ON CENTRAL LIMIT THEORY FOR RANDOM ADDITIVE FUNCTIONS UNDER WEAK DEPENDENCE RESTRICTIONS

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Random additive functions defined on intervals provide a general framework for varied applications including (dependent) array sums, and level-exceedance measures for stochastic sequences and processes. Central limit theory is developed in Leadbetter and Rootzén (1993) for families $\{\zeta_T(I) : T > 0\}$ of such functions under (array forms of) standard strong mixing conditions. One objective of the present paper is to introduce a potentially much weaker and more readily verifiable form of strong mixing under which the limiting distributional results are shown to apply. These lead to characterization of possible limits for such $\zeta_T(I)$ as those for independent array sums, i.e. the classical infinitely divisible types. The conditions and results obtained for one interval are then extended to apply to joint distributions of $\{\zeta_T(I_j) : 1 \leq j \leq p\}$ of (disjoint) intervals $I_1, I_2, \ldots I_p$, asymptotic independence of the components being shown under the extended conditions. Similar results are shown under even slightly weaker conditions for positive, additive families. Under *countable* additivity this leads in particular to distributional convergence of random measures under these mixing conditions, to infinitely divisible random measure limits having independent increments.

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1 Introduction

By a random additive function (r.a.f.) we mean a random function $\zeta(I)$ defined for subintervals $I = (a, b]$ of the unit interval and additive in the sense that $\zeta(a, b] + \zeta(b, c] = \zeta(a, c]$ when $0 < a < b < c \leq 1$. To our knowledge, such a framework was first used in proving a central limit theorem in the early (and pioneering) paper Volkonski and Rozanov (1959).

As described in Leadbetter and Rootzén (1993) and (1997), r.a.f. *families* $\{\zeta_T(I)\}$ (or $\{\zeta_n(I)\}$) provide a simple unifying framework for (array) central limit problems for both discrete and continuous parameter processes. This includes the general limiting distributional properties of array sums and exceedance measures, which are useful in a variety of areas such as environmental regulation and structural reliability (cf. Leadbetter and Huang (1996)). For example with obvious notation, for $I = (a, b] \subset (0, 1]$, $\zeta_n(I) = \sum_{i/n \in I} \xi_{n,i}$ gives general array sums, $\zeta_n(I) = \sum_{i/n \in I} 1(\xi_i > u_n)$ and $\zeta_T(I) =$

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$\int_{TI} 1(\xi_t > u_T)dt$ determine respectively the exceedance point process and the exceedance random measure of a level u_n (u_T). (cf. Leadbetter and Hsing (1990), Leadbetter et al (1983), Rootzén et al (1998)).

Dependent central limit theory was actively developed further in the 1960's and 1970's with the appearance of books (cf. Billingsley (1968), Ibragimov and Linnik (1971)) and numbers of papers. In particular, Philipp (1969) obtains the same results for the central limit problem for sums of random variables having bounded variances and satisfying a variety of mixing conditions, as apply to iid arrays (cf. Loève (1977)). Studies such as Philipp (1969) typically consider precise conditions under which a dependent array sum has the same (infinitely divisible) distribution as does an independent array sum with the same marginals. The conditions for such results can be very restrictive, involving both long and short term dependence behavior.

In Leadbetter and Rootzén (1993) the problem is considered in two parts: (a) the characterization of possible limits as the classical infinitely divisible laws and (b) domain of attraction criteria. It is there shown that the main condition required for the characterization (a) is just (an array form of) strong mixing (at the weak end of the spectrum of such dependence restrictions, in spite of its name). For (b) it is shown that the distributions of sums of relatively small groups of consecutive terms may be used in the classical domain of attraction criteria to determine which limit applies. This latter result is not as detailed as the use of marginal distributions in studies such as Philipp (1969), but requires much less by way of assumptions.

In this paper we improve the results of Leadbetter and Rootzén (1993) and extend them in detail to obtain multivariate limits under an at least formally much weaker type of strong mixing, through characteristic functions. As will be seen in the next section, for univariate limits this simply approximates $\mathcal{E}\{e^{it\zeta_T(I_1)}e^{it\zeta_T(I_2)}\}$ by $\mathcal{E}e^{it\zeta_T(I_1)}\mathcal{E}e^{it\zeta_T(I_2)}$ for the r.a.f. ζ_T and appropriate disjoint intervals I_1, I_2. A number of authors (e.g. Bulinskii and Zhurbenko (1976), Withers (1981)) have used related characteristic function conditions in obtaining Central Limit Theorems. Here we employ versions designed to be as efficient as possible for the more general discussion of limits (normal, Compound Poisson etc.) of r.a.f. *arrays*. The results apply to r.a.f.'s which are finitely but not necessarily countably additive. In the positive case it is natural to further consider countable additivity, extending the domain of definition from intervals to Borel sets and thus also obtaining limit theorems for random measures.

