A STATISTICAL APPROACH TO THE CAUCHY PROBLEM FOR THE LAPLACE EQUATION

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We study the problem of estimating an unknown solution of the Cauchy problem for the Laplace equation, with L_2-norm loss, when the initial conditions are observed in a white Gaussian noise with a small spectral density. It is shown in particular that asymptotically minimax estimators are as a rule nonlinear.

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1 Introduction

Hadamard (1912) proposed the famous example of the ill-posed boundary value problem

$$\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = 0, \quad u(x,0) = 0, \quad u'_y(x,0) = \varphi(x). \tag{1}$$

He noticed that $\sup_x |\varphi_n(x)| \to 0$, as $n \to \infty$ for $\varphi_n(x) = n^{-1}\sin(2\pi n x)$ whereas *sup*-norm of the solution $u_n(x,y) = \sinh(2\pi n y)\sin(2\pi n x)/(2\pi n^2)$ tends to infinity for any $y > 0$. So the problem is called ill-posed in the Hadamard sense. Nevertheless this problem is the important geophysical problem of interpreting the gravitational or magnetic anomalies (see Lavrentiev (1967) and Tarchanov (1995)).

The usual approach to ill-posed problems deals with the recovery of a solution based on a "noisy" data. In order to guarantee consistent recovering some additional information about the function $\varphi(x)$ is required. It is assumed as a rule that φ belongs to a compact set $\mathcal{K}$ in a suitable space. The performance of the optimal solution depends on $\mathcal{K}$ and on the definition of the noisy data. Usually (see Tichonov & Arsenin (1977), Engl & Groetsch (1987)) it is assumed that the observed data are

$$\varphi^\varepsilon(x) = \varphi(x) + n^\varepsilon(x),$$

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where $n^\varepsilon(x)$ is an unknown function from a Hilbert space with the norm $\|n^\varepsilon\| \le \varepsilon$.

On the other hand there exists another approach to ill-posed problems, which is based on the assumption that $n^\varepsilon(x)$ is a random process (see Sudakov & Khalfin (1964), Mair & Ruimgaart (1996), Sullivan (1996)). In the present paper we assume that $n^\varepsilon(x)$ is a white Gaussian noise with spectral density ε^2. Of course, $n^\varepsilon(x)$ does not belong to a Hilbert space and usually solutions of ill-posed problems obtained by deterministic and stochastic approaches are different. To illustrate, consider a simple ill-posed problem. Suppose we need to estimate the first derivative of a function $f(x)$, $x \in [0,1]$ based on the noisy data

$$z^\varepsilon(t) = f(x) + n^\varepsilon(x).$$

Assume also that f belongs to the Sobolev ball

$$\mathbf{W}_2^\beta(P) = \left\{ f : \int_0^1 f^2(u)\,du + \int_0^1 [f^{(\beta)}(u)]^2 du \le P \right\},$$

where $f^{(\beta)}(u)$ denotes the derivative of the order $\beta > 1$. It is not very difficult to show that as $\varepsilon \to 0$ the minimax rates of convergence are given by

$$\inf_{\widehat{f'}} \sup_{f \in \mathbf{W}_2^\beta(P)} \sup_{\|n^\varepsilon\| \le \varepsilon} \|f' - \widehat{f'}\|^2 \asymp \varepsilon^{2(\beta-1)/\beta},$$

and

$$\inf_{\widehat{f'}} \sup_{f \in \mathbf{W}_2^\beta(P)} \mathrm{E}_f \|f' - \widehat{f'}\|^2 \asymp \varepsilon^{4(\beta-1)/(2\beta+1)}$$

for the deterministic and stochastic approaches respectively.

Let us now return to the Cauchy problem (1), where as we will see later the minimax rates of convergence are the same for the deterministic and statistical approaches. Let $\varphi(x)$ be a periodic function with unit period. Assume that it admits an analytic continuation into a strip of length $L > y$ of the complex plane. Then one can easily check that

$$u(x,y) = \sum_{k=-\infty}^{\infty} \frac{\sinh(2\pi k y)}{2\pi k} \varphi_k e^{2\pi i k x}, \quad \varphi_k = \int_0^1 \varphi(x) e^{-2\pi i k x}\, dx \tag{2}$$

is a solution of (1). The problem is to recover a solution of (1) in the stochastic setting. We have at our disposal only the noisy data

$$dz^\varepsilon(x) = \varphi(x)dx + \varepsilon dw(x), \quad x \in [0,1], \tag{3}$$

where $w(x)$ is the standard Wiener process. Our goal is to estimate the function $u(x,y)$ based on these data. For a given y the risk of the estimator $\widehat{u}(x,y)$ is measured in the $\mathbf{L}_2$-norm, so

$$R_\varphi(\widehat{u},u) = \mathrm{E}_\varphi \int_0^1 [\widehat{u}(x,y) - u(x,y)]^2\,dx.$$

Notice that the data $z^\varepsilon(x)$ admit an equivalent representation in terms of their Fourier coefficients (cf. (3)), given by

$$z_k^\varepsilon = \varphi_k + \varepsilon\xi_k, \tag{4}$$

where z_k^ε are the Fourier coefficients of the data $z^\varepsilon(x)$

$$z_k^\varepsilon = \int_0^1 \mathrm{e}^{2\pi \mathrm{i} kx}\,dz^\varepsilon(x),$$

and ξ_k are iid $\mathcal{N}(0,1)$. Thus (2) determines a one-to-one correspondence between the estimator $\widehat{u}(x)$ based on $z^\varepsilon(t)$, $t \in [0,1]$ and the estimator $\widehat{\varphi}_k$ based on the data from (4). By the Parseval formula we can rewrite the risk $R_\varphi(\widehat{u},u)$ in the following equivalent, but more convenient form

$$R_\varphi(\widehat{u},u) = \mathrm{E}_\varphi \sum_{k=-\infty}^{\infty} b_k^2(y)|\widehat{\varphi}_k - \varphi_k|^2, \tag{5}$$

where $b_k(y) = \sinh(2\pi ky)/(2\pi k)$.

