Institute of Mathematical Statistics

LECTURE NOTES — MONOGRAPH SERIES

FIXED DESIGN REGRESSION UNDER ASSOCIATION

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Abstract

For $n = 1, \ldots, n$, let $x_{ni}, i = 1, \ldots, n$, be points in a compact subset in $\Re^d, d \geq 1$, at which observations Y_{ni} are taken. It is assumed that these observations have the structure $Y_{ni} = g(x_{ni}) + \varepsilon_{ni}$, where g is a real-valued unknown function, and the errors $(\varepsilon_{n1}, \ldots, \varepsilon_{nn})$ coincide with the segment $(\xi_1, \ldots, \xi_n)$ of a strictly stationary sequence of random variables $\xi_1, \xi_2, \ldots$. For each $x \in \Re^d$, the function $g(x)$ is estimated by $g_n(x; \boldsymbol{x}_n) = \sum_{i=1}^n w_{ni}(x; \boldsymbol{x}_n) Y_{ni}$, where $\boldsymbol{x}_n = (x_{n1}, \ldots, x_{nn})$ and $w_{ni}(\cdot;\cdot)$ are weight functions. Under suitable conditions on the underlying stochastic process $\xi_1, \xi_2, \ldots$ and the weights $w_{ni}(\cdot;\cdot)$, it is shown that the estimate $g_n(x; \boldsymbol{x}_n)$ is asymptotically unbiased, and consistent in quadratic mean. By adding the assumption of (positive or negative) association of the sequence $\xi_1, \xi_2, \ldots$, it is shown that $g_n(x; \boldsymbol{x}_n)$, properly normalized, is also asymptotically normal.

Key words and phrases: Fixed design regression, stationarity, weights, fixed design regression estimate, asymptotic unbiasedness, consistency in quadratic mean, association, asymptotic normality.

1 Introduction

For each natural number n, consider the design points x_{ni}, $i = 1, \ldots, n$ in $\Re^d$, $d \geq 1$, which, through a real-valued (Borel) function g defined on $\Re^d$, produce observations Y_{ni}, subject to errors ε_{ni}, $1 \leq i \leq n$. That is,

$$Y_{ni} = g(x_{ni}) + \varepsilon_{ni}, \qquad 1 \leq i \leq n. \tag{1.1}$$

It is eventually assumed that, for each n, $(\varepsilon_{n1}, \ldots, \varepsilon_{nn})$ is equal in distribution to $(\xi_1, \ldots, \xi_n)$, where $\{\xi_n\}$, $n \geq 1$, is a (strictly) stationary and (positively or negatively) associated (see Definition 1.1) sequence of random variables (r.v.s). The problem we are faced with here is that of estimating the function g in terms of the Y_{ni}s and x_{ni}s, and establishing optimal properties for the proposed estimate. Following established tradition in this line of work, for each $x \in \Re^d$, the contemplated estimate is $g_n(x; \boldsymbol{x_n})$ given by

$$g_n(x; \boldsymbol{x_n}) = \sum_{i=1}^{n} w_{ni}(x; \boldsymbol{x_n}) Y_{ni}, \tag{1.2}$$

where $\boldsymbol{x_n} = (x_{n1}, \ldots, x_{nn})$, and $w_{ni}(\,\cdot\,;\,\cdot\,)$, $1 \leq i \leq n$, are suitable weight functions. It will be shown that, under appropriate regularity conditions, the proposed estimate is asymptotically unbiased, consistent in quadratic mean, and asymptotically normal.

Properties of this nature and for specific choices of the weight functions were established by Priestly and Chao (1972), and Gasser and Müller (1979). This problem was also investigated by Georgiev and Greblicki (1986) and Georgiev (1988). In all of these cases, the errors ε_{ni}, $i = 1, \ldots, n$, were assumed to be independent identically distributed (i.i.d.). When independence is replaced by strong mixing, the above cited results were established in Roussas (1989) and Roussas et al. (1992). In the present contribution, independence is suppressed again and is replaced by association. For a brief review on the significance of the concept of association, some of its applications, and a summary of some (statistical) results under association, the interested reader is referred to the review paper Roussas (1999). Relevant are also the papers of Cai and Roussas (1999 a,b). Important results on some limit theorems for dependent r.v.s, and, in particular, negatively dependent r.v.s may be found in Bozorgnia et al. (1996), Patterson and Taylor (1997), Taylor and Patterson (1997), and Taylor et al. (1999a,b).

The paper is organized as follows. Asymptotic unbiasedness and consistency in quadratic mean are established in Section 2 after suitable assumptions are spelled out. Asymptotic mormality is proved in Section 3 along with a number of auxiliary results. Assumptions under which these results hold are also stated in this same section, and they are followed by some comments.

This section is concluded with the definition of association.

Definition 1.1. For a finite index set I, the r.v.s $\{X_i; i \in I\}$ are said to be *positively associated* (PA), if for any real-valued coordinatewise increasing functions G and H defined on $\Re^I$,

$$Cov\,[G(X_i, i \in I), H(X_j, j \in I)] \geq 0,$$

provided $\mathcal{E}G^2(X_i, i \in I) < \infty, \mathcal{E}H^2(X_j, j \in I) < \infty$. These r.v.s are said to be *negatively associated* (NA), if for any nonempty and disjoint subsets A and B of I, and any coordinatewise increasing functions G and H with G :

$\Re^A \to \Re$ and $H : \Re^B \to \Re$ with $\mathcal{E}G^2(X_i, i \in A) < \infty, \mathcal{E}H^2(X_j, j \in B) < \infty$,

$$Cov\,[G(X_i, i \in A), H(X_j, j \in B)] \leq 0.$$

If I is not finite, the r.v.s $\{X_i; i \in I\}$ are said to be PA or NA, if any finite subcollection is a set of PA or NA r.v.s, respectively.

Finally, it is mentioned at the outset that all limits are taken as $n \to \infty$ unless otherwise stated, and C stands for a generic (positive) constant.

2 Asymptotic Unbiasedness and Consistency in Quadratic Mean

Assumptions (A)

(A1) For a compact subset S of $\Re^d$, the function $g : S \to \Re$ is continuous.

(A2) For $1 \leq i \leq n$ and $n \geq 1$, the errors ε_{ni}s have expectation 0.