Basic framework and dependence assumptions are discussed in Section 2. In Section 3 the basic factorization lemma for the characteristic functions within that framework is obtained under a specific negligibility condition, and it is shown that this latter condition may be omitted if ζ_T is positive-valued and stationary. Section 4 provides central limit results and contains comments on the applications of classical domain of attraction criteria. Finally Section 5 extends the conditions and theory in obtaining multivariate results, i.e. limiting joint distributions of an r.a.f. evaluated for two or more intervals.

2 Framework

For an r.a.f. based on a stochastic sequence $\{\xi_j\}$ or process $\{\xi_t\}$ it is of course natural to consider the "observation period" $(1, 2, \ldots, n)$ or $(0, T]$ as the basic space. However the simple normalization by n or T allows use of the fixed space $(0, 1]$ which we employ here, defining r.a.f.'s for (semiclosed) subintervals of this space. For brevity by an *interval* we shall throughout mean specifically a semiclosed subinterval $(a, b]$ of the unit interval $(0, 1]$. Let $\{\zeta_T : T > 0\}$ be a family of r.a.f.'s, defined on such intervals and additive in the above sense, i.e. satisfying

$$\zeta_T(I \cup J) = \zeta_T(I) + \zeta_T(J) \text{ for each } T > 0,$$

whenever I, J are disjoint intervals, whose union $I \cup J$ is an interval (i.e. I and J abut). The domain of definition of r.a.f.'s may of course be extended by linearity to include finite unions of intervals, and the notation usage will reflect this where convenient.

This family of r.a.f.'s $\{\zeta_T : T > 0\}$ is assumed to satisfy a mixing condition Δ, which will be defined as follows: Write for $0 < r, l < 1$,

$$\Delta(r, l) = \sup |\mathcal{E} \exp\{it(\zeta_T(I_1) + \zeta_T(I_2))\} - \mathcal{E} \exp\{it\zeta_T(I_1)\} \, \mathcal{E} \exp\{it\zeta_T(I_2)\}|,$$

where the supremum is taken over pairs of disjoint intervals $I_1 = (a_1, b_1], I_2 = (a_2, b_2]$ satisfying $0 < a_1 < b_1 < a_2 < b_2 \leq 1$, $a_2 - b_1 > l$ and $b_2 - a_2 < r$.

Then $\{\zeta_T\}$ is said to be Δ-*mixing* if for each real t, $\Delta(r_T, l_T) \to 0$ for some $r_T = o(1)$ and $l_T = o(r_T)$, as $T \to \infty$. Note that $\Delta(r_T, l_T)$ depends on T and also on t.

This mixing condition has the same type of array form under which basic limiting theory for random additive functions is developed in Leadbetter and Rootzén (1993). However, it substantially weakens that in Leadbetter and Rootzén (1993) by considering only very special types of random variables, $\exp\{it\zeta_T(I)\}$ for intervals I, instead of all random variables which are measurable with respect to the σ-*field* $\mathcal{B}_I^T = \sigma\{\zeta_T(u, v) : u, v \in I\}$, or some substantial subclass thereof. The condition Δ allows consideration of the limiting distribution of $\zeta_T(I)$ for a single interval I. It will be extended in Section 5 to conditions Δ_p ($\Delta_1 = \Delta$) used in determining limiting joint distributions of $\zeta_T(I_1), \ldots, \zeta_T(I_p)$ for p (disjoint) intervals $I_1, I_2, \ldots, I_p$.

The following negligibility condition will be further assumed as needed, (using m for Lebesgue measure):

$$\sup\{P\{|\zeta_T(I)| > \epsilon\} : m(I) \leq l_T\} \to 0 \text{ as } T \to \infty, \text{ for each } \epsilon > 0,$$

which is readily shown to be equivalent to the condition

$$(1)\qquad \gamma_T = \sup\{1 - \mathcal{E}\exp(-|\zeta_T(I)|) : m(I) \leq l_T\} \to 0, \text{ as } T \to 0.$$

3 Asymptotic independence

In this section it will be shown under Δ-*mixing* that if an interval I is written as the union of appropriate disjoint abutting subintervals I_j, the characteristic function of $\zeta_T(I)$ is approximated by the product of those for the subintervals I_j. This substantially generalizes Lemma 2.1 in Leadbetter and Rootzén (1993). A simplified version of this lemma will also be shown for the positive and stationary case (e.g. where ζ_T is a random measure). These are key basic results leading to a classical central limit problem for $\zeta_T(I)$ for any fixed interval in $(0, 1]$. Extended conditions and results for joint distributions of $\zeta_T(I)$ for more than one interval I, are considered in Section 5.

