EFFICIENT COUPLING ON THE CIRCLE

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Abstract

We consider efficient coupling specifically treating reversible Markov chains on the circle. We also show that "most" Markov chains with an efficient coupling have an "asymptotic monotone function".

Introduction

This paper is prompted by a recent paper of [BK] concerning efficient couplings of irreducible reversible continuous time Markov chains on a finite state space S. (We note those authors also studied efficiency questions for reflecting Brownian motion.) The starting point for this paper is the eigenvector expansion for the density function $p_t(x,y) = P^x(X_t = y)$ where $(X_t)_{t\geq 0}$ is our reversible Markov chain

$$p_t(x,y) = \pi(y) \sum_i e^{-\lambda_i t} \phi_i(x) \phi_i(y) \, .$$

Here π is the (unique) invariant distribution and ϕ_i are the right eigenvectors satisfying

$$Q\phi_i = -\lambda_i \phi_i \qquad i = 1 \cdots n$$
$$\sum \phi_i(x) \pi(x) \phi_j(x) = \delta_{ij},$$

where Q is the Q-matrix or generator of X. We refer to [AF] for important background materials on reversible Markov chains. Of course if the ϕ_i are ordered according to increasing λ_i, then $\lambda_1 = 0$, $\phi_1 \equiv 1$, and we have that

$$\sum_y |p_t(x,y) - \pi(y)| = O(e^{-\lambda_2 t})$$

and $\sum_y |p_t(x,y) - \pi(y)| = o(e^{-\lambda_2 t})$ if and only if $\phi_i(x) = 0$ for each eigenvector corresponding to λ_2.

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Equally we have

$$\sum_y |p_t(x,y) - p_t(x',y)| = O(e^{-\lambda_2 t}) \quad \text{and}$$

$$\sum_y |p_t(x,y) - p_t(x',y)| = o(e^{-\lambda_2 t}) \quad \text{if and only if}$$

$\phi_i(x) = \phi_i(x')$ for each eigenvector ϕ_i corresponding to λ_2.

Now consider for $x \neq x'$ a coupling between the Markov chain started x and the Markov chain started at x'. This is a process $(X,X')_t$ on $S \times S$ so that the marginal processes X_t, X'_t are the Markov chains starting respectively at x and x'. The preceding discussion shows that for typical pairs (x,x') and any coupling (X,X')

$$P(X_t \neq X'_t) \geq Ce^{-\lambda_2 t}.$$

[BK] considered Markov couplings $(X,X')_t$, that is couplings $(X,X')_t$ so that (X_t,X'_t) is a Markov process on $S \times S$ and for which if $\tau = \inf\{t : X_t = X'_t\}$ then $X_S = X'_S$, $\forall\, S \geq \tau$. ¿From now on in this paper, all couplings will be such.

In this context we have: if for some second eigenvector ϕ_2, $\phi_2(x) \neq \phi_2(x')$, then for any Markov coupling, $P(\tau \geq t) \geq Ce^{-\lambda_2 t}$ for some $C > 0$ and all t. We say a coupling is efficient if

$$P(\tau \geq t) = O(e^{-\lambda_2 t})$$

and superefficient if $P(\tau \geq t) = o(e^{-\lambda_2 t})$. Superefficiency is only possible when $\phi_i(x) = \phi_i(x')$ for all eigenvectors corresponding to λ_2. Of course for a Markov chain it is possible that no efficient coupling exists for any initial x, x'. One nice criterion due to Burdzy and Kendall for an efficient coupling to exist is the following:

There exists a function f on S and a coupling $(X,X')_t$ so that a.s. for $\tau > t f(X_t) < f(X'_t)$.

Again it is easy to find examples where there is an efficient coupling but no monotone f, or indeed where there is no efficient coupling possible that would correspond to a monotone f.

Here one should note that the special function f is not so much relevent as the order on S it induces.

Much work has been done in recent years on estimating the first non-trivial eigenvalue of reversible Markov Processes (see e.g. [CW] , [W] for probablistic approaches and see [C1], [C2] and its references for an analytic perspective), so the question of whether coupling can provide a reasonable estimate arises.

