \frac{1}{4} \\ \epsilon_{Eg} = 1 & \epsilon_{Ew} = \frac{3}{4}. \end{array}$$

Equations (15) and (16) show that those for h are

$$\begin{array}{ll} \epsilon_{bg} = 0 & \epsilon_{bw} = \frac{1}{2} \\ \epsilon_{cg} = 1 & \epsilon_{cw} = \frac{1}{2}. \end{array}$$

The cyclic block design for eight treatments with initial block $\{0, 1, 2, 4\}$ is more efficient than $h \circ g$, but it cannot be arranged as the rows and columns of a 4×4 square.

A	C	E	G
C	B	G	F
F	H	A	D
H	E	D	B

Figure 7: Design for 8 treatments in L(2, 4)

Example 5

Houtman and Speed [20] discuss the design h in Figure 8, originally given by Kshirsagar [23]. They regard the association scheme on the 6×6 square Ω as $R(6,6)$, and show that h is partially balanced for $R(6,6)$ with respect to the association scheme $L(2,3)$ on Θ shown in Figure 9. However, if we regard the association scheme on Ω as $L(2,6)$ then the design h is balanced.

B	D	H	G	F	C
C	E	G	B	D	I
E	F	C	A	G	H
D	I	A	H	C	E
F	G	I	E	A	B
A	H	B	D	I	F

Figure 8: Design h on Ω in Ex. 5; Ω may carry $R(6,6)$ or $L(2,6)$

A	B	C
D	E	F
G	H	I

Figure 9: Treatment set Θ for Ex. 5; its association scheme may be $L(2,3)$ or trivial

In nested row-column designs the experimental units carry the orthogonal block structure $\underline{b}/R(n,m)$, which consists of b copies of $R(n,m)$. If $n = m$ then $R(n,m)$ can be replaced by $L(2,n)$. Some nested row-column designs with $n = m$ bear a double interpretation similar to the one in Example 5. A family of such examples consists of the lattice square designs of Yates [35].

8 Pair schemes

In a full diallel experiment without self-crosses, the experimental units are all ordered crosses between n parental lines; that is, the gender of the parent is deemed important. Similarly, an experiment on tasks may need ordered pairs of people if the two people play different roles.

Now the appropriate association scheme is $\text{Pair}(n)$, which was introduced by Nair [24] in the context of rectangular lattice designs, called the *square* association scheme in [5] and $\text{Pair}(n)$ in [6]. The set Ω consists of all ordered pairs of distinct elements from an n -set, where $n \geq 4$. For ω in Ω , if $\omega = (x,y)$ then put $\bar{\omega} = (y,x)$. The associate

classes are defined so that α and β are

- 0th associates if $\alpha = \beta$
- 1st associates if $\bar{\alpha} = \beta$
- 2nd associates if α and β are in the same row or column but $\alpha \neq \beta$
- 3rd associates if $\bar{\alpha}$ and β are in the same row or column but $\bar{\alpha} \neq \beta$
- 4th associates otherwise.

Call a vector in \mathbb{R}^Ω *symmetric* if $v_\omega = v_{\bar{\omega}}$ for all ω in Ω , and *antisymmetric* if $v_\omega = -v_{\bar{\omega}}$ for all ω in Ω . Then the strata are as follows.

- $W_0 = V_U$
- $W_1 =$ the space of symmetric vectors spanned by row and column contrasts (dimension $n - 1$)
- $W_2 =$ the space of antisymmetric vectors spanned by row and column contrasts (dimension $n - 1$)
- $W_s =$ the space of symmetric vectors orthogonal to row and column contrasts (dimension $n(n - 3)/2$)
- $W_a =$ the space of antisymmetric vectors orthogonal to row and column contrasts (dimension $(n - 1)(n - 2)/2$)

The stratum projectors are

$$\begin{aligned}
 S_0 &= \frac{1}{n(n-1)}J \\
 S_1 &= \frac{1}{2(n-2)}[2(I+A_1)+A_2+A_3-\frac{4}{n}J] \\
 S_2 &= \frac{1}{2n}[2(I-A_1)+A_2-A_3] \\
 S_s &= \frac{1}{2(n-2)}[(n-4)(I+A_1)-A_2-A_3+\frac{2}{n-1}J] \\
 S_a &= \frac{1}{2n}[(n-2)(I-A_1)-A_2+A_3].
 \end{aligned}$$

Put $R = I + A_1$. Then R is the relation matrix of the uniform partition B of Ω into mirror-image pairs $\{\omega, \bar{\omega}\}$. Let \mathcal{R} be the group-divisible association scheme on Ω defined by B .