For integers $k_T \to \infty$ as $T \to \infty$, a k_T-partition of the interval I will mean a partition of I into k_T disjoint subintervals I_j $(= I_{T,j})$ and (for convenience) $2l_T < m(I_j) \leq r_T$, $j = 1, \ldots, k_T$.

Lemma 3.1 *Let $\{\zeta_T,\ T > 0\}$ be a Δ-mixing family of r.a.f.'s for some constants $\{r_T\}$, $\{l_T\}$ and let an interval $I = (a, b]$ (which may depend on T), have a k_T-partition $\{I_j\}$ where*

$$(2)\qquad k_T(\Delta(r_T, l_T) + \gamma_T) \to 0, \text{ as } T \to \infty.$$

Then, uniformly in $|t| \leq M$, given $M < \infty$,

$$(3)\qquad \mathcal{E}\exp\{it\zeta_T(I)\} \ - \ \prod_{j=1}^{k_T} \mathcal{E}\exp\{it\zeta_T(I_j)\} \to 0, \text{ as } T \to \infty.$$

Proof Take $I_j = (a_{j-1},\ a_j], 1 \leq j \leq k_T$ for $a = a_0 < a_1 < \ldots < a_{k_T} = b$, without loss of generality. Write $I'_j{=}(a_{j-1}{+}l_T,\ a_j]$ and $I^*_j{=}I_j{-}I'_j$, $j = 1, ..., k_T$, and for simplicity suppress the subscript T in l_T, k_T, r_T.

Now, clearly since $\zeta_T(I) = \zeta_T(\cup_{j=1}^{k-1} I_j) + \zeta_T(I'_k) + \zeta_T(I^*_k)$,

$$|\mathcal{E}\exp\{it\zeta_T(I)\} - \mathcal{E}\exp\{it\zeta_T(\cup_{j=1}^{k-1} I_j) + \zeta_T(I'_k))\}| \ \leq \ \mathcal{E}|1 - \exp\{it\zeta_T(I^*_k)\}|,$$

and it follows from $\Delta(r, l)$ applied to the two intervals $\cup_{j=1}^{k-1} I_j, I'_k$ that

$$|\mathcal{E}\exp\{it(\zeta_T(\cup_{j=1}^{k-1} I_j) + \zeta_T(I'_k))\} - \mathcal{E}\exp\{it\zeta_T(\cup_{j=1}^{k-1} I_j)\}\ \mathcal{E}\exp\{it\zeta_T(I'_k)\}| \\ \leq \Delta(r,\ l).$$

Since $|\mathcal{E}\exp\{it\zeta_T(I_k)\} - \mathcal{E}\exp\{it\zeta_T(I_k')\}| \le \mathcal{E}|1-\exp\{it\zeta_T(I_k^*)\}|$, we obtain from this and above inequalities that

$$|\mathcal{E}\exp\{it\sum_{j=1}^{k}\zeta_T(I_j)\} \ - \ \mathcal{E}\exp\{it\sum_{j=1}^{k-1}\zeta_T(I_j)\}\ \mathcal{E}\exp\{it\zeta_T(I_k)\}| \le \ \Delta(r,l) + 2\mathcal{E}|1-\exp\{it\zeta_T(I_k^*)\}|.$$

Applying this repeatedly gives

$$|\mathcal{E}\exp\{it\sum_{j=1}^{k}\zeta_T(I_j)\} \ - \ \prod_{j=1}^{k}\mathcal{E}\exp\{it\zeta_T(I_j)\}| \le \ k\Delta(r,l) + 2\sum_{j=1}^{k}\mathcal{E}|1-\exp\{it\zeta_T(I_j^*)\}|.$$

Now the first term on the right tends to zero by (2) and since $|(1-e^{i\theta})/(1-e^{-|\theta|})| \ \le K$ for some $K>0$ and all real θ, the second term does not exceed $2K\sum_{j=1}^{k}\mathcal{E}|1-\exp\{-|t\zeta_T(I_j^*)|\}|$. This is clearly dominated by $K|t|k_T\gamma_T$ (with appropriately changed K) which tends to zero so that (3) follows. ∎

It will be further seen that more definitive results are obtainable under simple conditions if an r.a.f. ζ_T is assumed to be both positive and stationary, in the sense that $\zeta_T(I+h) \stackrel{d}{=} \zeta_T(I)$, for each h and interval I with $I,\ I+h \subset (0,1]$.