A natural example is Birth and Death chains on $\{0,1,\ldots,\ n-1\}$ with $f(i) = i$. Then any coalescing coupling that does not allow jumps from $(i,i+1) \to (i+1,i)$ preserves order and is, therefore, efficient (noted in [A])

In this paper we consider *a* "next simplest" class of reversible Markov chains: $S = C_n = \{0,1,\ldots,\ n-1\}$ with $q_{ij} > 0 \Leftrightarrow i = j \pm 1 \mod (n)$. We refer to such chains as Nearest Neighbour (NN) processes on C_n.

In Section One we give a unimodality property of second eigenvectors of such processes which basically involves adapting nodal arguments for "continuous" reversible processes. In Section Two we show that while the existence of an efficient coupling does not imply a "monotone" f, it does imply an "asymptotically monotone" f for most Markov chains on C_n.

We conclude this introduction with a discussion of which n always give an efficient coupling! If $n = 3$ then (as noted in [BK]) we have an efficient coupling: wlog (relabelling if necessary) suppose

$$q_{10} \geq q_{20}\,.$$

Then if in addition

$$q_{12} \geq q_{02}$$

then the ordering $0 < 1 < 2$ can be preserved by an appropraite coupling. This follows since if (X, X') is at $(0, 1)$ then we can stipulate for our coupling that X' jumps to 2 whenever X jumps to 2 and we also stipulate that if (X, X') is at (1,2) then X jumps to 0 whenever X' jumps to 0. On the other hand if $q_{12} < q_{02}$ then since (by reversibility) $q_{01}q_{12}q_{20} = q_{10}q_{02}q_{21}$ we must have $q_{01}q_{21}$ and we then have that $1 < 0 < 2$ can be preserved. However, there does not generally exist an efficient coupling for $n = 4$ (or $n \geq 4$) as we will see in Section Three.

Section One

We now establish an analogue of Courant's nodal domain theorem.

Theorem 1.1. *For a NN process on C_n a second eigenvector ψ has only 2 crossings from positive to negative (a crossing from positive to negative occurs at bond $(x, x+1)$ if $\psi(x) > 0$ and $\psi(x+1) \leq 0$).*

Lemma 1.1. *ψ cannot have consecutive zeros.*

Proof. If $\psi(x) = 0 = \psi(x+1)$ then as $Q\psi = -\lambda_2\psi$, we have

$$\begin{aligned} 0 &= -\lambda_2\psi(x+1) \\ &= Q\psi(x+1) \\ &= q_{x+1x}[\psi(x) - \psi(x+1)] \quad + q_{x+1x+2}[\psi(x+2) - \psi(x+1)] \\ &= q_{x+1x+2}[\psi(x+2)]. \end{aligned}$$

Thus $\psi(x+2) = 0$ and by repeating one obtains $\psi \equiv 0$. This is a contradiction as $\langle \psi, \pi\psi \rangle = 1$. □

Lemma 1.2. *ψ cannot have 3 or more zeros.*

Proof. We argue by contradiction and suppose that $x_0 < x_1 < x_2$ are 3 zeros. Let $A = (x_0, x_1)$, $B = (x_1, x_2)$, $C = (x_2, x_0)$. By Lemma 1.2 these are nonempty intervals.

Now ψ is a solution of

$$\min_{\langle \pi, f \rangle = 0} \frac{\xi(f, f)}{(\pi, f^2)}$$

where $\xi(f,f) = -\langle f, \pi Q f\rangle$.

But $\psi(x)\chi_A(x)$ satisfies

$$Q\psi(x)\chi_A(x) = -\lambda_2\psi(x)\chi_A(x), \quad \text{whenever} \quad \psi(x)\chi_A(x) \neq 0,$$

(and similarly for $\psi\chi_B, \psi\chi_C$). Thus

$$\frac{\xi(\psi\chi_A, \psi\chi_A)}{\langle(\psi\chi_A)^2, \pi\rangle} = \lambda_2$$

Since $\psi\chi_A$ and $\psi\chi_B$ are of disjoint support,

$$\begin{aligned}&\langle(\alpha\psi\chi_A + \beta\psi\chi_B)^2, \pi\rangle \\ &\qquad = \alpha^2\langle\psi^2\chi_A, \pi\rangle + \beta^2\langle\psi^2\chi_B, \pi\rangle\end{aligned}$$

and similarly

$$\begin{aligned}&\xi(\alpha\psi\chi_A + \beta\psi\chi_B, \alpha\psi\chi_A + \beta\psi\chi_B) \\ &\quad = \alpha^2\xi(\psi\chi_A, \psi\chi_A) + \beta^2\xi(\psi\chi_B, \psi\chi_B) \\ &\quad = \lambda_2\alpha^2\langle\psi^2\chi_A, \psi\rangle + \lambda_2\beta^2\langle\psi^2\chi_B, \psi\rangle.\end{aligned}$$

