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DEPENDENT BOOTSTRAP CONFIDENCE INTERVALS

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Abstract

A dependent bootstrap is shown to produce estimators which have smaller variances but which are still consistent and asymptotically valid. Simulated confidence intervals are used to examine possible gains in coverage probabilities and interval lengths.

1 Introduction

Inference for stochastic processes seeks to provide more appropriate dependent models to situations where the assumption of independence is not plausible. The use of independent models is prevalent in bootstrapping. Efron (1979) introduced the bootstrap as a tool to estimate the standard error of a statistic, and an enormous amount of applied and theoretical research on the bootstrap technique has followed in the past two decades. While much of this ensuing research has been methodological adaptions and theoretical validity verifications for different statistics, considerable research has been directed toward shortcomings and possible improvements to the basic bootstrap technique. The traditional resampling of the sample observations (with replacement) produces independent and identically distributed (bootstrap) random variables (conditional on the original sample), and many of the theoretical justifications of the bootstrap procedures are crucially related to techniques involving independent random variables. Resampling without replacement produces dependent random variables (actually negatively dependent) which are still identically distributed (and in fact has the desirable property of exchangeability). The purpose of this paper is to consider some estimation using a form of dependent bootstrapping. In particular, confidence interval comparisons will be given for the traditional bootstrap procedure and the dependent bootstrap procedure.

Resampling without replacement is not new. The majority of research on resampling without replacement has been for application in finite population sampling. Gross (1980) introduced the concept and many others (Bickel and Freedman, 1984; Chao and Lo, 1985; Sitter, 1992; Booth,Butler and Hall, 1994; and others) have extended this research.

In 1994 Politis and Romano examined resampling without replacement from a data set to approximate the sampling distribution of a statistic T_n. Under weak assumptions, they showed that the empirical distribution of the suitably normalized values of the statistic computed for all subsamples of size b from the original data is first order asymptotically valid for the true sampling distribution of T_n. This is a generalization from Wu (1990) who studied the same method in the i.i.d case for statistics which are asymptotically normal. Bertail (1997) showed second order correctness of this method for an adequately chosen resample size. Their investigations differ from this proposed research because they sample without replacement from the original data rather than an enriched collection with a fixed number of copies of each observation. Their procedure has been termed "m out of n" where $m(<< n)$ is the bootstrap sample size and n is the sample size of the original sample. Bickel, Götze, and Van Zwet (1997) investigated the gains and losses for "m out of n" resampling where $m = o(n)$. Praestgaard and Wellner (1993) showed that "m out of kn" could allow larger bootstrap sample sizes and some asymptotic results using exchangeability arguments. Babu and Singh (1985) and Babu and Bai (1996) showed that Edgeworth expansions could be used to obtain approximation results for estimators based on samples drawn without replacement from a finite population. Their approximation results provide for weak convergence of normalized absolute differences of original sample statistics and bootstrap statistics. This paper will compare the coverage probabilities and lengths of the more generally used bootstrap confidence intervals for the traditional bootstrap and the dependent bootstrap.

The formal definition of the dependent bootstrap procedure and the theoretical properties of consistency and asymptotic validity are listed in Section 2. Section 3 provides the description and results of the simulations for the confidence intervals.

2 Properties of the Dependent Bootstrap

Consider the random sample observations $X_1, X_2, \ldots, X_n$, that is, identically distributed random variables with distribution function F. Often the random variables $X_1, X_2, \ldots, X_n$ are also assumed to be independent, but may be dependent as when sampling from a finite population. A *dependent boot-*

strap is defined as the sample of size m, denoted by $X_{n1}^*, \ldots, X_{nm}^*$, drawn without replacement from the collection of kn items made up of k copies each of the sample observations, $X_1, \ldots, X_n$, where $m \leq kn$. This dependent bootstrap is proposed as a procedure to reduce variation of estimators and to obtain better confidence intervals.

Let E^*, Var^*, and P^* denote the conditional expectation, variance, and probablility given $X_1, X_2, \ldots, X_n$. It can be shown that

$$E^* \bar{X}_{nm}^* = \bar{X}_n, \tag{2.1}$$

$$Var^*(X_{nj}^*) = S_n^2, \quad j = 1, \ldots, m, and \tag{2.2}$$

$$Var^*(\bar{X}_{nm}^*) = \frac{kn - m}{kn - 1}\frac{S_n^2}{m}, \tag{2.3}$$

where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i, \bar{X}_{nm}^* = \frac{1}{m}\sum_{j=1}^m X_{nj}^*$ and $S_n^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)^2$. Notice that the multiplier in the variance of the dependent bootstrap sample mean, $\frac{kn-m}{kn-1}$, is similar to the finite population correction factor.