Note that for positive variables it is convenient to work with Laplace instead of Fourier transforms and hence it is natural to define a Δ-*mixing* coefficient with Laplace transforms. Specifically the same definition is used but $\mathcal{E}e^{-t\zeta_T(I)}$ replaces $\mathcal{E}e^{it\zeta_T(I)}$ for an interval I.

It is then possible to obtain the similar result to Lemma 3.1 without assuming the negligibility condition (1), using a k_T-partition which consists of intervals of presumably different lengths. However, it is more desirable to consider a "uniform " partition if the stationarity of an r.a.f. ζ_T is assumed. This yields a simple proof and it is sufficient to evaluate only one Laplace transform $\mathcal{E}e^{-t\zeta_T((0,r_T])}$ when approximating $\mathcal{E}e^{-t\zeta_T(I)}$.

Lemma 3.2 *Let the positive and stationary r.a.f. family $\{\zeta_T\}$ be Δ-mixing (defined with Laplace transforms) for some constants $\{l_T\}$, $\{r_T\}$ where $r_T^{-1}\Delta(r_T,l_T) \to 0$, as $T\to\infty$. Let I be a subinterval of $(0,1]$, which may depend on T, but with $k_T = [m(I)/r_T] \to \infty$. Then without assuming the negligibility condition (1), as $T\to\infty$, for any $t\ge 0$,*

$$\mathcal{E}\exp\{-t\zeta_T(I)\} \ - \ (\mathcal{E}\exp\{-t\zeta_T((0,r_T])\})^{k_T} \to 0. \tag{4}$$

Hence also

$$\mathcal{E}\exp\{-t\zeta_T(I)\} \ - \ (\mathcal{E}\exp\{-t\zeta_T((0,1])\})^{m(I)} \to 0 \ as \ T\to\infty. \tag{5}$$

Proof Again for simplicity we suppress the subscript T in l_T, k_T, r_T and take $I = (0, a]$ without loss of generality, since ζ_T is stationary. Write I_j for the interval $((j-1)r, jr]$, $j = 1, 2, \ldots$. First of all it is shown that (4) holds for the interval $I = (0, a]$ with $kr = a$. It is sufficient to show (4) as $T \to \infty$ through any sequence such that $(\mathcal{E}\exp\{-t\zeta_T(I_1^*)\})^k$ converges to some ρ, $0 \le \rho \le 1$. Consider separately the following two possibilities:

(i) $\rho = 1$. Following the same steps as in Lemma 3.1, we obtain

$$\begin{aligned}
|\mathcal{E}\exp\{-t\sum_{j=1}^{k}\zeta_T(I_j)\} - \prod_{j=1}^{k}\mathcal{E}\exp\{-t\zeta_T(I_j)\}| \\
&\le k\Delta(r,l) + 2\sum_{j=1}^{k}(1 - \mathcal{E}\exp\{-t\zeta_T(I_j^*)\}) \\
&= k\Delta(r,l) + 2k(1 - \mathcal{E}\exp\{-t\zeta_T(I_1^*)\}),
\end{aligned}$$

since the I_j^* all have the same length l and ζ_T is stationary.

Since $\rho = 1$, it follows that $k\log\mathcal{E}\exp\{-t\zeta_T(I_1^*)\} \to 0$, so that $k(1 - \mathcal{E}\exp\{-t\zeta_T(I_1^*)\}) \to 0$. Thus the right hand side of the above inequality tends to 0 as $T \to \infty$ and hence (4) holds.

(ii) $\rho < 1$. It is possible to choose $\theta = \theta_T \to \infty$ such that $k\theta\Delta(r,l) \to 0$ and $\theta l = o(r)$ since $k\Delta(r,l) \to 0$ and $l = o(r)$. Hence for sufficiently large T, $\theta + 1$ intervals $J_1, J_2, \ldots, J_{\theta+1}$ congruent to I_1^* may be chosen in I_1', all mutually separated by at least l. Let J_m^* be the interval separating J_m and J_{m+1}, $1 \le m \le \theta$.