Choosing $\alpha, \beta > 0$ so that $\langle\pi, \alpha\chi_A\psi + \beta\chi_B\psi\rangle = 0$, we see that $\alpha\chi_A\psi + \beta\chi_B\psi$ is a second eigenvector. This is a contradiction as this vector is zero on $[x_2, x_0]$ which has cardinality at least 2. $\square$

We now try to build on this. First we assume that our Markov chain has a unique second eigenvector. If not we can find a NN perturbation of our Markov chain with a given second eigenvector ψ as the unique second eigenvector.

The only problem we face is that a crossing from positive to negative value of ψ may not involve a site x with $\psi(x) = 0$.

We aim to construct a NN reversible chain on an augmented space obtained by adding a point between cross over points, so that the second eigenvector for the new chain is zero at the added point and agrees with ψ at the other points.

Let ψ (without loss of generality) be the unique second eigenvector of our Markov process and suppose $\psi(x) > 0$, $\psi(x+1) < 0$. Insert a new site x' between x and $x+1$ and choose the jump rates $q_{xx'}, q_{x+1x'}, q_{x'x}, q_{x'x+1}$ so that for the new process on $S' = S \cup \{x'\}$, ψ' defined by

$$\begin{aligned}\psi'(y) &= \psi(y) \qquad \text{for} \quad y \in S \\ \psi'(x') &= 0\end{aligned}$$

is a right eigenvector for the new jump rate matrix on S' with eigenvalue λ_2.

Obviously $\psi(y) = -\lambda_2 Q'\psi(y)$ yields the same equations for $y \neq x, x', x+1$. For x we require that

$$\begin{aligned}\psi'(x) = -\lambda_2 Q'\psi'(x) = &-\lambda_2 q_{xx-1}(\psi'(x-1) - \psi'(x)) \\ &- \lambda_2 q_{xx'}(\psi'(x') - \psi'(x))\end{aligned}$$

But as ψ is a $-\lambda_2$ eigenvector for Q we have

$$\begin{aligned}\psi(x) = &- \lambda_2 q_{xx-1}(\psi(x-1) - \psi(x)) \\ &- \lambda_2 q_{xx+1}(\psi(x+1) - \psi(x))\end{aligned}$$

Thus we require that

$$\begin{aligned}q_{xx+1}(\psi(x+1) - \psi(x)) = &\quad q_{xx'}(\psi'(x') - \psi'(x)) \\ &= -q_{xx'}\psi(x)\end{aligned}$$

since $\psi'(x') = 0$. So

$$q_{xx'} = q_{xx+1} \quad \frac{\psi(x) - \psi(x+1)}{\psi(x)}.$$

Similarly we require

$$q_{x+1x'} = -q_{x+1x} \quad \frac{\psi(x) - \psi(x+1)}{\psi(x+1)}.$$

Note this means that

(A)
$$\frac{q_{x+1x}}{q_{xx+1}} = -\frac{q_{x+1x'}}{q_{xx'}} \quad \frac{\psi(x+1)}{\psi(x)}.$$

We also require that

$$\begin{aligned}0 = -\lambda_2\psi'(x') = q_{x'x}(\psi'(x) - \psi'(x')) \\ + \quad q_{x'x+1}(\psi'(x+1) - \psi'(x'))\end{aligned}$$

i.e.

(B)
$$\frac{q_{x'x}}{q_{x'x+1}} = -\frac{\psi'(x+1)}{\psi'(x)}$$

and reversibility which requires

$$\frac{q_{x+1x'}q_{x'x}}{q_{xx'}q_{x'x+1}} \quad = \quad \frac{q_{x+1x}}{q_{xx+1}}$$

but this follows from (A) and (B).

Thus we have a new reversible NN Markov chains on S' having ψ' as an eigenvector corresponding to λ_2.