For each $x \in S$ and with $\boldsymbol{x_n} = (x_{n1}, \ldots, x_{nn}), x_{ni} \in \Re^d$, $i = 1, \ldots, n$, the weights $w_{ni}(x; \boldsymbol{x_n})$ are 0 for $i > n$, and satisfy the following requirements for $1 \leq i \leq n$:

(A3) $\sum_{i=1}^n |w_{ni}(x; \boldsymbol{x_n})| \leq B$, $n \geq 1$, for a positive constant B.

(A4) $\sum_{i=1}^n |w_{ni}(x; \boldsymbol{x_n})| \to 1$.

(A5) For any $c > 0$, $\sum_{i=1}^n |w_{ni}(x; \boldsymbol{x_n})| I_{(\|x_{ni} - x\| > c)}(x) \to 0$,

where $\|\cdot\|$ is any one of the familiar norms in $\Re^d$.

All results in this paper hold for all $x \in \Re^d$ and with $\boldsymbol{x_n}$ as defined above.

Theorem 2.1 *(asymptotic unbiasedness).* Under assumptions (A1) - (A5),

$$\mathcal{E}g_n(x; \boldsymbol{x_n}) \to g(x).$$

Proof. Writing $g_n(x)$ and $w_{ni}(x)$ instead of $g_n(x; \boldsymbol{x_n})$ and $w_{ni}(x; \boldsymbol{x_n})$, respectively, we have

$$
\begin{aligned}
\mid \mathcal{E}g_n(x) - g(x) \mid &=: \mid \sum_{i=1}^{n} w_{ni}(x) g(x_{ni}) - g(x) \mid \\
&\leq \sum_{i=1}^{n} \mid w_{ni}(x) \mid\mid g(x_{ni}) - g(x) \mid + \mid g(x) \mid\mid \sum_{i=1}^{n} w_{ni}(x) - 1 \mid \\
&= \sum_{i=1}^{n} \mid w_{ni}(x) \mid\mid g(x_{ni}) - g(x) \mid I_{(\|x_{ni}-x\|>c)}(x) \\
&\quad + \sum_{i=1}^{n} \mid w_{ni}(x) \mid\mid g(x_{ni}) - g(x) \mid I_{(\|x_{ni}-x\|\leq c)}(x) \\
&\quad + \mid g(x) \mid\mid \sum_{i=1}^{n} w_{ni}(x) - 1 \mid . \qquad (2.1)
\end{aligned}
$$

For every $\varepsilon > 0$ and sufficiently small $c = c(\varepsilon)$, consider those x_{ni}s for which $\| x_{ni} - x \| \leq c$. Then $\mid g(x_{ni}) - g(x) \mid < \varepsilon$, and therefore $\mid g(x_{ni}) - g(x) \mid I_{(\|x_{ni}-x\|\leq c)}(x) < \varepsilon$. Thus, for all sufficiently large n, (2.1) yields $\mid \mathcal{E}g_n(x) - g(x) \mid < 2C\varepsilon + \varepsilon C + \varepsilon C = 4\varepsilon C$, where C is a suitable bounding constant. This completes the proof. ∎

Before the formulation of the second main result, assumptions (A) are augmented as follows.

Assumptions (B)

(B1) For each $n \geq 1$, $(\varepsilon_{n1}, \ldots, \varepsilon_{nn})$ is equal in distribution to $(\xi_1, \ldots, \xi_n)$, where $\{\xi_n\}$, $n \geq 1$, is a (strictly) stationary sequence of r.v.s, $\mathcal{E}\xi_1^2 = \sigma^2 < \infty$, and $\sum_{j=1}^{\infty} |Cov(\xi_1, \xi_{j+1})| < \infty$.

(B2) For each $x \in \Re^d$ and $\mathbf{x_n}$ as above,

$$w_n = max\left\{|w_{ni}(x; \boldsymbol{x_n})|\, ; 1 \leq i \leq n\right\} \to 0.$$

Theorem 2.2 *(consistency in quadratic mean).* Under assumptions (A1) - (A5) and (B1) - (B2),

$$\mathcal{E}\left[g_n(x; \boldsymbol{x_n}) - g(x)\right]^2 \to 0.$$

Proof. For further notational simplification, write just w_{ni} instead of $w_{ni}(x) = w_{ni}(x; \boldsymbol{x_n})$, and recall that $w_n = max\{|w_{ni}|; 1 \leq i \leq n\}$. Then, by assumptions (A3) and (B2),

$$\sum_{i=1}^{n} w_{ni}^2 \leq w_n \sum_{i=1}^{n} |w_{ni}| \to 0. \qquad (2.2)$$

Next,

$$\mathcal{E}\left[g_n(x) - g(x)\right]^2 = \mathcal{E}\left[g_n(x) - \mathcal{E}g_n(x)\right]^2 + \left[\mathcal{E}g_n(x) - g(x)\right]^2,$$

and the second term on the right-hand side above tends to 0, under assumptions (A1) - (A5), by Theorem 2.1. So, it suffices to show that $Var(g_n(x)) \to 0$. To this end,

$$\begin{aligned} Var\left(g_n(x)\right) &= Var\left(\sum_{i=1}^{n} w_{ni}\varepsilon_{ni}\right) = \mathcal{E}\left(\sum_{i=1}^{n} w_{ni}\varepsilon_{ni}\right)^2 \\ &= \sum_{i=1}^{n} w_{ni}^2 \mathcal{E}\varepsilon_{ni}^2 + 2\sum_{1\le i<j\le n} w_{ni}w_{nj}\mathcal{E}\left(\varepsilon_{ni}\varepsilon_{nj}\right) \\ &= \sigma^2 \sum_{i=1}^{n} w_{ni}^2 + 2\sum_{1\le i<j\le n} w_{ni}w_{nj}\mathcal{E}\left(\varepsilon_{ni}\varepsilon_{nj}\right). \end{aligned} \tag{2.3}$$

Since the first term on the right-hand side of (2.3) tends to 0 by (2.2), it suffices to show that

$$\sum_{1\le i<j\le n} w_{ni}w_{nj}\mathcal{E}\left(\varepsilon_{ni}\varepsilon_{nj}\right) \to 0.$$