Since $\zeta_T(I_1') \ge \sum_{m=1}^{\theta}\{\zeta_T(J_m) + \zeta_T(J_m^*)\} + \zeta_T(J_{\theta+1})$ and ζ_T is positive,

$$\begin{aligned}
\mathcal{E}\exp\{-t\zeta_T(I_1)\} &\le \mathcal{E}\exp\{-t\zeta_T(I_1')\} \\
&\le \mathcal{E}\exp\{-t[\sum_{m=1}^{\theta-1}(\zeta_T(J_m) + \zeta_T(J_m^*)) + \zeta_T(J_\theta) + \zeta_T(J_{\theta+1})]\} \\
&\le \mathcal{E}\exp\{-t[\sum_{m=1}^{\theta-1}(\zeta_T(J_m) + \zeta_T(J_m^*)) + \zeta_T(J_\theta)]\}\ \mathcal{E}\exp\{-t\zeta_T(J_{\theta+1})\} + \Delta
\end{aligned}$$

by the mixing condition. This latter expression is equal to

$$\mathcal{E}\exp\{-t[\sum_{m=1}^{\theta-1}(\zeta_T(J_m) + \zeta_T(J_m^*)) + \zeta_T(J_\theta)]\}\ \mathcal{E}\exp\{-t\zeta_T(I_1^*)\} + \Delta$$

by the assumed stationarity. Applying this repeatedly gives

$$\mathcal{E}\exp\{-t\zeta_T(I_1)\} \le (\mathcal{E}\exp\{-t\zeta_T(I_1^*)\})^\theta + \theta\Delta. \tag{6}$$

so that

$$(7) \qquad \begin{aligned}(\mathcal{E}\exp\{-t\zeta_T(I_1)\})^k &\le (\mathcal{E}\exp\{-t\zeta_T(I_1^*)\})^{k\theta} + k\theta\Delta \\ &= (\rho + o(1))^\theta + o(1) \to 0,\end{aligned}$$

since $\rho < 1$. Hence the second term in the difference in (4) tends to zero. It may be similarly shown that

$$\mathcal{E}\exp\{-t\zeta_T(I)\} \le (\mathcal{E}\exp\{-t\zeta_T(I_j')\})^k + k\Delta$$

which tends to zero since (6) and hence (7) hold with I_j' in place of I_j. Hence both terms of (4) tend to zero if $\rho < 1$, and (4) again holds for the case $I = (0, a]$ satisfying $kr = a$.

Note that this result implies that $(\mathcal{E}\exp\{-t\zeta_T(I_1)\})^k \ge \delta$ for some $\delta > 0$ and hence that $\mathcal{E}\exp\{-t\zeta_T(I_1)\}) \to 1$, i.e. $\zeta_T(I_1) \xrightarrow{P} 0$.

Now to prove (4) for the interval $I = (0, a]$ with $kr \neq a$ it is sufficient to show that $\mathcal{E}\exp\{-t\zeta_T((0, kr])\} - \mathcal{E}\exp\{-t\zeta_T(I)\} \to 0$. But this difference does not exceed $1 - \mathcal{E}\exp\{-t\zeta_T((kr, a])\}$ which tends to zero since as noted $\zeta_T(I_1) \xrightarrow{P} 0$ and hence $\zeta_T((kr, a]) \xrightarrow{P} 0$. ($m((kr, a]) < m(I_1) = r$ and ζ_T is positive and stationary.) Hence (4) holds.

Since $k \sim m(I)/r$, it is readily seen that

$$(\mathcal{E}\exp\{-t\zeta_T(I_1)\})^k - (\mathcal{E}\exp\{-t\zeta_T(I_1)\})^{m(I)/r} \to 0.$$

Hence it follows from this and (4) that

$$(8) \qquad (\mathcal{E}\exp\{-t\zeta_T(I)\}) - (\mathcal{E}\exp\{-t\zeta_T(I_1)\})^{m(I)/r} \to 0.$$

Then (5) is readily obtained by applying (8) to the unit interval $(0, 1]$ and I. ∎

4 Limiting distributions

The results of Section 3 show (partial) asymptotic independence of $\zeta_T(I_j)$ for a *k_T-partition* of an interval I. These will now be used to show that classical central limit theory is obtained under *Δ-mixing* by considering an independent array with the same marginal distributions as $\zeta_T(I_j)$.