In order to get nontrivial results we suppose that $\varphi(x)$ admits an analytic continuation into a strip of length L of the complex plane. More precisely, we assume that $\varphi \in \Phi$, where

$$\Phi = \left\{\varphi(x) : \int_0^1 |\varphi(x+\mathrm{i}L)|^2\,dx = \sum_{k=-\infty}^{\infty} |\varphi_k|^2\cosh(4\pi kL) \le D\right\}. \tag{6}$$

It is easy to see from (2) that if $\varphi \in \Phi$, a solution of the problem (1) exists provided that $L > y$.

In order to find the rate of convergence of the risk to 0 we begin with the projection estimators (cf. (2))

$$\widehat{u}_N(x,y,z^\varepsilon) = \sum_{|k|<N} b_k(y) z_k^\varepsilon \mathrm{e}^{2\pi \mathrm{i} kx}.$$

The main problem related to this estimator is how to choose $N = N(\varepsilon)$ to minimize its risk. In view of (5)–(6), one easily computes the risk of

$\widehat{u}_N(x,y,z^\varepsilon)$ for $y>0$ as $\varepsilon\to 0$

$$(7)\qquad \begin{aligned}\sup_{\varphi\in\Phi} R_\varphi(\widehat{u}_N,u) &= \sup_{\varphi\in\Phi}\sum_{|k|\geq N} b_k^2(y)|\varphi_k|^2+\varepsilon^2\sum_{|k|<N} b_k^2(y)\\ &= Db_N^2(y)\cosh^{-1}(4\pi LN)+\varepsilon^2\sum_{|k|<N} b_k^2(y)\\ &= (1+o(1))\left[\frac{D}{8(\pi N)^2}\exp(4\pi(y-L)N)+2\varepsilon^2\sum_{k=0}^{N-1} b_k^2(y)\right]\\ &= (1+o(1))\frac{2D}{(4\pi N)^2}\left[\exp(4\pi(y-L)N)+\frac{\varepsilon^2\exp(4\pi yN)}{D(\exp(4\pi y)-1)}\right].\end{aligned}$$

Minimizing the right-hand side in the above equation with respect to N we get the optimal bandwidth

$$N(\varepsilon)=\arg\min_k\left\{\exp(4\pi(y-L)k)+\frac{\varepsilon^2\exp(4\pi yk)}{D(\exp(4\pi y)-1)}\right\},$$

where the minimum is taken over all integers. It follows from the above formula that

$$N(\varepsilon)=\frac{1}{4\pi L}\log\frac{D(L-y)(\exp(4\pi y)-1)}{\varepsilon^2 y}+c,$$

where $|c|\leq 0.5$. Substituting $N(\varepsilon)$ in (7) one arrives at the following formula for the risk of the projection estimator as $\varepsilon\to 0$:

$$\begin{aligned}\inf_N\sup_{\varphi\in\Phi} R_\varphi(\widehat{u}_N,u) &= (2+o(1))D\left[1+\frac{\varepsilon^2\exp(4\pi LN(\varepsilon))}{D(\exp(4\pi y)-1)}\right]e^{4\pi(y-L)N(\varepsilon)}\\ &\times\log^{-2}\frac{D(L-y)(\exp(4\pi y)-1)}{\varepsilon^2 y}\asymp D(\varepsilon^2/D)^{(1-y/L)}\log^{-2}(D/\varepsilon^2).\end{aligned}$$

We prefer to write D/ε^2 because it is dimensionless expression, and it is easy to see that the above equation is uniform in D such that $D/\varepsilon^2\to\infty$. It is not difficult also to show that the above rate of convergence cannot be improved in the class of all estimators. But the goal of the present paper is more delicate. We try to find the asymptotic minimax risk up to a constant. We will see that in the general case the asymptotically minimax estimator is nonlinear and only in some special cases one can use a linear or a projection estimators to achieve the optimal constant.

2 Main results

In this section we compute up to a constant the minimax risk

$$r_\varepsilon(y,L)=\inf_{\widehat{u}}\sup_{\varphi\in\Phi} R_\varphi(\widehat{u},u)=\inf_{\widehat{\varphi}}\sup_{\varphi\in\Phi}\mathrm{E}_\varphi\sum_{k=-\infty}^{\infty} b_k^2(y)|\widehat{\varphi}_k-\varphi_k|^2,$$

where *inf* is taken over all estimators. In order to describe the asymptotic behavior ($\varepsilon \to 0$) of this risk we consider an auxiliary statistical problem.

For convenience denote vectors from $\mathbf{R}^2$ by bold letters. Assume that one observes

$$\mathbf{z}_k = \boldsymbol{\theta}_k + \sigma\boldsymbol{\xi}_k, \tag{8}$$

where $\boldsymbol{\xi}_k$ are iid $\mathcal{N}(0, I)$ and I is the identity matrix. We also suppose that the unknown vectors $\boldsymbol{\theta}_k$ belong to the set

$$\Theta = \left\{ \boldsymbol{\theta}_k : \sum_{k=-\infty}^{\infty} \|\boldsymbol{\theta}_k\|^2 \mathrm{e}^{4\pi L k} \le 1 \right\}, \tag{9}$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbf{R}^2$. The goal is to find the best in the minimax sense estimator of $\boldsymbol{\theta}_k$, $k \in (-\infty, \infty)$ based on the data $\mathbf{z}_k$, when the risk is defined for $L > y$ as

$$\rho(\sigma, y, L) = \inf_{\widehat{\boldsymbol{\theta}}} \sup_{\boldsymbol{\theta} \in \Theta} \mathrm{E}_{\boldsymbol{\theta}} \sum_{k=-\infty}^{\infty} \|\widehat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k\|^2 \mathrm{e}^{4\pi y k}. \tag{10}$$

Let $\boldsymbol{\theta}^* = \{\dots, \boldsymbol{\theta}^*_{k-1}, \boldsymbol{\theta}^*_k, \boldsymbol{\theta}^*_{k+1}, \dots\}$ be the minimax estimator in (8)–(10) with the components

$$\theta^*_{1k} = \theta^*_{1k}\left(\dots, \begin{matrix} z_{1m} \\ z_{2m} \end{matrix}, \dots\right), \quad \theta^*_{2k} = \theta^*_{2k}\left(\dots, \begin{matrix} z_{1m} \\ z_{2m} \end{matrix}, \dots\right).$$