By assumption (B1),

$$\left|\sum_{1\le i<j\le n} w_{ni}w_{nj}\mathcal{E}\left(\varepsilon_{ni}\varepsilon_{nj}\right)\right| = \left|\sum_{1\le i<j\le n} w_{ni}w_{nj}\mathcal{E}\left(\xi_i\xi_j\right)\right|$$

$$\le \sum_{1\le i<j\le n} \left|w_{ni}w_{nj}\right| \left|Cov\left(\xi_i,\xi_j\right)\right| = \sum_{i=1}^{n-1} \left|w_{ni}\right| \sum_{j=i+1}^{n} \left|w_{nj}\right| \left|Cov\left(\xi_i,\xi_j\right)\right|$$

$$\le w_n \sum_{i=1}^{n-1} \left|w_{ni}\right| \Big[\left|Cov\left(\xi_i,\xi_{i+1}\right)\right| + \cdots + \left|Cov\left(\xi_i,\xi_n\right)\right|\Big]$$

$$= w_n \sum_{i=1}^{n-1} \left|w_{ni}\right| \Big[\left|Cov\left(\xi_1,\xi_2\right)\right| + \cdots + \left|Cov\left(\xi_1,\xi_{n-i+1}\right)\right|\Big]$$

(by stationarity)

$$= w_n \sum_{i=1}^{n-1} \left|w_{ni}\right| \sum_{j=1}^{n-i} \left|Cov\left(\xi_1,\xi_{j+1}\right)\right| \le w_n \left[\sum_{j=1}^{\infty} \left|Cov\left(\xi_1,\xi_{j+1}\right)\right|\right] \left(\sum_{i=1}^{n} \left|w_{ni}\right|\right)$$

$$\leq Bw_n \sum_{j=1}^{\infty} |Cov\,(\xi_1, \xi_{j+1})| \to 0,$$

by assumptions (A3), (B1) and (B2). ■

Remark 2.1. At this point, it is to be observed that Theorems 2.1 - 2.2 were established without reference to association. The property of association is used only in Theorem 3.1, stated and proved in Section 3.

3 Asymptotic Normality

Introduce the following notation by suppressing the argument x. Set

$$\left.\begin{array}{ll} Z_{ni} & = \sigma_n^{-1} w_{ni} \varepsilon_{ni}, \text{ equal in distribution to } \sigma_n^{-1} w_{ni} \xi_i, \\ \sigma_n^2 & = Var(g_n) = Var\left(\sum_{i=1}^n w_{ni} \xi_i\right). \end{array}\right\} \tag{3.1}$$

Also, for $m = 1, \ldots, k$, let

$$\left.\begin{array}{ll} I_m & = \{(m-1)(p+q)+1, \ldots, (m-1)(p+q)+p\}, \\ J_m & = \{(m-1)(p+q)+(p+1), \ldots, m(p+q)\}, \end{array}\right\} \tag{3.2}$$

and define y_{nm}, y'_{nm} and y''_n by:

$$y_{nm} = \sum_{i \in I_m} Z_{ni}, \quad y'_{nm} = \sum_{j \in J_m} Z_{nj}, \quad y''_n = \sum_{l=k(p+q)+1}^{n} Z_{nl}, \tag{3.3}$$

and let

$$T_n = \sum_{m=1}^{k} y_{nm}, \quad T'_n = \sum_{m=1}^{k} y'_{nm}, \quad T''_n = y''_n. \tag{3.4}$$

We wish to show that

$$S_n \xrightarrow{d} N(0,1), \quad \text{where } S_n = \sigma_n^{-1}(g_n - \mathcal{E} g_n). \tag{3.5}$$

Clearly,

$$S_n = T_n + T'_n + T''_n, \tag{3.6}$$

and (3.5) will be established by showing that

$$T_n \xrightarrow{d} N(0,1), \tag{3.7}$$

and

$$\mathcal{E}(T'_n)^2 + \mathcal{E}(T''_n)^2 \to 0. \tag{3.8}$$

These assertions hold true under the set of assumptions stated below.

Although some of the assumptions spelled out below coincide with assumptions previously made, we choose to gather all of them here for easy reference.

Assumptions (C)

(C1) The sequence $\{\xi_n\}$, $n \geq 1$, is (either positively or negatively) associated and (strictly) stationary.

(C2) $\mathcal{E}\xi_1 = 0$, $\mathcal{E}|\xi_1|^{2+\delta} < \infty$ for some $\delta > 0$, and $\sum_{j=1}^{\infty} |Cov(\xi_1, \xi_{j+1})| < \infty$.

(C3) For $1 \leq i \leq n$ and $n \geq 1$, $(\varepsilon_{n1}, \ldots, \varepsilon_{nn})$ is equal in distribution to $(\xi_1, \ldots, \xi_n)$.

With $w_n = max\{|w_{ni}(x; \boldsymbol{x_n})|, 1 \leq i \leq n\}$, it is assumed that:

(C4) (i) $w_n = O(n^{-1})$.

(ii) $w_n = O(\sigma_n^2)$, where $\sigma_n^2 = \sigma_n^2(x) = Var(g_n(x; \boldsymbol{x_n}))$.

Let $p = p_n$ and $q = q_n$ be positive integers with $q < p < n$ and tending to ∞, as $n \to \infty$, and let $k = k_n$ be the largest integer for which $k(p+q) \leq n$. Then select p and q as just described, and also to satisfy the requirements:

(C5) (i) $p = o(n^\rho)$, $\rho = \frac{\delta}{2(1+\delta)}$ (the same δ as in (C2)).

(ii) $\frac{pk}{n} \to 1$.

Comments on some assumptions

(a) The choice of p, q, and k as $0 < q < p < n$, and tending to ∞, and k being the largest integer for which $k(p+q) \leq n$ imply immediately that $\frac{k(p+q)}{n} \to 1$ and $\frac{k}{n} \to 0$ (since $\frac{k}{n} = \frac{k(p+q)}{n} \cdot \frac{1}{p+q}$).

(b) $\frac{pk}{n} \to 1$ implies $\frac{qk}{n} \to 0$ and $\frac{q}{p} \to 0$ (since $\frac{k(p+q)}{n} = \frac{pk}{n} + \frac{qk}{n}$ and both $\frac{k(p+q)}{n}, \frac{pk}{n} \to 1$, and $\frac{q}{p} = \frac{\frac{qk}{n}}{\frac{pk}{n}} \to 0$).