It does not follow that λ_2 is the second eigenvector. However, we will show that this will be the case if we choose rates sufficiently high.

Let π' be the unique stationary distribution for the new Markov chain on S'. Let $q_{x'} = q_{x'x} + q_{x'x+1}$ be the jump rate at site x'. As $q_{x'} \to \infty$ the value of $\pi'(x') \to 0$ and $\pi' \to \pi$ in total variation norm (with π considered as a measure on S').

We might hope for similar convergence for

(i) $\xi'(\psi', \psi')$ to $\xi(\psi, \psi)$

(ii) (ψ'^2, π') to (ψ^2, π)

(iii) (ψ', π') to (ψ, π).

This in fact turns out to be the case; essentially as $q_{x'} \to \infty$ the site x' becomes invisible. A jump from, say, x to x' "becomes" simply a jump to $x + 1$ with probability

$$\frac{\psi(x)}{\psi(x) - \psi(x+1)}$$

and with probability

$$\frac{-\psi(x+1)}{\psi(x) - \psi(x+1)}$$

no jump at all.

First note that π', the invariant distribution for Q', satisfies

$$\begin{aligned} \pi'(y) &= \lambda\pi(y) \qquad \forall y \in S, \\ \pi'(x') &= c, \end{aligned}$$

where as $q_{x'} \uparrow \infty$, $c \downarrow 0$ and $\lambda \uparrow 1$. This gives (ii) and (iii); for (i) note that

$$\begin{aligned} \xi'(\psi', \psi') &= \sum_{y \in S'} \pi'(y) q_{yy+1}(\psi'(y) - \psi'(y+1))^2 \\ &= \sum_{y \in S/\{x'\}} \pi'(y) q_{yy+1}(\psi(y) - \psi(y+1))^2 \\ &+ \pi'(x') q_{xx'}(\psi(x) - 0)^2 \\ &\quad \pi'(x') q_{x'x+1}(0 - \psi(x+1))^2 \\ &= \lambda \sum_{y \in S/\{x'\}} \pi(y) q_{yy+1}(\psi(y) - \psi(y+1))^2 \\ &\quad + \lambda\pi(x) q_{xx+1} \frac{\psi(x) - \psi(x+1)}{\psi(x)} \psi(x)^2 \\ &\quad \pi'(x') q_{x'x+1} \psi(x+1)^2 \end{aligned}$$

but by reversibility

$$\begin{aligned} \pi'(x') q_{x'x+1} &= \pi'(x+1) q_{x+1x'} \\ &= \pi'(x+1) q_{x+1x} \left(\frac{\psi(x) - \psi(x+1)}{-\psi(x+1)} \right) \end{aligned}$$

so

$$
\begin{aligned}
&\xi'(\psi',\psi') \\
&= \lambda \sum_{y\in S/\{x\}} \pi(y)q_{yy+1}(\psi(y)-\psi(y+1))^2 \\
&\qquad + \lambda\pi(x)q_{xx+1}(\psi(x)-\psi(x+1))\psi(x) \\
&\qquad - \lambda\pi(x+1)q_{x+1x}\left(\psi(x)-\psi(x+1)\right)\psi(x+1) \\
&= \lambda \sum_{y\in S/\{x\}} \pi(y)q_{yy+1}(\psi(y)-\psi(y+1))^2 \\
&\qquad + \lambda\pi(x)q_{xx+1}(\psi(x)-\psi(x+1))^2 \text{ by reversibility} \\
&= \lambda\xi(\psi,\psi)
\end{aligned}
$$

so (i) follows.

It is reasonable to expect that as $n = q_{x'} \to \infty$, ψ' becomes the unique second eigenvector for Q'.

Suppose not, then for all large $n = q_{x'}$, there is a distinct (from ψ') second eigenvector, denoted ψ'_n, for Q' with eigenvalue $\lambda_2^n \le \lambda_2$. That is, ψ'_n satisfies

$$
\begin{aligned}
\langle \psi'_n, \pi'_n\rangle &= 0 \\
\langle (\psi'_n)^2, \pi'_n\rangle &= 1 \\
\langle \psi'_n\psi', \pi'_n\rangle &= 0 \\
\xi'_n(\psi'_n,\psi'_n) &= \lambda_2^n \le \lambda_2,
\end{aligned}
$$

for all $n \ge N_0$ for some N_0.