As for independent r.v.'s, the array $\{\zeta_T(I_j)\}$ corresponding to a *k_T-partition* $\cup_{j=1}^{k_T} I_j = I$ of an interval I, will be termed *uniformly asymptotically negligible (uan)* if $\zeta_T(I_j) \xrightarrow{P} 0$ uniformly in j, i.e. for every $\epsilon > 0$

$$\max_{1 \le j \le k_T} \{P\{|\zeta_T(I_j)| > \epsilon\}\} \to 0 \text{ as } T \to \infty.$$

For each T let $\{\zeta_{T,j} : 1 \le j \le k_T\}$ be independent random variables with $\zeta_{T,j} \stackrel{\mathrm{d}}{=} \zeta_T(I_j), 1 \le j \le k_T$. Such a family will be called an *independent array (of size k_T) associated with $\zeta_T(I)$.*

Note that such a partition and independent array of course are not unique. However, as can be seen, the following result is independent of the choice of array and immediately obtained from Lemma 3.1.

Theorem 4.1 *Let $\{\zeta_T\}$ be Δ-mixing with $k_T(\Delta(r_T, l_T) + \gamma_T) \to 0$ and let $\{\zeta_{T,j}\}$ an independent array for $\{\zeta_T(I)\}$ based on a k_T-partition $\{I_j\}$ of an interval I. Then $\zeta_T(I)$ has the same limiting distribution (if any) as $\sum \zeta_{T,j}$. In particular if the array $\{\zeta_T(I_j)\}$ is uan, any limit is infinitely divisible.*

Proof Since $\zeta_T(I) = \sum_{j=1}^k \zeta_T(I_j)$ and by Lemma 3.1, for each t, as $T \to \infty$,

$$\begin{aligned}\mathcal{E}\exp\{it\zeta_T(I)\} &= \mathcal{E}\exp\{it\sum_{j=1}^{k}\zeta_T(I_j)\} = \prod_{j=1}^{k}\mathcal{E}\exp\{it\zeta_T(I_j)\} + o(1) \\ &= \prod_{j=1}^{k}\mathcal{E}\exp\{it\zeta_{T,j}\} + o(1) = \mathcal{E}\exp\{it\sum_{j=1}^{k}\zeta_{T,j}\} + o(1).\end{aligned}$$

Thus $\zeta_T(I) \xrightarrow{d} \eta$ for some r.v. η if and only if $\sum \zeta_{T,j} \xrightarrow{d} \eta$. ∎

General features of classical central limit theory apply under Δ-mixing to important cases such as normal and Compound Poisson convergence as follows. In these, $F_{T,j}$ will denote the distribution function of the contribution $\zeta_T(I_j)$ of the interval I_j in a k_T-partition of I and

$$\begin{aligned}\alpha_{T,j}(\tau) &= \int_{|x|<\tau} x\,dF_{T,j} \\ \sigma^2_{T,j}(\tau) &= \int_{|x|<\tau} x^2\,dF_{T,j} - \alpha^2_{T,j}(\tau)\end{aligned}$$

(1) Normal convergence. The general criterion (adapted from the form given by Loève (1977), Section 23 under independence) may be stated:

Let ζ_T be as in Theorem 4.1, and I a given interval. Then $\zeta_T(I) \xrightarrow{d} \eta$ for a normal r.v. $\eta = N(\alpha, \sigma^2)$, and $\{\zeta_T(I_j)\}$ is uan if and only if

(i) $\sum_j P\{|\zeta_T(I_j)| \ge \epsilon\} \to 0$ as $T \to 0$, each $\epsilon > 0$,

(ii) $\sum_j \alpha_{T,j}(\tau) \to \alpha$, $\sum \sigma^2_{T,j}(\tau) \to \sigma^2$ as $T \to \infty$, some $\tau > 0$.

It is interesting and potentially useful in applications to note that if it is known that $\zeta_T(I)$ converges in distribution to some r.v. η (not assumed normal) then normality of η actually follows from (i) alone and (ii) is automatically satisfied. This may be seen from the discussion of normal convergence in Loève (1977), Section 23.5 and the "Central Convergence Criterion" of Section 23.4 of that reference.

(2) Compound Poisson convergence. Let $X = CP(\lambda, F)$ denote a Compound Poisson random variable X i.e. $X = \sum_1^N X_i$ where X_i are independent with a distribution function F and N is Poisson with mean λ. Then with the above notation, $\zeta_T(I) \xrightarrow{d} CP(\lambda, F)$ if the following conditions hold:

(i) $\sum_j F_{T,j}(x) \to \lambda F(x)$, $x < 0$

$\sum_j (1 - F_{T,j}(x)) \to \lambda(1 - F(x))$, $x > 0$, at continuity points x of F

(ii) $\sum_j \alpha_{T,j}(\tau) \to \lambda \int_{(|x|<\tau)} x \, dF(x)$ for some fixed $\tau > 0$

(iii) $\limsup_{T\to\infty} \sum_j \sigma_{T,j}^2(\tau) \to 0$ as $\epsilon \to 0$.