We construct now a counterpart φ^* of $\boldsymbol{\theta}^*$ in the model (4). Let

$$W(\varepsilon) = \arg\min_k \left| 1 - \frac{\varepsilon}{\sqrt{2D}} \mathrm{e}^{2\pi L k} \right|, \tag{11}$$

where the minimum is taken over all integers. Denote for brevity

$$A(\varepsilon) = (2D)^{-1/2} \mathrm{e}^{2\pi L W(\varepsilon)}.$$

For $|k| \le \sqrt{W(\varepsilon)}$ we set

$$\begin{aligned} \varphi^*_{k+W(\varepsilon)} &= \frac{1}{A(\varepsilon)} \theta^*_{1k}\left(\dots, \begin{matrix} A(\varepsilon) z^\varepsilon_{m+W(\varepsilon)} \\ A(\varepsilon) z^\varepsilon_{-m-W(\varepsilon)} \end{matrix}, \dots\right), \\ \varphi^*_{-k-W(\varepsilon)} &= \frac{1}{A(\varepsilon)} \theta^*_{2k}\left(\dots, \begin{matrix} A(\varepsilon) z^\varepsilon_{m+W(\varepsilon)} \\ A(\varepsilon) z^\varepsilon_{-m-W(\varepsilon)} \end{matrix}, \dots\right). \end{aligned} \tag{12}$$

We continue this estimator over the set of all integers as follows

$$\varphi^*_k = \begin{cases} 0, & |k| \ge W(\varepsilon) + \sqrt{W(\varepsilon)}, \\ z^\varepsilon_k, & |k| < W(\varepsilon) - \sqrt{W(\varepsilon)}. \end{cases} \tag{13}$$

The following theorem determines relations between the estimators φ^*, $\boldsymbol{\theta}^*$ and the minimax risk $r_\varepsilon(y, L)$.

Theorem 2.1 *For $0 < y < L$ as $\varepsilon \to 0$*

$$\begin{aligned} r_\varepsilon(y,L) &= (1+o(1)) \sup_{\varphi\in\Phi} \mathrm{E}_\varphi \sum_{k=-\infty}^{\infty} b_k^2(y)|\varphi_k^* - \varphi_k|^2 \\ &= \frac{(1+o(1))D}{8[\pi W(\varepsilon)]^2} \mathrm{e}^{4\pi(y-L)W(\varepsilon)} \rho(\sigma_\varepsilon, y, L), \end{aligned}$$

where

(14) $$\sigma_\varepsilon = \varepsilon D^{-1/2} \exp(2\pi L W(\varepsilon)).$$

Proof The proof is based on a modification of renormalization arguments (see, for instance, Donoho and Low (1992)). It is divided into two steps. First, we obtain a lower bound for the minimax risk. For brevity we use from now on W instead of $W(\varepsilon)$ and $\sqrt{W}$ instead of $\sqrt{W(\varepsilon)}$. Let

(15) $$\Phi_\varepsilon = \Phi \cap \left\{\varphi_k : \varphi_k = 0,\ |k-W| > \sqrt{W},\ |k+W| > \sqrt{W}\right\}.$$

Since $\Phi_\varepsilon \subseteq \Phi$, it follows that

(16) $$\begin{aligned} r_\varepsilon(y,L) &\geq \inf_{\widehat{\varphi}} \sup_{\varphi\in\Phi_\varepsilon} \mathrm{E}_\varphi \sum_{k=-\infty}^{\infty} b_k^2(y)|\widehat{\varphi}_k - \varphi_k|^2 \\ &= \frac{1+o(1)}{16(\pi W)^2} \inf_{\widehat{\varphi}} \sup_{\varphi\in\Phi_\varepsilon} \mathrm{E} \sum_{|k-W|\leq\sqrt{W}} \mathrm{e}^{4\pi y|k|}\Big(|\widehat{\varphi}_k - \varphi_k|^2 \\ &\quad + |\widehat{\varphi}_{-k} - \varphi_{-k}|^2\Big) \\ &= \frac{1+o(1)}{16(\pi W)^2} \mathrm{e}^{4\pi y W} \inf_{\widehat{\varphi}} \sup_{\varphi\in\Phi_\varepsilon} \mathrm{E} \sum_{|l|\leq\sqrt{W}} \mathrm{e}^{4\pi y l}\Big(|\widehat{\varphi}_{l+W} - \varphi_{l+W}|^2 \\ &\quad + |\widehat{\varphi}_{-l-W} - \varphi_{-l-W}|^2\Big). \end{aligned}$$

Define the vectors $\boldsymbol{\theta}_l \in \mathbf{R}^2$ as follows

(17) $$\boldsymbol{\theta}_l = A(\varepsilon)(\varphi_{l+W},\ \varphi_{-l-W})^T.$$

In view of (15) and (17), it follows that $(\ldots, \boldsymbol{\theta}_{l-1}, \boldsymbol{\theta}_l, \boldsymbol{\theta}_{l+1}, \ldots)$ belongs to the set

$$\boldsymbol{\Theta}_\varepsilon = \left\{(\ldots, \boldsymbol{\theta}_l, \ldots) : \sum_{|l|\leq\sqrt{W}} \|\boldsymbol{\theta}_l\|^2 \mathrm{e}^{4\pi L l} \leq d_\varepsilon,\quad \boldsymbol{\theta}_p = 0,\ \text{for } |p| > \sqrt{W}\right\},$$

with $d_\varepsilon = \left[1 + \mathrm{e}^{-8\pi L(W+\sqrt{W})}\right]^{-1}$. Let $\mathrm{E}_{\boldsymbol{\theta}}^d$ be the expectation with respect to the measure generated by the observations (cf. (8))

$$\mathbf{z}_k = \boldsymbol{\theta}_k + \sigma_\varepsilon d_\varepsilon^{-1/2} \boldsymbol{\xi}_k,$$

and $\mathrm{E}_{\boldsymbol{\theta}}$ is the expectation, which corresponds to the observation from (8) with $\sigma = \sigma_\varepsilon$ (see (14)). Since $d_\varepsilon < 1$ we have