We will now show that this leads to a contradiction. Let ψ_n be the restriction of ψ'_n to S.

First we observe that $\sup_n |\psi'_n(x')| < \infty$. This follows from

$$
\begin{aligned}
\lambda_2^n &= \xi'_n(\psi'_n,\psi'_n) \\
&\ge (\psi'_n(x') - \psi'_n(x))^2\pi'_n(x)q_{xx'} \\
&\ge (\psi'_n(x') - \psi'_n(x))^2\lambda\pi(x)q_{xx+1}
\end{aligned}
$$

since $q_{xx'} = q_{xx+1}\frac{\psi(x)-\psi(x+1)}{\psi(x)} \ge q_{xx+1}$. This implies that for large n (recall that λ increases to 1 as n goes to infinity)

$$
(\psi'_n(x') - \psi'_n(x))^2 \le \frac{2\lambda_2}{\pi(x)q_{xx+1}}.
$$

Now the normalization $\langle(\psi'_n)^2, \pi'_n\rangle = 1$ gives $(\psi'_n(x))^2 \le \frac{2}{\pi(x)}$ for large n and thus $\sup_n |\psi'_n(x')| < \infty$. Thus $\sup_{n,z} |\psi'_n(z)| < \infty$. Therefore $\langle \psi'_n, \pi'_n\rangle = 0$, $\pi'_n \to \pi$, $\pi'_n(x') \to 0$ implies that $\langle \psi_n, \pi\rangle \to 0$ as n tends to infinity. Also since $\psi'(x') = 0$

and $\langle\psi_n'\psi', \pi_n'\rangle = 0$, we conclude that $\langle\psi_n\psi, \pi\rangle = 0$. In addition, $\langle\psi_n^2, \pi\rangle \to 1$ as n tends to infinity follows easily.

Finally,

$$\xi_n'(\psi_n', \psi_n') = \lambda\xi(\psi_n, \psi_n) + q_{xx'}\pi_n'(x)(\psi_n'(x) - \psi_n'(x'))^2$$
$$+\pi_n'(x')q_{x'x+1}(\psi_n'(x') - \psi_n'(x+1))^2 \quad - \lambda\pi(x)q_{xx+1}(\psi_n(x) - \psi_n(x+1))^2.$$

Applying

$$a(x-y)^2 + b(x-z)^2 \geq \frac{ab}{a+b}(y-z)^2,$$

to the second and third terms on the righthand side we get

$$\begin{aligned}\xi_n'(\psi_n', \psi_n') &\geq \lambda\xi(\psi_n, \psi_n) \\ &+ \frac{\pi_n'(x)\pi_n'(x+1)q_{xx'}q_{x+1x'}(\psi_n(x) - \psi_n(x+1))^2}{(\pi_n'(x)q_{xx'} + \pi_n'(x+1)q_{x+1x'})} \\ &- \lambda\pi(x)q_{xx+1}(\psi_n(x) - \psi_n(x+1))^2\end{aligned}$$

$$= \lambda\xi(\psi_n, \psi_n) + \lambda\pi(x)q_{xx+1}(\psi_n(x) - \psi_n(x+1))^2\left[\frac{q_{xx'}q_{x+1x'}}{q_{xx+1}} \Big/ \left(\frac{q_{xx'}q_{x+1x}}{q_{xx+1}} + q_{x+1x'}\right) - 1\right]$$
$$= \lambda\xi(\psi_n, \psi_n)$$

since $\frac{q_{xx'}}{q_{xx+1}} = \frac{\psi(x)-\psi(x+1)}{\psi(x)}$ and $\frac{q_{x+1x'}}{q_{x+1x}} = \frac{\psi(x+1)-\psi(x)}{\psi(x+1)}$. Thus, $\lambda_2 \geq \xi_n'(\psi_n', \psi_n') \geq \lambda\xi(\psi_n, \psi_n)$. Consequently if β is any limit function of ψ_n, it must satisfy

(1) $\langle\beta, \pi\rangle = 0$
(2) $\langle\beta, \pi\psi\rangle = 0$
(3) $\langle\beta^2, \pi\rangle = 1$
(4) $\xi(\beta, \beta) \leq \lambda_2$

However, this contradicts ψ being the unique second eigenvector. We conclude, therefore, that for some n $= q_{x'}$ sufficiently large, ψ' is the unique second eigenfunction for Q'.