(c) If $p = o(n)$ (which is implied by (C5)(i)), then $k \to \infty$ (since $\frac{1}{k} = \frac{\frac{p+q}{n}}{\frac{k(p+q)}{n}}$ and $\frac{p+q}{n} = \frac{p}{n}(1 + \frac{q}{p}) \to 0$ by (b)).

(d) Choices of p and q as described above and satisfying condition (C5) are readily available. Indeed, for $0 < \delta_2 < \delta_1 < \rho$, take $p \sim n^{\delta_1}$ and $q \sim n^{\delta_2}$ (where $x_n \sim y_n$ means $\frac{x_n}{y_n} \to 1$). This choice of p is consistent with (C5)(i) (since $\frac{p}{n^\rho} = \frac{p}{n^{\delta_1}} \cdot \frac{1}{n^{\rho-\delta_1}} \to 0$). Furthermore, $k \sim n^{1-\delta_1}$ (since $\frac{k}{n^{1-\delta_1}} = \frac{k(p+q)}{n} \cdot \frac{n^{\delta_1}}{p+q}$ and $\frac{n^{\delta_1}}{p+q} = \frac{1}{(\frac{p}{n^{\delta_1}} + \frac{q}{n^{\delta_2}} \cdot \frac{1}{n^{\delta_1-\delta_2}})}$ which tends to 1). Therefore $\frac{pk}{n} = \frac{k}{n^{1-\delta_1}} \cdot \frac{p}{n^{\delta_1}} \to 1$.

(e) That δ in assumptions (C2) and (C5)(i) must be the same stems from the proof of Lemma 3.2(iii).

(f) Assumption (C4)(ii) is borrowed from Roussas et al.(2000) (see Remark 2.1(ii), page 265).

Theorem 3.1. Under assumptions (C1) - (C5), the convergence asserted in (3.5) holds; that is,

$$S_n \xrightarrow{d} N(0,1),$$

where S_n is defined in (3.5), $g_n = g_n(x; \boldsymbol{x}_n)$ is given in (1.2), and $\sigma_n^2 = \sigma_n^2(x) = Var(g_n)$.

The proof of the theorem follows by combining the two propositions below. The propositions, as well as the three lemmas employed in this section, hold under all or parts only of assumptions (C1) - (C5). However, these lesser assumptions will not be explicitly stated.

Proposition 3.1. The convergence asserted in (3.8) holds; that is,

$$\mathcal{E}(T_n')^2 + \mathcal{E}(T_n'')^2 \to 0,$$

where T_n' and T_n'' are given in (3.4).

Proposition 3.2. The convergence asserted in (3.7) holds; that is,

$$T_n \xrightarrow{d} N(0,1),$$

where T_n is given in (3.4).

Assuming for a moment that Propositions 3.1 and 3.2 have been established, we have

Proof of Theorem 3.1. It follows from Propositions 3.1 - 3.2 and relation (3.6). ■

The following three lemmas will be required in various parts of the proofs of Propositions 3.1 - 3.2.

Lemma 3.1. Let y_{nm} and y'_{nm} be defined by (3.3). Then:

(i)

$$\sum_{1\le l<r\le k}\left|Cov(y_{nl},y_{nr})\right|\le 2\frac{w_n^2pk}{\sigma_n^2}\sum_{j=q}^{\infty}\left|Cov(\xi_1,\xi_{j+1})\right|,$$

and

(ii)

$$\sum_{1\le l<r\le k}\left|Cov(y'_{nl},y'_{nr})\right|\le 2\frac{w_n^2qk}{\sigma_n^2}\sum_{j=q}^{\infty}\left|Cov(\xi_1,\xi_{j+1})\right|,$$

where, it is recalled from assumption (B2) that, $w_n = max\{|w_{ni}|; 1 \le i \le n\}$, and σ_n^2 is given in (3.1).

Proof. (i) From the definition of the y_{nm}s, and with the I_ms as defined in (3.2), it is clear that

$$\sum_{1\le l<r\le k}\left|Cov(y_{nl},y_{nr})\right| = \sum_{1\le l<r\le k}\left|Cov\left(\sum_{i\in I_l}Z_{ni},\sum_{j\in I_r}Z_{nj}\right)\right|$$

$$\le \sum_{1\le l<r\le k}\sum_{i\in I_l}\sum_{j\in I_r}\left|Cov\left(Z_{ni},Z_{nj}\right)\right|$$

$$\le \frac{w_n^2}{\sigma_n^2}\sum_{1\le l<r\le k}\sum_{i\in I_l}\sum_{j\in I_r}\left|Cov(\xi_i,\xi_j)\right| = \frac{w_n^2}{\sigma_n^2}A_n, \tag{3.9}$$

where

$$A_n = \sum_{1\le l<r\le k}\sum_{i\in I_l}\sum_{j\in I_r}\left|Cov(\xi_i,\xi_j)\right|. \tag{3.10}$$

However, by stationarity of the ξ_is,

$$A_n = (k-1)\sum_{i\in I_1}\sum_{j\in I_2}\left|Cov(\xi_i,\xi_j)\right| + (k-2)\sum_{i\in I_1}\sum_{j\in I_3}\left|Cov(\xi_i,\xi_j)\right| +$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots +$$

$$+2\sum_{i\in I_1}\sum_{j\in I_{k-1}}\left|Cov(\xi_i,\xi_j)\right| + \quad 1\sum_{i\in I_1}\sum_{j\in I_k}\left|Cov(\xi_i,\xi_j)\right|$$