Random variables X and Y are said to be *negatively dependent* (ND) if

$$P[X \leq x, Y \leq y] \leq P[X \leq x]P[Y \leq y] \tag{2.4}$$

for all $x, y \in R$. Negative dependence includes independence, and the terminology of negative relates to (2.4) which (by dividing by $P[Y \leq y]$) implies

$$P[X \leq x | Y \leq y] \leq P[X \leq x]$$

Examples and properties of negative dependent random variables and the applications to the dependent bootstrap are given by Smith and Taylor (2001). In particular, they showed that the dependent bootstrap produces negatively dependent random variables. A collection of random variables $\{X_1, X_2, \ldots, X_n\}$ is said to be *exchangeable* if the joint distribution of $(X_1, X_2, \ldots, X_n)$ is invariant with respect to permutations of the indices $1, \ldots, n$. It can be shown (cf: Smith and Taylor (2001)) that the dependent bootstrap random variables, $X_{n1}^*, X_{n2}^*, \ldots, X_{nm}^*$, are exchangeable in addition to being negative dependent.

Theorem 2.1 gives the consistency of the bootstrap mean for this dependent bootstrap procedure established by Smith and Taylor (2001), while Theorem 2.2 establishes the consistency of the dependent bootstrap variance. The technique of the proof for Theorem 2.1 follows similar techniques for obtaining consistency in the i.i.d. bootstrap given by Hu and Taylor (1997) and Bozorgnia, Patterson and Taylor (1997). Moreover, it is important to observe that required moment conditions in Theorem 2.1 are identical to the traditional i.i.d. bootstrapping procedure (cf: Athreya, Ghosh, Low and Sen (1984)).

Theorem 2.1 *Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d. random variables with mean μ and $E|X_1|^{1+\delta} < \infty$ for some $\delta > 0$. Along almost all sample sequences $X_1, X_2, \ldots$ given $(X_1, \ldots, X_n)$,*

$$\bar{X}^*_{nm} \to \mu \text{ with conditional probability 1.} \tag{2.5}$$

Theorem 2.2 *Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d. random variables with mean μ, variance σ^2 and $E|X_1|^{2+\delta} < \infty$ for some $\delta > 0$. Along almost all sample sequences $X_1, X_2, \ldots$ given $(X_1, \ldots, X_n)$,*

$$S^{*2}_{nm} \to \sigma^2 \text{ with conditional probability 1,} \tag{2.6}$$

where $S^{*2}_{nm} = \frac{1}{m}\sum_{j=1}^{m}(X^*_{nj} - \bar{X}^*_{nm})^2$.

Theorems 2.3 and 2.4 are also from Smith and Taylor (2001) and provide the asymptotic validity for the dependent Kolmogorov-Smirnov bootstrap statistic.

Theorem 2.3 *Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d. random variables with distribution function F. Along almost all sample sequences $X_1, X_2, \ldots$ given $(X_1, \ldots, X_n)$,*

$$F^*_m(x) \to F(x) \text{ with conditional probability 1,} \tag{2.7}$$

where $F^*(x) = \frac{1}{m}\sum_{j=1}^{m} I_{[X^*_{nj} \le x]}$.

Theorem 2.4 *Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d. random variables with distribution function F. Then,*

$$D^*_m = \sup_{-\infty < x < \infty} |F^*_m(x) - F(x)| \to 0 \text{ with conditional probability 1.} \tag{2.8}$$

Theorems 2.1 - 2.4 were obtained using negative dependent limit theorems and are valid for all k and for all m such that $\liminf_n \frac{m}{n} > 0$. Finite population versions of these results were also obtained (cf: Smith and Taylor (2001)). For the validity (asymptotic normality) for the dependent bootstrap a stronger (more restrictive) form of negative dependence is needed, namely negative association.

Random variables $\{X_n\}$ are said to be *negatively associated* (NA) if for each $k \ge 2$ and every pair of disjoint subsets A_1, A_2 of $\{1, 2, \ldots, k\}$

$$Cov(f(X_i, i \in A_1), g(X_j, j \in A_2)) \le 0 \tag{2.9}$$

whenever f and g are monotone increasing, Borel functions. Using a combination of exchangeable and negative association results, Patterson, Smith, Taylor, and Bozorgnia (2001) obtained the following central limit thoerem for the dependent bootstrap.

Theorem 2.5 *If $\{X_n\}$ are i.i.d. random variables such that $E|X_1|^{2+\delta} < \infty$ for some $\delta > 0$, then conditionally on almost all sample paths*

$$\frac{\sum_{i=1}^{n}(X_{ni}^* - \bar{X}_n)}{\sqrt{n}s_n} \text{ converges in distribution} \tag{2.10}$$

to a $N(0,1)$ random variable where $s_n = \sqrt{\frac{kn-n}{kn-1}S_n^2}$.