$$\inf_{\widehat{\boldsymbol{\theta}}} \sup_{\boldsymbol{\theta}\in\Theta} \mathrm{E}^d_{\boldsymbol{\theta}} \sum_{|l|\le\sqrt{W}} \mathrm{e}^{4\pi y l}\|\widehat{\boldsymbol{\theta}}_l - \boldsymbol{\theta}_l\|^2 \ge \inf_{\widehat{\boldsymbol{\theta}}} \sup_{\boldsymbol{\theta}\in\Theta} \mathrm{E}_{\boldsymbol{\theta}} \sum_{|l|\le\sqrt{W}} \mathrm{e}^{4\pi y l}\|\widehat{\boldsymbol{\theta}}_l - \boldsymbol{\theta}_l\|^2$$

and hence

$$\begin{aligned}\inf_{\widehat{\boldsymbol{\theta}}} \sup_{\boldsymbol{\theta}\in\Theta_\varepsilon} \mathrm{E}_{\boldsymbol{\theta}} \sum_{|l|\le\sqrt{W}} \mathrm{e}^{4\pi y l}\|\widehat{\boldsymbol{\theta}}_l - \boldsymbol{\theta}_l\|^2 &= d_\varepsilon \inf_{\widehat{\boldsymbol{\theta}}} \sup_{\boldsymbol{\theta}\in\Theta} \mathrm{E}^d_{\boldsymbol{\theta}} \sum_{|l|\le\sqrt{W}} \mathrm{e}^{4\pi y l}\|\widehat{\boldsymbol{\theta}}_l - \boldsymbol{\theta}_l\|^2 \\ &\ge d_\varepsilon \inf_{\widehat{\boldsymbol{\theta}}} \sup_{\boldsymbol{\theta}\in\Theta} \mathrm{E}_{\boldsymbol{\theta}} \sum_{|l|\le\sqrt{W}} \mathrm{e}^{4\pi y l}\|\widehat{\boldsymbol{\theta}}_l - \boldsymbol{\theta}_l\|^2.\end{aligned}$$

Therefore noting that $d_\varepsilon \to 1$, we obtain by (16) and (17)

$$\begin{aligned} r_\varepsilon(y,L) &\ge \frac{D(2+o(1))}{8(\pi W)^2}\mathrm{e}^{4\pi(y-L)W} \inf_{\widehat{\boldsymbol{\theta}}} \sup_{\boldsymbol{\theta}\in\Theta_\varepsilon} \mathrm{E}_{\boldsymbol{\theta}} \sum_{|l|\le\sqrt{W}} \mathrm{e}^{4\pi y l}\|\widehat{\boldsymbol{\theta}}_l - \boldsymbol{\theta}_l\|^2 \\ &= \frac{D(2+o(1))}{8(\pi W)^2}\mathrm{e}^{4\pi(y-L)W} \inf_{\widehat{\boldsymbol{\theta}}} \sup_{\boldsymbol{\theta}\in\Theta} \mathrm{E}_{\boldsymbol{\theta}} \sum_{|l|\le\sqrt{W}} \mathrm{e}^{4\pi y l}\|\widehat{\boldsymbol{\theta}}_l - \boldsymbol{\theta}_l\|^2.\end{aligned}$$

Thus to complete the proof of the lower bound it remains to check that

$$\begin{aligned}(18)\qquad & \inf_{\widehat{\boldsymbol{\theta}}} \sup_{\boldsymbol{\theta}\in\Theta} \mathrm{E}_{\boldsymbol{\theta}} \sum_{|l|\le\sqrt{W}} \mathrm{e}^{4\pi y l}\|\widehat{\boldsymbol{\theta}}_l - \boldsymbol{\theta}_l\|^2 \\ &\quad \ge \inf_{\widehat{\boldsymbol{\theta}}} \sup_{\boldsymbol{\theta}\in\Theta} \mathrm{E}_{\boldsymbol{\theta}} \sum_{l} \mathrm{e}^{4\pi y l}\|\widehat{\boldsymbol{\theta}}_l - \boldsymbol{\theta}_l\|^2 + o(1).\end{aligned}$$

By the triangle inequality, which guarantees that *inf* taken over all estimators coincides with *inf* over $\widehat{\boldsymbol{\theta}} \in \Theta$, we have

$$\begin{aligned}(19)\quad \inf_{\widehat{\boldsymbol{\theta}}} \sup_{\boldsymbol{\theta}\in\Theta} \mathrm{E}_{\boldsymbol{\theta}} \sum_{l} \mathrm{e}^{4\pi y l}\|\widehat{\boldsymbol{\theta}}_l - \boldsymbol{\theta}_l\|^2 &= \inf_{\widehat{\boldsymbol{\theta}}\in\Theta} \sup_{\boldsymbol{\theta}\in\Theta} \mathrm{E}_{\boldsymbol{\theta}} \sum_{l} \mathrm{e}^{4\pi y l}\|\widehat{\boldsymbol{\theta}}_l - \boldsymbol{\theta}_l\|^2 \\ \le \inf_{\widehat{\boldsymbol{\theta}}\in\Theta} \sup_{\boldsymbol{\theta}\in\Theta} \mathrm{E}_{\boldsymbol{\theta}} \sum_{l\le\sqrt{W}} \mathrm{e}^{4\pi y l}\|\widehat{\boldsymbol{\theta}}_l - \boldsymbol{\theta}_l\|^2 &+ \sup_{\widehat{\boldsymbol{\theta}}\in\Theta} \sup_{\boldsymbol{\theta}\in\Theta} \mathrm{E}_{\boldsymbol{\theta}} \sum_{l>\sqrt{W}} \mathrm{e}^{4\pi y l}\|\widehat{\boldsymbol{\theta}}_l - \boldsymbol{\theta}_l\|^2 \\ \le \inf_{\widehat{\boldsymbol{\theta}}\in\Theta} \sup_{\boldsymbol{\theta}\in\Theta} \mathrm{E}_{\boldsymbol{\theta}} \sum_{l\le\sqrt{W}} \mathrm{e}^{4\pi y l}\|\widehat{\boldsymbol{\theta}}_l - \boldsymbol{\theta}_l\|^2 &+ O(\exp(4\pi(y-L)\sqrt{W}) \\ \le \inf_{\widehat{\boldsymbol{\theta}}} \sup_{\boldsymbol{\theta}\in\Theta} \mathrm{E}_{\boldsymbol{\theta}} \sum_{l\le\sqrt{W}} \mathrm{e}^{4\pi y l}\|\widehat{\boldsymbol{\theta}}_l - \boldsymbol{\theta}_l\|^2 &+ o(1).\end{aligned}$$