$$\le \Bigg\{(k-1)\Big[p\left|Cov(\xi_1,\xi_{(p+q)+1})\right| + (p-1)\left|Cov(\xi_1,\xi_{(p+q)+2})\right| +$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad \ldots + \left|Cov(\xi_1, \xi_{(p+q)+p})\right|\Big] + \\
&(k-2)\Big[p\left|Cov(\xi_1, \xi_{2(p+q)+1})\right| + (p-1)\left|Cov(\xi_1, \xi_{2(p+q)+2})\right| + \\
&\qquad\qquad\qquad\qquad \ldots + \left|Cov(\xi_1, \xi_{2(p+q)+p})\right|\Big] + \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\
&2\Big[p\left|Cov(\xi_1, \xi_{(k-2)(p+q)+1})\right| + (p-1)\left|Cov(\xi_1, \xi_{(k-2)(p+q)+2})\right| + \\
&\qquad\qquad\qquad\qquad \ldots + \left|Cov(\xi_1, \xi_{(k-2)(p+q)+p})\right|\Big] + \\
&1\Big[p\left|Cov(\xi_1, \xi_{(k-1)(p+q)+1})\right| + (p-1)\left|Cov(\xi_1, \xi_{(k-1)(p+q)+2})\right| + \\
&\qquad\qquad\qquad\qquad \ldots + \left|Cov(\xi_1, \xi_{(k-1)(p+q)+p})\right|\Big] \Bigg\} + \\
&\Bigg\{(k-1)\Big[(p-1)\left|Cov(\xi_2, \xi_{(p+q)+1})\right| + \ldots + \left|Cov(\xi_p, \xi_{(p+q)+1})\right|\Big] + \\
&(k-2)\Big[(p-1)\left|Cov(\xi_2, \xi_{2(p+q)+1})\right| + \ldots + \left|Cov(\xi_p, \xi_{2(p+q)+1})\right|\Big] + \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\
&2\Big[(p-1)\left|Cov(\xi_2, \xi_{(k-2)(p+q)+1})\right| + \ldots + \left|Cov(\xi_p, \xi_{(k-2)(p+q)+1})\right|\Big] + \\
&1\Big[(p-1)\left|Cov(\xi_2, \xi_{(k-1)(p+q)+1})\right| + \ldots + \left|Cov(\xi_p, \xi_{(k-1)(p+q)+1})\right|\Big]\Bigg\} \\
&= A_{n1} + A_{n2}, \qquad (3.11)
\end{aligned}
$$

where A_{n1} and A_{n2} stand for the first and second bracket in (3.11), respectively.

However,

$$
\begin{aligned}
A_{n1} &\leq pk\left[\sum_{j\in I_2}\left|Cov(\xi_1, \xi_j)\right| + \sum_{j\in I_3}\left|Cov(\xi_1, \xi_j)\right| + \ldots + \sum_{j\in I_k}\left|Cov(\xi_1, \xi_j)\right|\right] \\
&\leq pk \sum_{j=p+q+1}^{(k-1)(p+q)+p}\left|Cov(\xi_1, \xi_j)\right| \\
&\leq pk \sum_{j=p+q}^{\infty}\left|Cov(\xi_1, \xi_{j+1})\right|, \qquad (3.12)
\end{aligned}
$$

and

$$A_{n2} \leq pk\left[\sum_{j=(p+q)-(p-1)}^{p+q} \left|Cov(\xi_1, \xi_{j+1})\right| + \sum_{j=2(p+q)-(p-1)}^{2(p+q)} \left|Cov(\xi_1, \xi_{j+1})\right| + \right.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots +$$

$$\left. \sum_{j=(k-2)(p+q)-(p-1)}^{(k-2)(p+q)} \left|Cov(\xi_1, \xi_{j+1})\right| + \sum_{j=(k-1)(p+q)-(p-1)}^{(k-1)(p+q)} \left|Cov(\xi_1, \xi_{j+1})\right|\right]$$

$$\leq pk\left[\sum_{j=q}^{p+q} \left|Cov(\xi_1, \xi_{j+1})\right| + \sum_{j=(p+q)+q}^{2(p+q)} \left|Cov(\xi_1, \xi_{j+1})\right| + \right.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots +$$

$$\left. \sum_{j=(k-3)(p+q)+q}^{(k-2)(p+q)} \left|Cov(\xi_1, \xi_{j+1})\right| + \sum_{j=(k-2)(p+q)+q}^{(k-1)(p+q)} \left|Cov(\xi_1, \xi_{j+1})\right|\right]$$

$$\leq pk \sum_{j=q}^{\infty} \left|Cov(\xi_1, \xi_{j+1})\right|. \tag{3.13}$$

Relations (3.11) - (3.13) imply that $A_n \leq 2pk \sum_{j=q}^{\infty} |Cov(\xi_1, \xi_{j+1})|$, so that (3.9) and (3.10) yield

$$\sum_{1 \leq l < r \leq k} \left|Cov(y_{nl}, y_{nr})\right| \leq 2\frac{w_n^2 pk}{\sigma_n^2} \sum_{j=q}^{\infty} \left|Cov(\xi_1, \xi_{j+1})\right|,$$

which is what part(i) asserts.

(ii) It follows as in part (i) upon replacing the y_{nm}s by the y'_{nm}s. ■

Corollary 3.1. It holds:

(i) $\sum_{1 \leq l < r \leq k} \left|Cov(y_{nl}, y_{nr})\right| \to 0$, and (ii) $\sum_{1 \leq l < r \leq k} \left|Cov(y'_{nl}, y'_{nr})\right| \to 0$.

Proof. (i) The right-hand side of the expression in Lemma 3.1(i) is written as: $(\frac{w_n}{\sigma_n^2})(\frac{pk}{n})(nw_n) \sum_{j=q}^{\infty} |Cov(\xi_1, \xi_{j+1})|$, and this is bounded by: $C \sum_{j=q}^{\infty} |Cov(\xi_1, \xi_{j+1})|$, on account of assumptions (C4)(ii), (C5)(ii) and (C4)(i). However, this last expression tends to 0 by assumption (C2).

(ii) Likewise, the right-hand side of the expression in Lemma 3.1(ii) is written as: $(\frac{w_n}{\sigma_n^2})(\frac{qk}{n})(nw_n) \sum_{j=p}^{\infty} |Cov(\xi_1, \xi_{j+1})| \to 0$ as in part (i). ■

We may now proceed with the proof of Proposition 3.1.