It is important to observe that the normalizing factor s_n in (2.10) includes k, and that the result holds for all choices of $k = k(n) \geq 2$ as $n \to \infty$. Moreover, Theorem 2.5 can be extended to include sample observations from finite populations and bootstrap sample sizes $m = m(n)$ such that

$$0 < \inf_n \frac{m}{kn} \leq \sup_n \frac{m}{kn} < 1. \tag{2.11}$$

In addition, the asymptotic validity of the dependent bootstrap distribution function estimator $F_m^*(x)$ follows from these techniques. This result is stated for i.i.d. random variables but is also obtainable when sampling from finite populations.

Theorem 2.6 *Let $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{[X_i \leq x]}$ and $F_m^*(x) = \frac{1}{m}\sum_{i=1}^{m} I_{[X_{ni}^* \leq x]}$, and let (2.11) hold, then along almost all samle sequences $X_1, X_2, \ldots$ given $(X_1, \ldots, X_n)$*

$$\frac{F_m^*(x) - F_n(x)}{s_n(x)} \text{ converges in distribution to a N(0,1) random variable,}$$

where $s_n^2(x) = Var^*(F_m^*(x)) = \frac{1}{m}\frac{kn-m}{kn-1}F_n(x)(1 - F_n(x))$.

3 Simulations

In this section the simulation results are given to compare the coverage probabilities and average lengths of confidence intervals calculated using two methods of calculating CI's for both traditional and dependent bootstrap methods. Also included in these simulation is the computation of the traditional normal theory confidence intervals for comparison, where the limits are computed using

$$\bar{X}_n \pm Z_{\frac{\alpha}{2}}\frac{S_n}{\sqrt{n}}. \tag{3.1}$$

While there are many methods of computing bootstrap confidence intervals (cf: DiCiccio and Efron (1996)), these simulations employed the general methods of percentile and bootstrap-t. Percentile method confidence limits are

$$[\bar{X}_{nm}^{*(\frac{\alpha}{2})}, \bar{X}_{nm}^{*(1-\frac{\alpha}{2})}], \tag{3.2}$$

where $\bar{X}_{nm}^{*(p)}$ is the $B \cdot p$th ordered value in the list of the B bootstrap sample means, $\bar{X}_{nm}^{*(b)}$, $b = 1, \ldots, B$.

For the bootstrap-t method the limits are given by

$$[\bar{X}_n - \hat{z}^{(1-\frac{\alpha}{2})} \frac{S_n}{\sqrt{n}}, \bar{X}_n - \hat{z}^{(\frac{\alpha}{2})} \frac{S_n}{\sqrt{n}}] \tag{3.3}$$

where $\hat{z}^{(\frac{\alpha}{2})}$ is the $B \cdot \frac{\alpha}{2}$th ordered value in the list of the B standardized bootstrap sample means, $Z^{*(b)} = \frac{\bar{X}_{nm}^{*(b)} - \bar{X}_n}{\hat{se}^{*(b)}}$, $b = 1, \ldots, B$ and $\hat{se}^{*(b)}$ is the estimated standard error of $\bar{X}^{*(b)}$. When resampling using the traditional bootstrap method, $\hat{se}^{(b)} = \frac{S_b^*}{\sqrt{m}}$, where S_b^* is the sample standard deviation of the bth bootstrap sample, while for the dependent bootstrap, $\hat{se}^{(b)} = S_b^* \sqrt{\frac{kn-1}{m(kn-m)}}$ as provided in Theorem 2.5.

For these simulations 1500 samples of size $n = 20, 40, 100, 200$ were generated from the following distributions, each having mean 4 and variance 8: normal ($\mu = 4, \sigma^2 = 8$), χ^2 with 4 degrees of freedom, double exponential with $\mu = 4$ and $\sigma = 2$, and mixture of two normal distributions: 80 % $N(4.8, 6.45)$ and 20 % $N(0.8, 1.4)$.

For each original sample, the traditional normal theory 90% confidence interval was calculated and the coverage probability and length of the 1500 intervals was computed. The 90% confidence level was chosen (rather than the usual 95%) so that there would be sufficient number of confidence intervals not including the mean of 4 for interesting comparisons.

Next, for each original sample the dependent bootstrap 90% CI was formed by drawing 2000 dependent bootstrap samples of size $m = n$ from each of the original samples for varying replication factors, k, specifically, $k = 2, 4, 6, 8, 10, 20$. Using the 2000 bootstrap samples, the percentile and bootstrap-t confidence intervals were obtained. The same procedure was followed using the traditional bootstrap. The estimated coverage probabilities and average lengths are reported in Tables 1-4.