Define

$$\Theta^+_\varepsilon = \Big\{\boldsymbol{\theta}_k : \sum_{|k|\le\sqrt{W}} \|\boldsymbol{\theta}_k\|^2 \mathrm{e}^{4\pi L k} \le 1\Big\}.$$

Since evidently $\Theta \subset \Theta_\varepsilon^+$ we have

$$
\begin{aligned}
(20)\quad & \inf_{\widehat{\boldsymbol\theta}} \sup_{\boldsymbol\theta\in\Theta} \mathrm{E}_{\boldsymbol\theta} \sum_{l\le\sqrt{W}} \mathrm{e}^{4\pi yl}\|\widehat{\boldsymbol\theta}_l - \boldsymbol\theta_l\|^2 \le \inf_{\widehat{\boldsymbol\theta}} \sup_{\boldsymbol\theta\in\Theta_\varepsilon^+} \mathrm{E}_{\boldsymbol\theta} \sum_{l\le\sqrt{W}} \mathrm{e}^{4\pi yl}\|\widehat{\boldsymbol\theta}_l - \boldsymbol\theta_l\|^2 \\
&= \inf_{\widehat{\boldsymbol\theta}} \sup_{\boldsymbol\theta\in\Theta_\varepsilon^+} \mathrm{E}_{\boldsymbol\theta} \Big\{ \sum_{|l|\le\sqrt{W}} \mathrm{e}^{4\pi yl}\|\widehat{\boldsymbol\theta}_l - \boldsymbol\theta_l\|^2 + \sum_{l<-\sqrt{W}} \mathrm{e}^{4\pi yl}\|\widehat{\boldsymbol\theta}_l - \boldsymbol\theta_l\|^2 \Big\} \\
&= \inf_{\widehat{\boldsymbol\theta}} \sup_{\boldsymbol\theta\in\Theta_\varepsilon^+} \mathrm{E}_{\boldsymbol\theta} \sum_{|l|<\sqrt{W}} \mathrm{e}^{4\pi yl}\|\widehat{\boldsymbol\theta}_l - \boldsymbol\theta_l\|^2 \\
&\quad + \inf_{\widehat{\boldsymbol\theta}} \sup_{\boldsymbol\theta_l\in\mathbf{R}^2} \mathrm{E}_{\boldsymbol\theta} \sum_{l\le-\sqrt{W}} \mathrm{e}^{4\pi yl}\|\widehat{\boldsymbol\theta}_l - \boldsymbol\theta_l\|^2.
\end{aligned}
$$

On the other hand it is clear that

$$
\inf_{\widehat{\boldsymbol\theta}} \sup_{\boldsymbol\theta_l\in\mathbf{R}^2} \mathrm{E}_{\boldsymbol\theta} \sum_{l\le-\sqrt{W}} \mathrm{e}^{4\pi yl}\|\widehat{\boldsymbol\theta}_l - \boldsymbol\theta_l\|^2 = \sigma_\varepsilon^2 \sum_{l\le-\sqrt{W}} \mathrm{e}^{4\pi yl} = o(1).
$$

The above equality together with (19) and (20) completes the proof (18).

Let us turn to the upper bound for the minimax risk. In proving of the lower bound we used the following relations between the estimated parameters and the data in the models (8) and (4)

$$
\begin{pmatrix} \theta_{1k} \\ \theta_{2k} \end{pmatrix} = A(\varepsilon) \begin{pmatrix} \varphi_{k+W} \\ \varphi_{-k-W} \end{pmatrix}, \quad \begin{pmatrix} z_{1k} \\ z_{2k} \end{pmatrix} = A(\varepsilon) \begin{pmatrix} z^\varepsilon_{k+W} \\ z^\varepsilon_{-k-W} \end{pmatrix}.
$$

The above equations provide a motivation for the estimator (12). By (12), (13) and simple algebra we have

$$
\begin{aligned}
(21)\quad & \sup_{\varphi\in\Phi} \mathrm{E}_\varphi \sum_{|k|\ge W+\sqrt{W}} b_k^2(y)|\varphi_k^* - \varphi_k|^2 &=& \; o(1) r_\varepsilon(y,L), \\
(22)\quad & \sup_{\varphi\in\Phi} \mathrm{E}_\varphi \sum_{|k|\le W-\sqrt{W}} b_k^2(y)|\varphi_k^* - \varphi_k|^2 &=& \; o(1) r_\varepsilon(y,L).
\end{aligned}
$$

On the other hand we obtain

$$
\begin{aligned}
(23)\quad & \sup_{\varphi\in\Phi} \mathrm{E}_\varphi \sum_{|k-W|<\sqrt{W}} b_k^2(y)|\varphi_k^* - \varphi_k|^2 \\
&\le \frac{D(2+o(1))}{8(\pi W)^2} \mathrm{e}^{4\pi(y-L)W} \sup_{\boldsymbol\theta\in\widetilde\Theta_\varepsilon} \mathrm{E}_{\boldsymbol\theta} \sum_{|k|\le\sqrt{W}} \mathrm{e}^{4\pi yk}\|\boldsymbol\theta_k^* - \boldsymbol\theta_k\|^2,
\end{aligned}
$$

where

$$
\widetilde\Theta_\varepsilon = \Big\{ \boldsymbol\theta_k : \sum_{|k|\le\sqrt{W}} \|\boldsymbol\theta_k\|^2 \mathrm{e}^{4\pi Lk} \le \Big[1 + \mathrm{e}^{-8\pi L(W-\sqrt{W})}\Big]^{-1} \Big\}.
$$

It is also clear that

$$\sup_{\theta\in\widetilde{\Theta}_\varepsilon} \mathrm{E}_\theta \sum_{|k|\le\sqrt{W}} \mathrm{e}^{4\pi yk}\|\theta_k^* - \theta_k\|^2 \le \sup_{\theta\in\Theta} \mathrm{E}_\theta \sum_k \mathrm{e}^{4\pi yk}\|\theta_k^* - \theta_k\|^2.$$

This inequality together with (21)–(23) completes the proof of the upper bound. ■

Remark 2.1 As $\varepsilon \to 0$

$$r_\varepsilon(y,L) \asymp D\left(\varepsilon^2/D\right)^{1-y/L}\log^{-2}(D/\varepsilon^2).$$

Proof easily follows from Theorem 2.1 since by (11)

$$(4\pi L)^{-1}\log(2D/\varepsilon^2) - 0.5 \le W_\varepsilon \le (4\pi L)^{-1}\log(2D/\varepsilon^2) + 0.5$$

and hence by (14) $\exp(-\pi L) \le \sigma_\varepsilon \le \exp(\pi L)$, so that the limiting risk $\rho(\sigma_\varepsilon, y, L)$ is $O(1)$.