Proof of Proposition 3.1. First,

$$\mathcal{E}(T_n')^2 = \mathcal{E}\left(\sum_{m=1}^{k} y_{nm}'\right)^2 = \sum_{m=1}^{k} Var(y_{nm}') + 2 \sum_{1 \le l < r \le k} Cov\left(y_{nl}', y_{nr}'\right), \quad (3.14)$$

and

$$Var(y_{nm}') = Var\left(\sum_{i \in J_m} Z_{ni}\right) = \sum_{i \in J_m} Var(Z_{ni}) + 2 \sum_{\substack{i,j \in J_m \\ i<j}} Cov(Z_{ni}, Z_{nj})$$

$$\le \frac{\sigma^2 q w_n^2}{\sigma_n^2} + \frac{2w_n^2}{\sigma_n^2} \sum_{1 \le i < j \le q} \left|Cov(\xi_i, \xi_j)\right|$$

$$\le \frac{\sigma^2 q w_n^2}{\sigma_n^2} + \frac{2q w_n^2}{\sigma_n^2} \sum_{j=1}^{\infty} \left|Cov(\xi_1, \xi_{j+1})\right| \le C \frac{q w_n^2}{\sigma_n^2},$$

so that, by assumptions (C4)(ii),

$$\sum_{m=1}^{k} Var(y_{nm}') \le C \frac{w_n^2}{\sigma_n^2} qk = C\left(\frac{qk}{n}\right)(n w_n) \to 0; \quad (3.15)$$

this is so because $n w_n = O(1)$ and $\frac{qk}{n} \to 0$, which is implied by $\frac{pk}{n} \to 1$ (see Comment (b) after Assumptions (C)). Relations (3.14) - (3.15) and Corollary 3.1(ii) show that $\mathcal{E}(T_n')^2 \to 0$.

Next,

$$\mathcal{E}(T_n'')^2 = Var\left(\sum_{i=k(p+q)+1}^{n} Z_{ni}\right)$$

$$\le \sum_{i=k(p+q)+1}^{n} Var(Z_{ni}) + 2 \sum_{k(p+q)+1 \le i < j \le n} \left|Cov(Z_{ni}, Z_{nj})\right|, \quad (3.16)$$

and

$$\sum_{i=k(p+q)+1}^{n} Var(Z_{ni}) \leq \frac{\sigma^2 w_n^2}{\sigma_n^2}\Big[n - k(p+q)\Big] \leq \frac{\sigma^2 w_n^2}{\sigma_n^2}(p+q)$$

$$= \frac{\sigma^2 w_n^2}{\sigma_n^2} p\left(1 + \frac{q}{p}\right) \leq C\frac{w_n^2}{\sigma_n^2} p \text{ (since } \frac{q}{p} \to 0, \text{ by (Comment(b))}$$

$$\leq C w_n p = C\frac{p}{n} \to 0 \text{ (by (C4)(i) and (C5)(i)).} \quad (3.17)$$

Thus, relation (3.17) implies that

$$\sum_{i=k(p+q)+1}^{n} Var(Z_{ni}) \to 0. \quad (3.18)$$

Finally,

$$\sum_{k(p+q)+1 \leq i < j \leq n} \Big|Cov(Z_{ni}, Z_{nj})\Big| \leq \frac{w_n^2}{\sigma_n^2} \sum_{k(p+q)+1 \leq i < j \leq n} \Big|Cov(\xi_i, \xi_j)\Big|$$

$$= \frac{w_n^2}{\sigma_n^2} \sum_{1 \leq i < j \leq n-k(p+q)} \Big|Cov(\xi_i, \xi_j)\Big|$$

$$= \frac{w_n^2}{\sigma_n^2} \sum_{j=1}^{n-k(p+q)-1} \Big\{[n - k(p+q)] - j\Big\} \Big|Cov(\xi_1, \xi_{j+1})\Big|$$

$$\leq \frac{w_n^2}{\sigma_n^2} [n - k(p+q)] \sum_{j=1}^{\infty} \Big|Cov(\xi_1, \xi_{j+1})\Big|$$

$$\leq C\frac{w_n^2}{\sigma_n^2}(p+q) = C\frac{w_n^2}{\sigma_n^2} p\left(1 + \frac{q}{p}\right) \leq C\frac{w_n^2}{\sigma_n^2} p \text{ (by Comment (b))}$$

$$\leq C\frac{p}{n} \to 0 \qquad \text{(by (3.17)).} \quad (3.19)$$

Relations (3.16) - (3.19) show that $\mathcal{E}(T_n'')^2 \to 0$. The proof of the proposition is completed. ■

For the formulation of the second lemma, introduce the following notation. Let Y_{nm}, $m = 1, \ldots, k$ be independent r.v.s with Y_{nm} having the distribution of y_{nm}, set $s_n^2 = \sum_{m=1}^k Var(Y_{nm})$, and let $X_{nm} = \frac{Y_{nm}}{s_n}$ with distribution function F_{nm}, $m = 1, \ldots, k$. Then the r.v.s X_{nm}, $m = 1, \ldots, k$ are independent with $\mathcal{E}X_{nm} = 0$ and $\sum_{m=1}^k Var(X_{nm}) = 1$. Finally, for $\varepsilon > 0$, set

$$g_n(\varepsilon) = \sum_{m=1}^{k} \int_{(|x|\geq\varepsilon)} x^2 dF_{nm}(x). \tag{3.20}$$

Then we have

Lemma 3.2. Let T_n and $g_n(\varepsilon)$ be given by (3.4) and (3.20), respectively, and recall that $s_n^2 = \sum_{m=1}^k Var(Y_{nm})$.

Then:

(i) $\mathcal{E}T_n^2 \to 1$, (ii) $s_n^2 \to 1$, and (iii) $g_n(\varepsilon) \to 0$.