It is reasonable to assume that ξ_1 and ξ_2 are much bigger than ξ_s and ξ_a , so that only W_s and W_a are used for estimation. (There is also a randomization argument for using only these two strata: see Section 12 of [3].) Thus we want efficiency factors in W_1 and W_2 to be as small as possible.

One way to achieve this is to use a unipotent Latin square of order n and omit its main diagonal: recall that a Latin square is unipotent if it has the same letter throughout

	A	B	C	D	E	F	G
A		C	D	E	F	G	B
B	C		E	F	G	A	D
C	D	E		G	A	B	F
D	E	F	G		B	C	A
E	F	G	A	B		D	C
F	G	A	B	C	D		E
G	B	D	F	A	C	E	

Figure 10: Orthogonal balanced design for 7 treatments in Pair(8), obtained from a symmetric unipotent Latin square

	A	B	C	D	E	F
A		C	D	E	F	G
B	C		E	F	G	A
C	D	E		G	A	B
D	E	F	G		B	C
E	F	G	A	B		D
F	G	A	B	C	D	

Figure 11: Balanced design for 7 treatments in Pair(7), obtained from a symmetric idempotent Latin square

its main diagonal. Examples are shown in Figures 10, 12 and 14. Then there are $n - 1$ treatments, each replicated n times.

For such a design,

$$X'(2I + A_2)X = X'(2A_1 + A_3)X = 2nJ,$$

and $X'JX = n^2J$, so $L_1 = L_2 = 0$. Moreover, $X'IX = nI$, so

$$L_s = \frac{1}{2}X'RX - nT_0$$

$$L_a = nI - \frac{1}{2}X'RX.$$

(Here T_0 denotes the projector onto stratum V_U in Q .) Therefore, such a design h on Ω is partially balanced for Pair(n) with respect to Q if and only if it is partially balanced for \mathcal{R} with respect to Q . Moreover, $\epsilon_{sf} = \epsilon_{Bf}$ and $\epsilon_{af} = \epsilon_{Ef}$ for all strata f in Q .

There are three obvious ways to construct h as a block design for n treatments in $n(n - 1)/2$ blocks of size 2. The first is to apply each treatment to both experimental units in each of $n/2$ blocks. Then $X'RX = 2nI$, so $L_a = 0$ and $L_s = n(I - T_0)$. The design is orthogonal and balanced, with all estimation taking place in stratum W_s . A unipotent Latin square gives such a block design if and only if it is symmetric. Such a square exists if and only if n is even. An example with $n = 8$ is in Figure 10.

The second is to have each pair of treatments occurring together in a single block, and each treatment occurring on both experimental units in one block. Then $X'RX = (n + 1)I + J$, so the design is balanced with $\epsilon_B = \epsilon_s = (n + 1)/2n$ and $\epsilon_E = \epsilon_a = (n - 1)/2n$. Construction of a unipotent Latin square with this property is possible when n is even and 3 does not divide $n - 1$. An example with $n = 8$ is in Figure 12.

The third is to divide the $n - 1$ treatments into $(n - 1)/2$ groups of two and ensure that each pair $\{\omega, \bar{\omega}\}$ is allocated one of these groups. Then the design is orthogonal and group divisible, with contrasts between groups estimated in stratum W_s and contrasts

	A	B	C	D	E	F	G
B		D	E	F	G	A	C
D	E		G	A	B	C	F
F	G	A		C	D	E	B
A	B	C	D		F	G	E
C	D	E	F	G		B	A
E	F	G	A	B	C		D
G	C	F	B	E	A	D	

Figure 12: Balanced design for 7 treatments in Pair(8), obtained from a unipotent Latin square

	A	B	C	D	E	F
B		D	E	F	G	A
D	E		G	A	B	C
F	G	A		C	D	E
A	B	C	D		F	G
C	D	E	F	G		B
E	F	G	A	B	C	

Figure 13: Balanced design for 7 treatments in Pair(7), obtained from an idempotent Latin square

	A	B	C	D	E	F
F		A	B	C	D	E
E	F		A	B	C	D
D	E	F		A	B	C
C	D	E	F		A	B
B	C	D	E	F		A
A	B	C	D	E	F	

Figure 14: Orthogonal group-divisible design for 6 treatments in Pair(7), obtained from a unipotent Latin square

	A	B	C	D	E
F		A	B	C	D
E	F		A	B	C
D	E	F		A	B
C	D	E	F		A
B	C	D	E	F	

Figure 15: Group-divisible design for 6 treatments in Pair(6)

within groups estimated in stratum W_a . A unipotent Latin square with this property exists if and only if n is odd. One construction for odd n is to label the rows and columns of Ω by the integers modulo n , and put $y - x \pmod n$ in cell (x, y) . An example with $n = 7$ is in Figure 14.