3 Asymptotic behavior of the limit minimax risk

The minimax estimator in the problem (8)–(10) is of course nonlinear. Unfortunately in the general case it is impossible to find it in an explicit form. Therefore, we study in this section the behavior of the minimax risk $\rho(\sigma, y, L)$ and respectively the minimax estimator for $L \ll 1$ and for $L \gg 1$. In the first case we recover the solution in the vicinity of the boundary and show that the minimax estimator can be well approximated by a linear smoother. The second case is related to estimation of "very smooth" functions. There are two possibilities in this situation. The asymptotically minimax estimator may be either a simple nonlinear estimator or a linear projection estimator. Denote $[x]_+ = \max(0,x)$.

Theorem 3.1 *Uniformly in y such that*

$$\frac{2\pi L(L+y)}{L-y} < 1-\delta, \quad \delta > 0 \tag{24}$$

the following assertion holds

$$\lim_{L\to 0}\frac{\rho(\sigma_\varepsilon, y, L)}{\rho^{lin}(y,L)} = 1, \tag{25}$$

where

$$\rho^{lin}(y,L) = \frac{L}{2\pi y}\left[\frac{L-y}{2\pi L(L+y)}\right]^{1-y/L} \tag{26}$$

is the risk of the best in minimax sense linear estimator

$$\widetilde{\theta}_k = \left[1 - \mathrm{e}^{2\pi(L-y)(k-w)}\right]_+ \mathbf{z}_k, \quad w = \left\lfloor \frac{1}{4\pi L}\log\frac{2\pi L(L+y)}{L-y}\right\rfloor \tag{27}$$

in the model (8–10) with $\sigma = 1$.

Proof The proof consists of two steps. First we compute the risk of $\widetilde{\boldsymbol{\theta}}_k$ thus proving the upper bound for the minimax risk and then we show by constructing a prior distribution on Θ, that our upper bound cannot be improved. Notice that by (14) $\sigma_\varepsilon^2 \to 1$ as $L \to 0$, so that

$$\begin{aligned}
(28) \qquad \rho(\sigma^\varepsilon, y, L) &\le \sup_{\boldsymbol{\theta}\in\Theta} \mathrm{E}_{\boldsymbol{\theta}} \sum_{k=-\infty}^{\infty} \|\widetilde{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k\|^2 \mathrm{e}^{4\pi yk} \\
&\le \mathrm{e}^{-4\pi w(L-y)} + 2\sigma_\varepsilon^2 \sum_{k=-\infty}^{\infty} \mathrm{e}^{4\pi yk} \left[1 - \mathrm{e}^{2\pi(L-y)(k-w)}\right]_+^2 \\
&\le \mathrm{e}^{-4\pi w(L-y)} + (2+o(1))\mathrm{e}^{4\pi yw} \left[\frac{1}{4\pi y} - \frac{1}{\pi(L+y)} + \frac{1}{4\pi L}\right] \\
&= (1+o(1))\rho^{lin}(y, L).
\end{aligned}$$

Our next step is to prove that the above upper bound is precise. Here we follow the idea of Pinsker (1982). Let $\boldsymbol{\zeta}_k \in \mathbf{R}^2$ be iid $\mathcal{N}(0, I)$. Define

$$\sigma_k^2 = \left[1 - \mathrm{e}^{2\pi(L-y)(k-w)}\right]_+ \mathrm{e}^{-2\pi(L-y)(k-w)}.$$

We shall show that

$$(29) \qquad \lim_{L\to 0} \mathrm{E}\left[\sum_k \mathrm{e}^{4\pi kL} \sigma_k^2 \left(\|\boldsymbol{\zeta}_k\|^2 - 2\right)\right]^2 = 0.$$

We have by simple algebra

$$\begin{aligned}
(30) \qquad \mathrm{E}\left[\sum_k \mathrm{e}^{4\pi kL} \sigma_k^2 (\|\boldsymbol{\zeta}_k\|^2 - 2)\right]^2 &= 4\sum_k \mathrm{e}^{8\pi kL} \sigma_k^4 \\
&= \mathrm{e}^{8\pi wL} \sum_{k=-\infty}^{0} \mathrm{e}^{8\pi kL} \left[\mathrm{e}^{-4\pi k(L-y)} - 2\mathrm{e}^{-2\pi k(L-y)} + 1\right] \\
&= \frac{2(1+o(1))\mathrm{e}^{8\pi wL}}{\pi} \left[\frac{1}{2(L+y)} - \frac{2}{3L+y} + \frac{1}{4L}\right] \\
&= (1+o(1)) \frac{\mathrm{e}^{8\pi wL}(L-y)^2}{2\pi L(L+y)(3L+y)} \\
&= (1+o(1)) \frac{2\pi L(L+y)}{3L+y} < 2\pi L
\end{aligned}$$

thus proving (29). Let Θ_0 be the subset in Θ

$$\Theta_0 = \left\{\boldsymbol{\theta} : \sum_{k=-\infty}^{w} \|\boldsymbol{\theta}_k\|^2 \mathrm{e}^{4\pi Lk} \le 1, \quad \boldsymbol{\theta}_k = 0, \quad k > w\right\}.$$