The results show that for all distributions there is little difference between the coverage probabilities and the lengths of the normal theory CI, the traditional bootstrap and the dependent bootstrap procedure, when the bootstrap-t method of CI computation is used. However, for the percentile method of confidence interval computation, the dependent bootstrap performs poorly when compared to the traditional bootstrap and the normal theory method (cf: Tables 5-8). For all distributions, the dependent bootstrap confidence intervals displayed far lower coverage probabilities than the other methods while yielding much shorter lengths. This result is to be expected since the variance of the dependent bootstrap mean estimator (cf: (2.3)) is smaller than that of the sample mean or the traditional bootstrap mean, and hence the distribution of the dependent bootstrap mean is narrower. Thus, an adjustment to the percentile method is needed for the dependent bootstrap confidence intervals to achieve desired coverage probabilities while trying to maintain shorter lengths. Specifically, since the percentiles are functionally related to the standard deviation, instead of using (3.2) in obtaining the confidence interval limits, use

$$[\bar{X}_{nm}^{*(\frac{p*}{2})}, \bar{X}_{nm}^{*(1-\frac{p^*}{2})}], \tag{3.4}$$

where p^* satisfies $1 - \Phi(Z_{(\alpha,\theta)}) = \frac{p^*}{2}$ with

$$Z_{(\alpha,\theta)} = \theta\sqrt{\frac{kn-1}{kn-m}}Z_{\frac{\alpha}{2}} + (1-\theta)Z_{\frac{\alpha}{2}}, \tag{3.5}$$

for $0 < \theta < 1$ and where $Z_{\frac{\alpha}{2}}$ is a standard normal $(1 - \frac{\alpha}{2})$ percentile. Using $\alpha = .1$, this becomes

$$Z_{(.1,\theta)} = \theta\sqrt{\frac{kn-1}{kn-m}}1.645 + (1-\theta)1.645,$$

Note that $\theta = 0$ is the usual percentile method described above whereas for $0 < \theta < 1$ the attempt is to improve the coverage probabilies of the confidence intervals while maintaining shorter lengths. Using the same distributions and sampling plan as before, these new simulations were carried out with results for $\theta = 0, 0.5., 0.75, 1$ given in Tables 9-12. For all distributions, even for moderate k ($k = 4, 6, 8$) and $\theta = 0.75$, the dependent bootstrap confidence intervals achieve comparable coverage probabilities while retaining shorter lengths.

The simulation studies presented in this paper suggest some directions for future research. While $\theta = 0.75$ appears to be the best choice from these simulations, more investigations might determine if θ is dependent on k, n, or the distribution the original sample was drawn from, or some combination of these factors.

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Table 1: 90% Bootstrap-t Confidence Intervals for Normal($\mu = 4, \sigma^2 = 8$)

Coverage Probabilities

	Normal	Traditional	Dependent Bootstrap					
n	Approx.	Bootstrap	$k = 2$	$k = 4$	$k = 6$	$k = 8$	$k = 10$	$k = 12$
20	0.877	0.901	0.892	0.898	0.896	0.899	0.899	0.897
40	0.907	0.911	0.914	0.915	0.912	0.913	0.914	0.918
100	0.892	0.895	0.894	0.890	0.890	0.897	0.891	0.891
200	0.912	0.912	0.914	0.913	0.912	0.911	0.917	0.916

Interval Lengths

	Normal	Traditional	Dependent Bootstrap					
n	Approx.	Bootstrap	$k = 2$	$k = 4$	$k = 6$	$k = 8$	$k = 10$	$k = 12$
20	2.049	2.181	2.137	2.157	2.164	2.168	2.172	2.173
40	1.463	1.503	1.491	1.497	1.499	1.501	1.502	1.501
100	0.930	0.940	0.938	0.939	0.939	0.939	0.939	0.940
200	0.658	0.661	0.661	0.661	0.661	0.661	0.661	0.661

Table 2: Bootstrap-t method 90% Confidence Intervals for Chi Square($df = 4$)

Coverage Probabilities

	Normal	Traditional	Dependent Bootstrap					
n	Approx.	Bootstrap	$k = 2$	$k = 4$	$k = 6$	$k = 8$	$k = 10$	$k = 12$
20	0.867	0.852	0.887	0.886	0.891	0.891	0.895	0.889
40	0.863	0.883	0.881	0.885	0.886	0.881	0.883	0.883
100	0.901	0.911	0.910	0.911	0.913	0.909	0.911	0.909
200	0.893	0.895	0.889	0.892	0.895	0.893	0.893	0.897

Interval Lengths

	Normal	Traditional	Dependent Bootstrap					
n	Approx.	Bootstrap	$k = 2$	$k = 4$	$k = 6$	$k = 8$	$k = 10$	$k = 12$
20	2.028	2.295	2.234	2.262	2.274	2.277	2.282	2.283
40	1.449	1.538	1.521	1.530	1.534	1.534	1.536	1.536
100	0.925	0.949	0.944	0.946	0.947	0.947	0.947	0.948
200	0.654	0.663	0.661	0.662	0.663	0.662	0.663	0.663