Suppose now that $\{\zeta_T\}$ is a family of positive and stationary random additive functions. Lemma 3.2 gives the following more definitive result (without negligibility conditions).

Theorem 4.2 *Let the positive and stationary r.a.f. family $\{\zeta_T\}$ be Δ-mixing (defined with Laplace transforms). If $\zeta_T(I) \xrightarrow{d} \eta_I$, a random variable for some (nondegenerate) interval I, then such convergence occurs for all intervals I and η_I is infinitely divisible with Laplace transform $\mathcal{E}\exp(-t\eta_I) = \phi(t)^{m(I)}$ where ϕ is the Laplace transform of $\eta_{(0,1]}$.*

5 Multivariate limits

It follows simply from the above results that if $I_1, I_2, \ldots, I_p$ are disjoint abutting intervals, then

$$\mathcal{E}\exp\{it\sum_{j=1}^{p} \zeta_T(I_j)\} - \prod_{j=1}^{p} \mathcal{E}\exp\{it\zeta_T(I_j)\} \to 0$$

under Δ-*mixing* and negligibility assumptions. Hence if $\zeta_T(I_j) \xrightarrow{d} \eta_{I_j}$ a r.v. for each j then

$$\sum_{j=1}^{p} \zeta_T(I_j) \xrightarrow{d} \sum_{j=1}^{p} \eta_{I_j} \text{ as } T \to \infty,$$

and the $\{\eta_{I_j}\}$ may be taken to be independent. However convergence of this sum does not necessarily imply joint convergence in distribution of the components $(\zeta_T(I_1), \ldots, \zeta_T(I_p))$ to $(\eta_{I_1}, \ldots, \eta_{I_p})$, which requires the more general relation

$$(9) \qquad \mathcal{E} \exp\{i \sum_{j=1}^{p} t_j \zeta_T(I_j)\} \; - \; \prod_{j=1}^{p} \mathcal{E} \exp\{it_j \zeta_T(I_j)\} \to 0.$$

This latter convergence requires a more detailed version of the Δ-*mixing* condition, which may be tailored to the number of intervals I_j involved. Specifically, for each $p \geq 1$, $\{\zeta_T\}$ is said to be Δ_p-*mixing* if there exist some constants $r_T = o(1)$ and $l_T = o(r_T)$ for each real $t_1, \ldots, t_p$, such that

$$\begin{aligned}\Delta_p(r_T, l_T) \; &= \; \sup\{|\mathcal{E} \exp\{\sum_{j=1}^{p} it_j \zeta_T(I_j) + it_p \zeta_T(I_{p+1})\} \\ &\quad - \mathcal{E}\exp\{\sum_{j=1}^{p} it_j \zeta_T(I_j)\} \; \mathcal{E}\exp\{it_p \zeta_T(I_{p+1})\}| \to 0 \text{ as } T \to \infty,\end{aligned}$$

where the supremum is taken over any $(p+1)$ disjoint intervals $I_j = (a_j, b_j]$, $j = 1, ..., (p+1)$ with $0 < a_1 < b_1 \leq a_2 < \ldots \leq 1 \;\; a_{p+1} - b_p > l$ and $b_{p+1} - a_{p+1} < r$.

Remark 5.1 Note that Δ_1 is the previous Δ-condition and putting selected $t_j = 0$ shows that Δ_p-mixing implies Δ_m-mixing for $1 \leq m \leq p$.

The following result is then readily shown along now familiar lines.

Theorem 5.1 *Let the r.a.f. family $\{\zeta_T\}$ satisfy the condition Δ_p for some $\{r_T\}, \{l_T\}$. Then (9) holds for given disjoint intervals $I_1, I_2, \ldots, I_p$ and any t_j which are uniformly bounded in j and T, if the interval I_j has a $k_{T,j}$-partition $j = 1, ..., p$ such that $\sum_{j=1}^{p} k_{T,j} = k_T$ and*

$$(10) \qquad k_T(\Delta_p(r_T, l_T) + \gamma_T) \to 0 \text{ as } T \to \infty.$$

Moreover, if $\zeta_T(I_j) \xrightarrow{d} \eta_{I_j}$ a r.v. for each j, then $(\zeta_T(I_1), \ldots, \zeta_T(I_p)) \xrightarrow{d} (\eta_{I_1}, \ldots, \eta_{I_p})$ where η_{I_j} are independent.