Then by the triangle inequality we have

$$
\begin{aligned}
(31)\qquad \rho(\sigma,y) &\geq \inf_{\widehat{\boldsymbol{\theta}}} \sup_{\boldsymbol{\theta}\in\Theta_0} \mathrm{E}_{\boldsymbol{\theta}} \sum_{k=-\infty}^{w} \|\widehat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k\|^2 \mathrm{e}^{4\pi yk} \\
&= \inf_{\widehat{\boldsymbol{\theta}}\in\Theta_0} \sup_{\boldsymbol{\theta}\in\Theta_0} \mathrm{E}_{\boldsymbol{\theta}} \sum_{k=-\infty}^{w} \|\widehat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k\|^2 \mathrm{e}^{4\pi yk}.
\end{aligned}
$$

Assuming that for some $\delta > 0$

$$
\boldsymbol{\theta}_k = \begin{cases} (1-\delta)\sigma_k\boldsymbol{\zeta}_k, & k \leq w \\ 0, & k > w \end{cases}
$$

we get

$$
\mathrm{E}\sum_k \mathrm{e}^{4\pi kL}\boldsymbol{\theta}_k^2 = (1-\delta)^2(1+o(1)),
$$

and hence by the Markov inequality and (30) $\mathrm{P}\Big\{\boldsymbol{\theta} \notin\Theta_0\Big\} = o(1)$. Therefore, by the Cauchy-Schwartz inequality and (29–31) we obtain

$$
\begin{aligned}
(32)\quad \rho(\sigma,y) &\geq \inf_{\widehat{\boldsymbol{\theta}}\in\Theta_0} \mathrm{E} \sum_{k=-\infty}^{w} \|\widehat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k\|^2 \mathrm{e}^{4\pi yk}\mathbf{1}\{\boldsymbol{\theta} \in\Theta_0\} \\
&\geq \inf_{\widehat{\boldsymbol{\theta}}\in\Theta_0} \mathrm{E} \sum_{k=-\infty}^{w} \|\widehat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k\|^2 \mathrm{e}^{4\pi yk} \\
&\quad - \sup_{\widehat{\boldsymbol{\theta}}\in\Theta_0} \mathrm{E} \sum_{k=-\infty}^{w} \|\widehat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k\|^2 \mathrm{e}^{4\pi yk}\mathbf{1}\{\boldsymbol{\theta} \notin \Theta_0\} \\
&\geq \inf_{\widehat{\boldsymbol{\theta}}\in\Theta_0} \mathrm{E} \sum_{k=-\infty}^{w} \|\widehat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k\|^2 \mathrm{e}^{4\pi yk} - 2\mathrm{e}^{4\pi w(L-y)}\mathrm{P}\Big\{\boldsymbol{\theta} \notin\Theta_0\Big\} \\
&\quad -2\mathrm{E}^{1/2}\Big(\sum_{k=-\infty}^{w} \|\boldsymbol{\theta}_k\|^2 \mathrm{e}^{4\pi yk}\Big)^2 \mathrm{P}^{1/2}\Big\{\boldsymbol{\theta} \notin \Theta_0\Big\} \\
&\geq \inf_{\widehat{\boldsymbol{\theta}}\in\Theta_0} \mathrm{E} \sum_{k=-\infty}^{w} \|\widehat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k\|^2 \mathrm{e}^{4\pi yk} - o(1)\mathrm{e}^{4\pi w(L-y)} \\
&\geq \inf_{\widehat{\boldsymbol{\theta}}} \mathrm{E} \sum_{k=-\infty}^{w} \|\widehat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k\|^2 \mathrm{e}^{4\pi yk} - o(1)\rho^{lin}(\sigma,y).
\end{aligned}
$$

Since $\boldsymbol{\theta}_k$ are independent Gaussian random variables and $\sigma_\varepsilon = 1 + O(L)$ we have

$$
\inf_{\widehat{\boldsymbol{\theta}}} \mathrm{E} \sum_{k=-\infty}^{w} \|\widehat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k\|^2 \mathrm{e}^{4\pi yk} = 2\sum_{k=-\infty}^{w} \frac{\sigma_k^2\sigma_\varepsilon^2}{\sigma_\varepsilon^2+\sigma_k^2}\mathrm{e}^{4\pi yk} = (1+o(1))\rho^{lin}(\sigma,y).
$$

Equation (32) together with the above formula complete the proof. ∎

Remark 3.1 It follows from Theorems 2.1, 3.1 that the asymptotically minimax estimator in the initial problem is linear as $L \to 0$ and condition (24) is fulfilled.

Let us look at the asymptotic behavior of the minimax risk $\rho(\sigma, y, L)$ as $L \to \infty$. There is a simpler statistical problem, which determines the asymptotics. Assume that we are given the sample

$$\mathbf{z} = \boldsymbol{\theta} + \sigma \boldsymbol{\xi}, \tag{33}$$

where $\boldsymbol{\xi}$ is $\mathcal{N}(0, I)$, and we want to estimate the unknown vector $\boldsymbol{\theta}$ provided that it belongs to the ball

$$\Theta_0 = \left\{ \boldsymbol{\theta} : \|\boldsymbol{\theta}\|^2 \le 1 \right\}. \tag{34}$$

Let

$$\rho^\infty(\sigma) = \inf_{\widehat{\boldsymbol{\theta}}} \sup_{\boldsymbol{\theta} \in \Theta_0} \mathrm{E}\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\|^2 \tag{35}$$

be the minimax risk in the considered problem.

Theorem 3.2 *Uniformly in σ and $y < L$ as $L \to \infty$*

$$\begin{aligned} \rho(\sigma, y, L) &= \left[1 + O\left(\sigma \mathrm{e}^{-\pi L}\right)\right]\left[\rho^\infty(\sigma) + \frac{2\sigma^2}{\mathrm{e}^{4\pi y} - 1}\right] \\ &\quad + O\left(D \mathrm{e}^{-4\pi(L-y)}\right). \end{aligned} \tag{36}$$

Proof In order to obtain an upper bound for the risk $\rho(\sigma, y, L)$ we denote by $\boldsymbol{\theta}^*$ the minimax estimator in the problem (33)–(35) and consider the following estimator

$$\widetilde{\boldsymbol{\theta}}_k = \begin{cases} \mathbf{z}_k, & k < 0, \\ \boldsymbol{\theta}^*(\mathbf{z}_0), & k = 0, \\ 0, & k > 0. \end{cases}$$