A similar family of three types of design is available for n treatments with replication $n - 1$. This time we start with an idempotent Latin square of order n ; that is, one in which every letter occurs once on the main diagonal. Omitting the diagonal leaves each letter in all rows except one and in all columns except one; the exceptional row does not meet the exceptional column. Therefore

$$X'(2I + A_2)X = X'(2A_1 + A_3)X = 2I + 2(n - 2)J,$$

$$X'IX = (n - 1)I \text{ and } X'JX = (n - 1)^2J, \text{ so}$$

$$L_1 = \frac{2}{n - 2}(I - T_0),$$

which has efficiency factor $2/(n - 1)(n - 2)$ on all treatment contrasts. Meanwhile,

$$L_2 = 0,$$

$$L_s = \frac{1}{2}X'RX - \frac{2}{n-2}I - \frac{n(n-3)}{n-2}T_0$$

and

$$L_a = (n-1)I - \frac{1}{2}X'RX.$$

Once again, the design is partially balanced for $\text{Pair}(n)$ with respect to Q if and only if it is partially balanced for \mathcal{R} with respect to Q . This time, $\varepsilon_{sf} = \varepsilon_{Bf} - 2/(n-1)(n-2)$ and $\varepsilon_{af} = \varepsilon_{Ef}$ for all strata f in Q .

If the Latin square is symmetric then $X'RX = 2(n-1)I$ so the design is balanced with $\varepsilon_1 = 2/(n-1)(n-2)$, $\varepsilon_2 = 0$, $\varepsilon_s = n(n-3)/(n-1)(n-2)$ and $\varepsilon_a = 0$. A symmetric idempotent Latin square exists if and only if n is odd. One construction for odd n is to label the rows and columns of Ω by the integers modulo n , and put $x+y \pmod{n}$ in cell (x,y) . An example with $n = 7$ is in Figure 11. A unipotent Latin square can be obtained from this by moving the letter on cell (x,x) to the cell in row x of an additional column and column x of a new row, and putting a new letter on the main diagonal. The design in Figure 10 is obtained in this way from the design in Figure 11.

In the second type of design, each pair of treatments occurs together in a single block. Thus $X'RX = (n-2)I + J$ so the design is balanced with $\varepsilon_1 = 2/(n-1)(n-2)$, $\varepsilon_2 = 0$, $\varepsilon_s = n(n-4)/2(n-1)(n-2)$ and $\varepsilon_a = n/2(n-1)$. If n is odd and not divisible by 3 then such an idempotent Latin square can be constructed by labelling the rows and columns of Ω by the integers modulo n and putting $2x+y \pmod{n}$ in cell (x,y) . An example is in Figure 13. A unipotent Latin square of order $n+1$ can be constructed from this just as in the previous case. Thus the design in Figure 12 is obtained from the design in Figure 13.

For the third type of design, we do not use an idempotent Latin square. If treatments A and B always occur on mirror-image pairs then the row which omits A passes through the same diagonal cell as the column which omits B . Hence

$$X'(2I + A_2)X = 2I + 2(n-2)J$$

but

$$X'(2A_1 + A_3)X = \frac{2}{n-1}X'RX - 2I + 2(n-2)J.$$

Therefore

$$\begin{aligned}
 L_1 &= \frac{1}{n-2} \left[\frac{1}{n-1} X'RX - 2T_0 \right] = \frac{2}{n-2} T_g \\
 L_2 &= \frac{1}{n} \left[2I - \frac{1}{n-1} X'RX \right] = \frac{2}{n} T_w \\
 L_s &= \frac{n(n-3)}{2(n-2)} \left[\frac{1}{n-1} X'RX - 2T_0 \right] = \frac{n(n-3)}{n-2} T_g \\
 L_a &= \frac{(n-2)(n+1)}{2n} \left[2I - \frac{1}{n-1} X'RX \right] = \frac{(n-2)(n+1)}{n} T_w.
 \end{aligned}$$

The design is group divisible. Such a design can be constructed from the third type of design based on unipotent squares, by simply omitting the final row and column. An example is in Figure 15.

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