Table 3: Bootstrap-t method 90% Confidence Intervals for Double Exponential($\mu = 4, \sigma = 2$)

Coverage Probabilities

n	Normal Approx.	Traditional Bootstrap	Dependent Bootstrap $k=2$	$k=4$	$k=6$	$k=8$	$k=10$	$k=12$
20	0.877	0.865	0.865	0.863	0.863	0.867	0.867	0.863
40	0.894	0.887	0.891	0.894	0.889	0.889	0.891	0.888
100	0.905	0.899	0.897	0.902	0.902	0.901	0.899	0.900
200	0.902	0.896	0.901	0.897	0.898	0.896	0.897	0.899

Interval Lengths

n	Normal Approx.	Traditional Bootstrap	Dependent Bootstrap $k=2$	$k=4$	$k=6$	$k=8$	$k=10$	$k=12$
20	2.008	1.954	2.184	2.187	2.191	2.193	2.193	2.195
40	1.458	1.438	1.517	1.515	1.514	1.514	1.515	1.516
100	0.927	0.922	0.941	0.940	0.941	0.939	0.940	0.940
200	0.656	0.654	0.662	0.661	0.661	0.661	0.661	0.661

Table 4: Bootstrap-t method 90% Confidence Intervals for mixture of two normal distributions

Coverage Probabilities

n	Normal Approx.	Traditional Bootstrap	Dependent Bootstrap $k=2$	$k=4$	$k=6$	$k=8$	$k=10$	$k=12$
20	0.885	0.905	0.895	0.903	0.902	0.904	0.904	0.905
40	0.888	0.902	0.896	0.897	0.899	0.901	0.903	0.902
100	0.901	0.907	0.905	0.906	0.906	0.907	0.905	0.907
200	0.903	0.902	0.904	0.905	0.905	0.904	0.905	0.902

Interval Lengths

n	Normal Approx.	Traditional Bootstrap	Dependent Bootstrap $k=2$	$k=4$	$k=6$	$k=8$	$k=10$	$k=12$
20	2.069	2.197	2.145	2.168	2.179	2.183	2.185	2.188
40	1.469	1.509	1.491	1.501	1.504	1.506	1.506	1.507
100	0.929	0.940	0.936	0.938	0.938	0.940	0.939	0.939
200	0.656	0.660	0.659	0.659	0.660	0.659	0.660	0.660

Table 5: Percentile 90% Confidence Intervals for Normal($\mu = 4, \sigma^2 = 8$)

Coverage Probabilities

n	Normal Approx.	Traditional Bootstrap	Dependent Bootstrap $k=2$	$k=4$	$k=6$	$k=8$	$k=10$	$k=12$
20	0.877	0.869	0.728	0.814	0.830	0.840	0.849	0.852
40	0.907	0.901	0.767	0.855	0.879	0.887	0.886	0.893
100	0.892	0.892	0.775	0.835	0.853	0.865	0.867	0.876
200	0.912	0.907	0.783	0.861	0.875	0.889	0.893	0.893

Interval Lengths

n	Normal Approx.	Traditional Bootstrap	Dependent Bootstrap $k=2$	$k=4$	$k=6$	$k=8$	$k=10$	$k=12$
20	2.049	1.997	1.433	1.743	1.834	1.877	1.903	1.920
40	1.463	1.443	1.029	1.256	1.323	1.355	1.374	1.386
100	0.930	0.925	0.657	0.803	0.846	0.867	0.878	0.888
200	0.658	0.656	0.465	0.569	0.600	0.614	0.623	0.629

Table 6:
Percentile method 90% Confidence Intervals for Chi Square($df = 4$)

Coverage Probabilities

n	Normal Approx.	Traditional Bootstrap	Dependent Bootstrap $k=2$	$k=4$	$k=6$	$k=8$	$k=10$	$k=12$
20	0.867	0.852	0.726	0.806	0.824	0.833	0.840	0.840
40	0.863	0.859	0.735	0.813	0.830	0.841	0.845	0.852
100	0.901	0.903	0.763	0.850	0.867	0.878	0.883	0.885
200	0.893	0.887	0.732	0.833	0.852	0.864	0.868	0.875

Interval Lengths

n	Normal Approx.	Traditional Bootstrap	Dependent Bootstrap $k=2$	$k=4$	$k=6$	$k=8$	$k=10$	$k=12$
20	2.028	1.973	1.423	1.725	1.813	1.855	1.881	1.895
40	1.449	1.428	1.021	1.244	1.311	1.341	1.362	1.373
100	0.925	0.920	0.653	0.799	0.842	0.862	0.874	0.882
200	0.654	0.653	0.463	0.566	0.597	0.613	0.620	0.626

Table 7: Percentile method 90% Confidence Intervals for Double Exponential($\mu = 4, \sigma = 2$)