Then we have

$$\begin{aligned} \sup_{\boldsymbol{\theta} \in \Theta} \mathrm{E} \sum_{k=-\infty}^{\infty} \|\widetilde{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k\|^2 \mathrm{e}^{4\pi y k} &\le \sup_{\boldsymbol{\theta} \in \Theta} \mathrm{E}\|\widetilde{\boldsymbol{\theta}}_0 - \boldsymbol{\theta}_0\|^2 \\ &\quad + 2\sigma^2 \sum_{k=-\infty}^{-1} \mathrm{e}^{4\pi y k} + \sup_{\boldsymbol{\theta} \in \Theta} \sum_{k=1}^{\infty} \|\boldsymbol{\theta}_k\|^2 \mathrm{e}^{4\pi y k} \\ &\le \sup_{\boldsymbol{\theta} \in \Theta_0} \mathrm{E}\|\boldsymbol{\theta}^* - \boldsymbol{\theta}\|^2 + \frac{2\sigma^2}{\mathrm{e}^{4\pi y} - 1} + D \mathrm{e}^{-4\pi(L-y)}. \end{aligned}$$

Let us turn to the lower bound. The idea is to construct a suitable prior distribution on Θ. First of all, we assume that $\boldsymbol{\theta}_k = 0$ for $k > 0$. Next we suppose that $\|\boldsymbol{\theta}_0\|^2 \leq 1 - \delta$. For $k < 0$ the unknown vectors are defined as follows $\boldsymbol{\theta}_k = S(\eta_{1k}, \eta_{2k})^T$, where $\eta_{jk} \in [-1, 1]$ are iid with the density $p(x) = \cos^2(\pi x/2)$, and S is such that

$$2S^2 \sum_{k \leq -1} e^{4\pi L k} = \delta.$$

Noting that the Fisher information of $\boldsymbol{\theta}_k$ for $k < 0$ is π^2/S^2 we obtain from the Van Trees (1968) inequality

$$\begin{aligned}
\rho(\sigma, y, L) &\geq \inf_{\widehat{\boldsymbol{\theta}}_0} \sup_{\|\boldsymbol{\theta}_0\|^2 \leq 1-\delta} \mathrm{E}\|\widehat{\boldsymbol{\theta}}_0 - \boldsymbol{\theta}_0\|^2 + \frac{2}{\pi^2 S^{-2} + \sigma^{-2}} \sum_{k \leq -1} e^{4\pi y k} \\
&= (1-\delta) \inf_{\widehat{\boldsymbol{\theta}}_0} \sup_{\|\boldsymbol{\theta}_0\|^2 \leq 1} \mathrm{E}\|\widehat{\boldsymbol{\theta}}_0 - \boldsymbol{\theta}_0\|^2 + \frac{2\sigma^2 S^2}{\pi^2 \sigma^2 + S^2} \sum_{k \leq -1} e^{4\pi y k} \\
&= (1-\delta)\rho^{\infty}(\sigma) + \frac{2\sigma^2}{e^{4\pi y} - 1} \cdot \frac{1}{1 + 0.5\pi^2\sigma^2\delta^{-1}(e^{4\pi L} - 1)^{-1}} \\
&\geq \rho^{\infty}(\sigma) + \frac{2\sigma^2}{e^{4\pi y} - 1} - \delta\rho^{\infty}(\sigma) - \frac{\pi^2\sigma^4}{\delta(e^{4\pi y} - 1)e^{4\pi L}} .
\end{aligned}$$

Maximizing the right-hand side of the above equation with respect to δ we arrive at

$$\begin{aligned}
\rho(\sigma, y, L) &\geq \rho^{\infty}(\sigma) + \frac{2\sigma^2}{e^{4\pi y} - 1} - 2\pi\sigma^2 e^{-2\pi L} \left[\frac{\rho^{\infty}(\sigma)}{e^{4\pi y} - 1}\right]^{1/2} \\
&\geq \left(1 - \pi\sigma e^{-\pi L}\right)\left(\rho^{\infty}(\sigma) + \frac{2\sigma^2}{e^{4\pi y} - 1}\right).
\end{aligned}$$

This concludes the proof. ■

In proving Theorem 2.1 we have assumed that y is fixed. On the other hand we see from Theorem 3.2 that $\rho(\sigma, y, L) \approx O(y^{-1})$ for small y, thus indicating that the rate of convergence of the minimax risk (see Theorem 2.1) is not uniform with respect to y. To cover this gap we describe in the next theorem the asymptotics of the minimax risk and an asymptotically minimax estimator, when y is in a vicinity of 0.

Theorem 3.3 *Let y^{ε} be such that*

$$\lim_{\varepsilon \to 0} y^{\varepsilon} \log D/\varepsilon^2 = 0.$$

Then uniformly in $y \in [0, y^{\varepsilon}]$ as $\varepsilon \to 0$

$$r^{\varepsilon}(y, L) = (1 + o(1)) \frac{y^2}{2\pi L} \varepsilon^2 \log \frac{D}{\varepsilon^2}$$

and the projection estimator

$$\widehat{u}_W(x,y,z^\varepsilon) = \sum_{|k|<W(\varepsilon)} b_k(y) z_k^\varepsilon e^{2\pi ikx}, \qquad W(\varepsilon) = \frac{1}{4\pi L}\log\frac{D}{\varepsilon^2}$$

is asymptotically minimax.

Proof The upper bound for the minimax risk follows from the Taylor formula for $b_k(y)$, when $y \in [0, y^\varepsilon]$

$$b_k(y) = y + (2\pi k)y^2 \sinh(2\pi k \zeta_y),$$

where $\zeta_y \in [0, y^\varepsilon]$. Therefore, we have (see (7))

$$\begin{aligned} r^\varepsilon(y,L) &\le \sup_{\varphi\in\Phi} R_\varphi(\widehat{u}_W, u) \le D b^2_{W(\varepsilon)}(y)\cosh^{-1}(4\pi W(\varepsilon)L) \\ &\quad + \varepsilon^2 \sum_{|k|<W(\varepsilon)} b_k^2(y) = (1+o(1))\frac{y^2}{2\pi L}\varepsilon^2 \log\frac{D}{\varepsilon^2}. \end{aligned}$$

The proof of the lower bound is based on arguments used in Golubev & Levit (1996). ∎

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