Proof. (i) From (3.5), $\mathcal{E}S_n^2 = 1$, whereas from (3.6),

$$\begin{aligned}
\mathcal{E}T_n^2 &= \mathcal{E}\Big[S_n - (T_n' + T_n'')\Big]^2 \\
&= \mathcal{E}S_n^2 + \mathcal{E}(T_n' + T_n'')^2 - 2\mathcal{E}\Big[S_n(T_n' + T_n'')\Big] \\
&= 1 + \mathcal{E}(T_n' + T_n'')^2 - 2\mathcal{E}\Big[S_n(T_n' + T_n'')\Big].
\end{aligned}$$

But by Proposition 3.1,

$$\mathcal{E}^{\frac{1}{2}}(T_n' + T_n'')^2 \leq \mathcal{E}^{\frac{1}{2}}(T_n')^2 + \mathcal{E}^{\frac{1}{2}}(T_n'')^2 \to 0, \tag{3.21}$$

and

$$\begin{aligned}
&\Big|\mathcal{E}\Big[S_n(T_n' + T_n'')\Big]\Big| \leq \mathcal{E}\Big|S_n(T_n' + T_n'')\Big| \\
&\quad \leq \Big(\mathcal{E}^{\frac{1}{2}}S_n^2\Big)\mathcal{E}^{\frac{1}{2}}\Big(T_n' + T_n''\Big)^2 = \mathcal{E}^{\frac{1}{2}}\Big(T_n' + T_n''\Big)^2 \to 0 \text{ (by (3.21))}.
\end{aligned}$$

Thus,

$$\mathcal{E}T_n^2 \to 1.$$

(ii) From (3.4) again,

$$\begin{aligned}
\mathcal{E}T_n^2 &= Var(\sum_{m=1}^{k} y_{nm}) = \sum_{m=1}^{k} Var(y_{nm}) + 2 \sum_{1\leq l<r\leq k} Cov(y_{nl}, y_{nr}) \\
&= s_n^2 + 2 \sum_{1\leq l<r\leq k} Cov(y_{nl}, y_{nr}),
\end{aligned} \tag{3.22}$$

and $\sum_{1\leq l<r\leq k} Cov(y_{nl}, y_{nr}) \to 0$ by Corollary 3.1(i). Then relation (3.22) and part(i) yield $s_n^2 \to 1$.

(iii) We have

$$\int_{(|x|\geq\varepsilon)} x^2 dF_{nm}(x) = \mathcal{E}\left[X_{nm}^2 I\left(|X_{nm}| \geq \varepsilon\right)\right] = s_n^{-2}\mathcal{E}\left[y_{nm}^2 I\left(|y_{nm}| \geq \varepsilon s_n\right)\right]$$

$$\leq s_n^{-2}\mathcal{E}^{\frac{1}{s}} |y_{nm}|^{2s} P^{\frac{1}{t}}\left(|y_{nm}| \geq \varepsilon s_n\right) \qquad (\text{where } s,\, t>1 \text{ with } \frac{1}{s}+\frac{1}{t}=1\ )$$
$$\leq s_n^{-2}\mathcal{E}^{\frac{1}{s}} |y_{nm}|^{2s}\left(\varepsilon^{-2s}s_n^{-2s}\mathcal{E}\,|y_{nm}|^{2s}\right)^{\frac{1}{t}} = \varepsilon^{-\frac{2s}{t}} s_n^{-\frac{2s}{t}-2}\mathcal{E}\,|y_{nm}|^{2s}. \tag{3.23}$$

At this point, take $s = \frac{2+\delta}{2}$ and $t = \frac{2+\delta}{\delta}$, so that $2s = 2+\delta = \nu$, and $\frac{2s}{t} = \delta$. Then (3.23) becomes

$$\int_{(|x|\geq\varepsilon)} x^2 dF_{nm}(x) \leq \frac{1}{\varepsilon^\delta}\cdot\frac{1}{s_n^\nu}\mathcal{E}\,|y_{nm}|^\nu. \tag{3.24}$$

However, by assumption (B1)

$$\mathcal{E}\,|y_{nm}|^\nu = \mathcal{E}\left|\sum_{i\in I_m} Z_{ni}\right|^\nu \leq \mathcal{E}\left(\sum_{i\in I_m}|Z_{ni}|\right)^\nu \leq \frac{w_n^\nu}{\sigma_n^\nu}\mathcal{E}\left(\sum_{i=1}^p|\xi_i|\right)^\nu, \tag{3.25}$$

and

$$\mathcal{E}^{\frac{1}{\nu}}\left(\sum_{i=1}^p|\xi_i|\right)^\nu \leq \sum_{i=1}^p \mathcal{E}^{\frac{1}{\nu}}\,|\xi_i|^\nu = p\mathcal{E}^{\frac{1}{\nu}}\,|\xi_1|^\nu,$$

so that

$\mathcal{E}(\sum_{i=1}^p|\xi_i|)^\nu \leq p^\nu\mathcal{E}|\xi_1|^\nu$, and therefore, by (3.24) - (3.25),

$$\int_{(|x|\geq\varepsilon)} x^2 dF_{nm}(x) \leq \left(\frac{1}{\varepsilon^\delta}\mathcal{E}\,|\xi_1|^\nu\right)\frac{w_n^\nu p^\nu}{\sigma_n^\nu s_n^\nu}.$$

Hence, with $C = \varepsilon^{-1}\mathcal{E}\mid\xi_1\mid^\nu$,

$$g_n(\varepsilon) \leq C\frac{w_n^\nu p^\nu k}{\sigma_n^\nu s_n^\nu}. \tag{3.26}$$

However,

$$\frac{w_n^\nu p^\nu k}{\sigma_n^\nu s_n^\nu} = \left(\frac{w_n}{\sigma_n^2}\right)^\nu\cdot\frac{1}{s_n^\nu}\cdot\left(\frac{pk}{n}\right)(\sigma_n^\nu p^{\nu-1}n) \leq C(\sigma_n^\nu p^{\nu-1}n), \tag{3.27}$$

by assumption (C4)(ii), part(ii) of the present lemma and assumption (C5)(ii), and

$$\sigma_n^2 = \mathcal{E}\left(\sum_{i=1}^{n} w_{ni}\xi_i\right)^2 = \sigma^2 \sum_{i=1}^{n} w_{ni}^2 + 2 \sum_{1\leq i<j\leq n} w_{ni}w_{nj}Cov(\xi_i,\xi_j)$$

$$\leq \sigma^2 n w_n^2 + 2nw_n^2 \sum_{i=1}^{\infty} |Cov(\xi_1,\xi_{j+1})| = Cnw_n^2,$$

so that

$$\sigma_n^\nu \leq C n^{\frac{\nu}{2}} w_n^\nu. \tag{3.28}$$

Then, by (3.26) - (3.28), and assumption (C4(i)),

$$g_n(\varepsilon) \leq C\frac{p^{\nu-1}}{n^{\frac{\nu}{2}-1}} = C\frac{p^{1+\delta}}{n^{\frac{\delta}{2}}}. \tag{3.29}$$