Coverage Probabilities

n	Normal Approx.	Traditional Bootstrap	Dependent Bootstrap					
			$k=2$	$k=4$	$k=6$	$k=8$	$k=10$	$k=12$
20	0.877	0.859	0.721	0.802	0.822	0.833	0.837	0.840
40	0.894	0.885	0.738	0.833	0.855	0.866	0.869	0.874
100	0.905	0.900	0.754	0.839	0.855	0.870	0.876	0.878
200	0.902	0.897	0.755	0.849	0.869	0.876	0.881	0.885

Interval Lengths

n	Normal Approx.	Traditional Bootstrap	Dependent Bootstrap					
			$k=2$	$k=4$	$k=6$	$k=8$	$k=10$	$k=12$
20	2.008	1.954	1.408	1.708	1.797	1.838	1.862	1.879
40	1.458	1.438	1.027	1.253	1.318	1.349	1.370	1.382
100	0.927	0.922	0.655	0.801	0.845	0.863	0.876	0.884
200	0.656	0.654	0.464	0.568	0.598	0.613	0.622	0.627

Table 8: Percentile method 90% Confidence Intervals for Mixture of Two Normal Distributions

Coverage Probabilities

n	Normal Approx.	Traditional Bootstrap	Dependent Bootstrap					
			$k=2$	$k=4$	$k=6$	$k=8$	$k=10$	$k=12$
20	0.885	0.875	0.733	0.812	0.838	0.847	0.854	0.856
40	0.888	0.885	0.749	0.837	0.854	0.862	0.866	0.873
100	0.901	0.901	0.759	0.849	0.868	0.875	0.877	0.886
200	0.903	0.900	0.759	0.844	0.865	0.879	0.884	0.885

Interval Lengths

n	Normal Approx.	Traditional Bootstrap	Dependent Bootstrap					
			$k=2$	$k=4$	$k=6$	$k=8$	$k=10$	$k=12$
20	2.069	2.018	1.448	1.759	1.852	1.895	1.921	1.938
40	1.469	1.451	1.032	1.262	1.330	1.361	1.379	1.393
100	0.929	0.925	0.656	0.803	0.846	0.868	0.879	0.886
200	0.656	0.654	0.464	0.568	0.599	0.613	0.622	0.628

Table 9: Modified Percentile 90% Confidence Intervals: Normal($\mu = 4, \sigma^2 = 8$)

Coverage Probabilities

θ	n	Normal Approx.	Traditional Bootstrap	Dependent Bootstrap $k=2$	$k=4$	$k=6$	$k=8$	$k=10$
0	20	0.877	0.901	0.892	0.898	0.896	0.899	0.899
	40	0.907	0.911	0.914	0.915	0.912	0.913	0.914
	100	0.892	0.895	0.894	0.890	0.890	0.897	0.891
	200	0.912	0.912	0.914	0.913	0.912	0.911	0.917
.5	20	0.877	0.869	0.801	0.842	0.855	0.858	0.861
	40	0.907	0.901	0.853	0.883	0.893	0.894	0.895
	100	0.892	0.892	0.830	0.866	0.878	0.877	0.880
	200	0.912	0.907	0.858	0.887	0.896	0.904	0.903
0.75	20	0.877	0.869	0.839	0.855	0.865	0.863	0.865
	40	0.907	0.901	0.881	0.892	0.898	0.899	0.901
	100	0.892	0.892	0.863	0.876	0.881	0.883	0.883
	200	0.912	0.907	0.887	0.904	0.905	0.907	0.909
1	20	0.877	0.869	0.864	0.869	0.873	0.868	0.869
	40	0.907	0.901	0.901	0.901	0.903	0.901	0.902
	100	0.892	0.892	0.890	0.889	0.891	0.891	0.887
	200	0.912	0.907	0.913	0.912	0.910	0.915	0.911

Interval Lengths

θ	n	Normal Approx.	Traditional Bootstrap	Dependent Bootstrap $k=2$	$k=4$	$k=6$	$k=8$	$k=10$
0	20	2.049	2.181	2.137	2.157	2.164	2.168	2.172
	40	1.463	1.503	1.491	1.497	1.499	1.501	1.502
	100	0.930	0.940	0.938	0.939	0.939	0.939	0.939
	200	0.658	0.661	0.661	0.661	0.661	0.661	0.661
0.5	20	2.049	1.997	1.713	7.874	1.919	1.938	1.954
	40	1.463	1.443	1.240	1.356	1.385	1.400	1.411
	100	0.930	0.925	0.792	0.867	0.889	0.899	0.903
	200	0.658	0.656	0.562	0.614	0.630	0.637	0.640
0.75	20	2.049	1.997	1.849	1.939	1.963	1.968	1.980
	40	1.463	1.443	1.343	1.396	1.416	1.423	1.428
	100	0.930	0.925	0.861	0.897	0.907	0.913	0.915
	200	0.658	0.656	0.610	0.636	0.643	0.647	0.648
1	20	2.049	1.997	1.982	1.997	2.003	2.002	2.004
	40	1.463	1.443	1.448	1.447	1.446	1.446	1.448
	100	0.930	0.925	0.927	0.929	0.929	0.929	0.926
	200	0.658	0.656	0.658	0.659	0.659	0.659	0.657