Proof It will be convenient to write a partition of I_j as $U_{j,1}, U_{j,2}, \ldots, U_{j,k_j}$ and define $U'_{j,m}$ and $U^*_{j,m}$ for $j = 1, \ldots, p$ and $m = 1, \ldots, k_j$ as I'_j and I^*_j were defined in Lemma 3.1, where $\sum_{j=1}^{p} k_j = k_T$. Again for simplicity suppress the subscript T in k_T, l_T, r_T.

We readily obtain from Lemma 3.1 and Remark 5.1 that for $j = 1, \dots, p$

$$|\mathcal{E}\exp\{it_j\zeta_T(I_j)\} \; - \; \prod_{m=1}^{k_j}\mathcal{E}\exp\{it_j\zeta_T(U_{j,m})\}| \\ \leq \; k_j\Delta_p(r,l) + 2\sum_{m=1}^{k_j}\mathcal{E}|1-\exp\{it_j\zeta_T(U^*_{j,m})\}|.$$

Hence, using the inequality

$$|\prod_1^n x_i - \prod_1^n y_i| \leq \sum_1^n |x_i - y_i|, \quad 0 \leq |x_i|, \; |y_i| \leq 1,$$

it follows that

$$|\prod_{j=1}^{p}\mathcal{E}\exp\{it_j\zeta_T(I_j)\} \; - \; \prod_{j=1}^{p}\prod_{m=1}^{k_j}\mathcal{E}\exp\{it_j\zeta_T(U_{j,m})\}|$$

$$\leq \; \sum_{j=1}^{p}k_j\Delta_p(r,l) + 2\sum_{j=1}^{p}\sum_{m=1}^{k_j}\mathcal{E}|1-\exp\{it_j\zeta_T(U^*_{j,m})\}| \tag{11}$$

Again, the same reasoning as in the proof of Lemma 3.1 and the definition of $\Delta_p(r,l)$ yields

$$|\mathcal{E}\exp\{i\sum_{j=1}^{p}t_j\zeta_T(I_j)\} \; - \; \prod_{j=1}^{p}\prod_{m=1}^{k_j}\mathcal{E}\exp\{it_j\zeta_T(U_{j,m})\}| \\ \leq \; \sum_{j=1}^{p}k_j\Delta_p(r,l) + 2\sum_{j=1}^{p}\sum_{m=1}^{k_j}\mathcal{E}|1-\exp\{it_j\zeta_T(U^*_{j,m})\}|.$$

It follows from this and (11) that

$$|\mathcal{E}\exp\{i\sum_{j=1}^{p}t_j\zeta_T(I_j)\} \; - \; \prod_{j=1}^{p}\mathcal{E}\exp\{it_j\zeta_T(I_j)\}| \\ \leq \; 2\sum_{j=1}^{p}k_j\Delta_p(r,l) + 4\sum_{j=1}^{p}\sum_{m=1}^{k_j}\mathcal{E}|1-\exp\{it_j\zeta_T(U^*_{j,m})\}| \\ = \; 2k\Delta_p(r,l) + 4\sum_{j=1}^{p}\sum_{m=1}^{k_j}\mathcal{E}|1-\exp\{it_j\zeta_T(U^*_{j,m})\}|$$

Note that the second term on the right does not exceed

$$K\sum_{j=1}^{p}\sum_{m=1}^{k_j}\mathcal{E}|1-\exp\{-|t_j\zeta_T(U^*_{j,m})|\}| \; \leq \; K\sum_{j=1}^{p}|t_j|\gamma_T \; \leq \; K'k_T\gamma_T \to 0,$$

for some constants K, K' and the first term tends to zero by (10). Thus (9) holds.

The conclusion regarding joint convergence of $(\zeta_T(I_1), \dots, \zeta_T(I_p))$ follows immediately from (9). ∎

This result thus shows the independence of the distributional limits of $\zeta_T(I_1), \dots, \zeta_T(I_p)$ for disjoint I_j under Δ_p-mixing and negligibility assumptions. Further the component limits will be infinitely divisible under uan assumptions for arrays corresponding to k_T-partitions for each term (Theorem 4.1).

Finally we note that again corresponding results hold in the positive case without negligibility assumptions. These lead in particular to "full" convergence theorems for Δ-mixing random measures $\{\zeta_T\}$ to random measures with independent increments.

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