However, the right-hand side in (3.29) tends to 0 on account of the choice of p in assumption (C5)(i); namely, $p = o(n^{\frac{\delta}{2(1+\delta)}})$ or $p^{1+\delta} = o(n^{\frac{\delta}{2}})$. The proof of the lemma is completed.■

Lemma 3.3. With the y_{nm}s defined by (3.3) and for any $t \in \mathbb{R}$, it holds

$$\left|\mathcal{E}e^{it\sum_{m=1}^{k} y_{nm}} - \prod_{m=1}^{k} \mathcal{E}e^{ity_{nm}}\right| \to 0. \tag{3.30}$$

Proof. Clearly,

$$\begin{aligned}
\left|\mathcal{E}e^{it\sum_{m=1}^{k} y_{nm}} - \prod_{m=1}^{k}\mathcal{E}e^{ity_{nm}}\right| &= \left|\left(\mathcal{E}e^{it\sum_{m=1}^{k} y_{nm}} - \mathcal{E}e^{it\sum_{m=1}^{k-1} y_{nm}}\cdot\mathcal{E}e^{ity_{nk}}\right)\right.\\
&\quad + \left.\left(\mathcal{E}e^{it\sum_{m=1}^{k-1} y_{nm}}\cdot\mathcal{E}e^{ity_{nk}} - \prod_{m=1}^{k}\mathcal{E}e^{ity_{nm}}\right)\right|\\
&\le \left|\mathcal{E}e^{it\sum_{m=1}^{k} y_{nm}} - \mathcal{E}e^{it\sum_{m=1}^{k-1} y_{nm}}\cdot\mathcal{E}e^{ity_{nk}}\right|\\
&\quad + \left|\mathcal{E}e^{it\sum_{m=1}^{k-1} y_{nm}} - \prod_{m=1}^{k-1}\mathcal{E}e^{ity_{nm}}\right|\\
&= \left|Cov\left(e^{it\sum_{m=1}^{k-1} y_{nm}}, e^{ity_{nk}}\right)\right|\\
&\quad + \left|\mathcal{E}e^{it\sum_{m=1}^{k-1} y_{nm}} - \prod_{m=1}^{k-1}\mathcal{E}e^{ity_{nm}}\right|,
\end{aligned}$$

and, by a repetition of the argument, inequality (3.30) becomes

$$\begin{aligned}
\left|\mathcal{E}e^{it\sum_{m=1}^{k} y_{nm}} - \prod_{m=1}^{k}\mathcal{E}e^{ity_{nm}}\right| &\le \left|Cov\left(e^{it\sum_{m=1}^{k-1} y_{nm}}, e^{ity_{nk}}\right)\right|\\
&\quad + \left|Cov\left(e^{it\sum_{m=1}^{k-2} y_{nm}}, e^{ity_{n,k-1}}\right)\right|\\
&\quad + \ldots + \left|Cov\left(e^{ity_{n2}}, e^{ity_{n1}}\right)\right|. \qquad (3.31)
\end{aligned}$$

At this point, use relations (3.1) and (3.3) to define the functions

$$f_m(x_i, i\in I_m) = e^{it\sigma_n^{-1}\sum_{i\in I_m} w_{ni}x_i}, m = 1,\ldots,k,$$

and observe that, for each $i \in I_m$, $\left|\frac{\partial}{\partial x_i} f_m(x_i, i\in I_m)\right| \le \left|t\sigma_n^{-1}w_{ni}\right| \le 2\left|t\sigma_n^{-1}\right| w_n$. It follows that, for each $i\in I_m$, $m = 1,\ldots,l$, and $l = 1,\ldots,k$,

$$\left|\frac{\partial}{\partial x_i}\prod_{m=1}^{l} f_m(x_i, i\in I_m)\right| \le 2\left|t\sigma_n^{-1}\right| w_n. \qquad (3.32)$$

Therefore, applying Lemma 1 in Bulinski(1996) to each one of the terms on the right-hand side of (3.31), we obtain by way of (3.32)

$$\left|\mathcal{E}e^{it\sum_{m=1}^{k} y_{nm}} - \prod_{m=1}^{k}\mathcal{E}e^{ity_{nm}}\right| \leq \frac{4t^2w_n^2}{\sigma_n^2}\Big[\sum_{j\in I_1}\sum_{l\in I_2}|Cov\,(\xi_j,\xi_l)|$$
$$+\sum_{j\in(I_1+I_2)}\sum_{l\in I_3}|Cov\,(\xi_j,\xi_l)|$$
$$+\ldots+\sum_{j\in(I_1+\ldots+I_{k-1})}\sum_{l\in I_k}|Cov\,(\xi_j,\xi_l)|\Big]. \quad (3.33)$$

By utilizing stationarity of the ξ_is and repeating the arguments used in relations (4.3) - (4.5) in Roussas(2000), inequality (3.33) becomes

$$\left|\mathcal{E}e^{it\sum_{m=1}^{k} y_{nm}} - \prod_{m=1}^{k}\mathcal{E}e^{ity_{nm}}\right| \leq 4t^2\frac{w_n^2}{\sigma_n^2}pk\sum_{j=(p+q)+1}^{(k-1)(p+q)+p}|Cov\,(\xi_1,\xi_j)|$$
$$\leq 4t^2(nw_n)\left(\frac{w_n}{\sigma_n^2}\right)\left(\frac{pk}{n}\right)\sum_{j=q}^{\infty}|Cov\,(\xi_1,\xi_j)|$$
$$\to 0, \quad (3.34)$$

by assumptions (C4), (C5)(ii) and (C2). The proof of the lemma is completed. ■

Proof of Proposition 3.2. Lemma 3.2(ii) implies that $\sum_{m=1}^{k} Y_{nm}\xrightarrow{d}N(0,1)$ by the Feller-Lindeberg Criterion (see, for example, Loève (1963), page 280). This fact along with (3.34) yield the result $\sum_{m=1}^{k} y_{nm}\xrightarrow{d}N(0,1)$ or $T_n\xrightarrow{d}N(0,1)$, as the proposition asserts. ■

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