Table 10: Modified Percentile 90% Confidence Intervals: Chi-Square (df = 4)

Coverage Probabilities

θ	n	Normal Approx.	Traditional Bootstrap	Dependent Bootstrap $k=2$	$k=4$	$k=6$	$k=8$	$k=10$
0	20	0.867	0.852	0.726	0.806	0.824	0.833	0.840
	40	0.863	0.859	0.735	0.813	0.830	0.841	0.845
	100	0.901	0.903	0.763	0.850	0.867	0.878	0.883
	200	0.893	0.887	0.732	0.833	0.852	0.864	0.868
0.5	20	0.867	0.852	0.798	0.831	0.841	0.841	0.847
	40	0.863	0.859	0.805	0.841	0.848	0.851	0.855
	100	0.901	0.903	0.844	0.879	0.888	0.890	0.893
	200	0.893	0.887	0.823	0.863	0.875	0.877	0.877
0.75	20	0.867	0.852	0.827	0.845	0.849	0.850	0.852
	40	0.863	0.859	0.833	0.851	0.853	0.855	0.857
	100	0.901	0.903	0.876	0.893	0.894	0.897	0.899
	200	0.893	0.887	0.862	0.880	0.881	0.887	0.888
1	20	0.867	0.852	0.849	0.859	0.860	0.857	0.855
	40	0.863	0.859	0.860	0.861	0.863	0.862	0.863
	100	0.901	0.903	0.899	0.901	0.899	0.903	0.902
	200	0.893	0.887	0.890	0.893	0.894	0.888	0.895

Interval Lengths

θ	n	Normal Approx.	Traditional Bootstrap	Dependent Bootstrap $k=2$	$k=4$	$k=6$	$k=8$	$k=10$
0	20	2.028	1.973	1.423	1.725	1.813	1.855	1.881
	40	1.449	1.428	1.021	1.244	1.311	1.341	1.362
	100	0.925	0.920	0.653	0.799	0.842	0.862	0.874
	200	0.654	0.653	0.463	0.566	0.597	0.613	0.620
0.5	20	2.028	1.973	1.688	1.850	1.897	1.917	1.962
	40	1.449	1.428	1.225	1.342	1.372	1.386	1.398
	100	0.925	0.920	0.787	0.862	0.884	0.894	0.898
	200	0.654	0.653	0.558	0.611	0.628	0.634	0.638
0.75	20	2.028	1.973	1.819	1.914	1.939	1.947	1.958
	40	1.449	1.428	1.326	1.384	1.403	1.408	1.415
	100	0.925	0.920	0.854	0.893	0.902	0.909	0.909
	200	0.654	0.653	0.607	0.633	0.639	0.644	0.645
1	20	2.028	1.973	1.947	1.972	1.978	1.980	1.980
	40	1.449	1.428	1.427	1.432	1.432	1.433	1.434
	100	0.925	0.920	0.920	0.924	0.924	0.923	0.921
	200	0.654	0.653	0.654	0.655	0.655	0.655	0.654

Table 11: Modified Percentile 90% Confidence Intervals: DE($\mu = 4, \sigma = 2$)

Coverage Probabilities

		Normal	Traditional	Dependent Bootstrap				
θ	n	Approx.	Bootstrap	$k = 2$	$k = 4$	$k = 6$	$k = 8$	$k = 10$
	20	0.877	0.859	0.721	0.802	0.822	0.833	0.837
0	40	0.894	0.885	0.738	0.833	0.855	0.866	0.869
	100	0.905	0.900	0.754	0.839	0.855	0.870	0.876
	200	0.902	0.897	0.755	0.849	0.869	0.876	0.881
	20	0.877	0.859	0.793	0.830	0.839	0.845	0.849
0.5	40	0.894	0.885	0.829	0.870	0.877	0.879	0.885
	100	0.905	0.900	0.833	0.872	0.884	0.889	0.889
	200	0.902	0.897	0.845	0.877	0.885	0.889	0.893
	20	0.877	0.859	0.827	0.850	0.856	0.853	0.855
0.75	40	0.894	0.885	0.867	0.876	0.883	0.884	0.885
	100	0.905	0.900	0.869	0.889	0.893	0.895	0.895
	200	0.902	0.897	0.874	0.893	0.893	0.896	0.897
	20	0.877	0.859	0.860	0.861	0.861	0.859	0.857
1	40	0.894	0.885	0.890	0.888	0.887	0.885	0.885
	100	0.905	0.900	0.901	0.903	0.899	0.901	0.903
	200	0.902	0.897	0.901	0.899	0.899	0.898	0.898

Interval Lengths

		Normal	Traditional	Dependent Bootstrap				
θ	n	Approx.	Bootstrap	$k = 2$	$k = 4$	$k = 6$	$k = 8$	$k = 10$
	20	2.008	1.954	1.408	1.708	1.797	1.838	1.862
0	40	1.458	1.438	1.027	1.253	1.318	1.349	1.370
	100	0.927	0.922	0.655	0.801	0.845	0.863	0.876
	200	0.656	0.654	0.464	0.568	0.598	0.613	0.622
	20	2.008	1.954	1.672	1.834	1.880	1.900	1.914
0.5	40	1.458	1.438	1.233	1.351	1.380	1.396	1.408
	100	0.927	0.922	0.789	0.863	0.886	0.897	0.900
	200	0.656	0.654	0.560	0.612	0.629	0.636	0.639
	20	2.008	1.954	1.801	1.898	1.922	1.928	1.937
0.75	40	1.458	1.438	1.335	1.394	1.412	1.418	1.425
	100	0.927	0.922	0.857	0.895	0.903	0.911	0.912
	200	0.656	0.654	0.608	0.635	0.641	0.647	0.646
	20	2.008	1.954	1.925	1.953	1.961	1.961	1.961
1	40	1.458	1.438	1.434	1.442	1.443	1.443	1.442
	100	0.927	0.922	0.923	0.926	0.927	0.926	0.923
	200	0.656	0.654	0.656	0.657	0.656	0.657	0.655

Table 12: Modified Percentile 90% Confidence Intervals: Mixture of Two Normals

Coverage Probabilities

		Normal	Traditional	Dependent Bootstrap				
θ	n	Approx.	Bootstrap	$k=2$	$k=4$	$k=6$	$k=8$	$k=10$
0	20	0.885	0.875	0.733	0.812	0.838	0.847	0.854
	40	0.888	0.885	0.749	0.837	0.854	0.862	0.866
	100	0.901	0.901	0.759	0.849	0.868	0.875	0.877
	200	0.903	0.900	0.759	0.844	0.865	0.879	0.884
0.5	20	0.885	0.875	0.803	0.847	0.862	0.863	0.896
	40	0.888	0.885	0.831	0.861	0.873	0.875	0.880
	100	0.901	0.901	0.840	0.875	0.883	0.889	0.889
	200	0.903	0.900	0.843	0.877	0.886	0.894	0.891
0.75	20	0.885	0.875	0.843	0.857	0.871	0.871	0.872
	40	0.888	0.885	0.856	0.875	0.879	0.877	0.881
	100	0.901	0.901	0.868	0.885	0.891	0.898	0.893
	200	0.903	0.900	0.871	0.893	0.897	0.895	0.899
1	20	0.885	0.875	0.869	0.874	0.875	0.878	0.876
	40	0.888	0.885	0.889	0.889	0.885	0.885	0.887
	100	0.901	0.901	0.901	0.899	0.896	0.897	0.905
	200	0.903	0.900	0.897	0.900	0.903	0.902	0.904

Interval Lengths

		Normal	Traditional	Dependent Bootstrap				
θ	n	Approx.	Bootstrap	$k=2$	$k=4$	$k=6$	$k=8$	$k=10$
0	20	2.069	2.018	1.448	1.759	1.852	1.895	1.921
	40	1.469	1.451	1.032	1.262	1.330	1.361	1.379
	100	0.929	0.925	0.656	0.803	0.846	0.868	0.879
	200	0.656	0.654	0.464	0.568	0.599	0.613	0.622
0.5	20	2.069	2.018	1.732	1.891	1.939	1.959	1.974
	40	1.469	1.451	1.246	1.360	1.392	1.407	1.418
	100	0.929	0.925	0.792	0.866	0.889	0.900	0.902
	200	0.656	0.654	0.560	0.613	0.629	0.636	0.639
0.75	20	2.069	2.018	1.872	1.956	1.983	1.991	1.998
	40	1.469	1.451	1.350	1.404	1.423	1.429	1.434
	100	0.929	0.925	0.861	0.897	0.907	0.913	0.914
	200	0.656	0.654	0.608	0.635	0.642	0.646	0.647
1	20	2.069	2.018	2.008	2.016	2.023	2.022	2.023
	40	1.469	1.451	1.455	1.453	1.453	1.452	1.455
	100	0.929	0.925	0.930	0.929	0.928	0.930	0.926
	200	0.656	0.654	0.657	0.657	0.657	0.657	0.655