We can now finish the proof of Theorem 1.1:

Suppose ψ, the unique second eigenvector, has 3 crossings of zero at $x_0 < x_1 < x_2$. Then as shown above, we may (if necessary) augment the state space S by adding

$$\begin{array}{llll} x_0' & \text{between} & x_0 & \text{and} & x_0+1 \\ x_1' & \text{between} & x_1 & \text{and} & x_1+1 \\ x_2' & \text{between} & x_2 & \text{and} & x_2+1. \end{array}$$

Define a new transition rate matrix Q' on $S' = S \cup \{x_0', x_1', x_2'\}$ so that for $x, y \in S$, $Q'(x,y) = Q(x,y)$ for $\{x,y\} \neq \{x_i, x_i+1\}$ and $Q'(x, x_i'),\quad Q(x_i', x)$ are as detailed above. Then if we choose the jump rates

$$Q'(x_i', x_i) + Q'(x_i', x_i+1)$$

all sufficiently large, we will have a NN Markov process on S' with a second eigenvector having 3 zeros at least.

By Lemma 1.3 this is a contradiction . □

Corollary 1.1. *Let ψ be a second eigenvector for a NN Markov process on C_n. If x is a non extremal site for ψ, lying in the interval (y_0, z_0) where y_0 takes the minimum ψ value and z_0 takes the maximum value, then $\psi(x-1) < \psi(x) < \psi(x+1)$.*

Proof. Assume without loss of generality that $\psi(x) \geq 0$. Then we have $\psi > 0$ on (x, z_0) by Theorem 1.1. Also, assume without loss of generality that $\psi(z_0 - 1) < \psi(z_0)$

Then $-\lambda_2 \psi(z_0 - 1) \leq 0$ but

$$\begin{aligned} -\lambda_2 \psi(z_0 - 1) = Q\psi(z_0 - 1) \\ &= q_{z_0 - 1 z_0}(\psi(z_0) - \psi(z_0 - 1)) \\ &\quad + q_{z_0 - 1 z_0 - 2}(\psi(z_0 - 2) - \psi(z_0 - 1)) \end{aligned}$$

so that $\psi(z_0 - 2) < \psi(z_0 - 1)$. Continuing, we have

$$\psi(z_0) > \psi(z_0 - 1) > \psi(z_0 - 2) > \cdots > \psi(x)$$

but again

$$-2\lambda_2 \psi(x) \leq 0$$

which implies

$$q_{xx+1}(\psi(x+1) - \psi(x)) + q_{xx-1}(\psi(x-1) - \psi(x)) \leq 0.$$

Thus $\psi(x-1) - \psi(x) < 0$. $\square$

Section Two

We now define Markov chains on C_n which are *Reflections*: A Markov chain on C_n is a reflection if there exists a reflection θ of S (considering S as n equally spaced points on a circle) so that $\forall i, j$

$$q_{ij} = q_{\theta(i)\theta(j)} \quad \text{unless} \quad \theta(i) = i \,\text{or} \quad i = \theta(j) \quad \text{and} \quad j = \theta(i).$$

Lemma 2.1. *If a NN Markov chain on C_n is not a reflection, then for any Markov coupling (X, X') with $X_0 \neq X'_0$ there is positive probability of (X, X') reaching some (x_0, y_0) with $\psi(x_0) \neq \psi(y_0)$ for a second eigenvector ψ.*

Proof. If $\psi(X_0) \neq \psi(X'_0)$ there is nothing to prove. Suppose without loss of generality that the initial sites $(x_0,\, x'_0)$ are not extremal and that $\psi(x_0) = \psi(x'_0)$. In this case it follows from Corollary 1.1 that either

$$\begin{aligned} &\psi(x_0 - 1) < \psi(x_0) < \psi(x_0 + 1) \quad \text{and} \\ &\psi(x'_0 + 1) < \psi(x'_0) < \psi(x'_0 - 1) \end{aligned}$$

or

$$\begin{aligned} &\psi(x_0 - 1) > \psi(x_0) > \psi(x_0 + 1) \text{and} \\ &\psi(x'_0 + 1) > \psi(x'_0) > \psi(x'_0 - 1) \end{aligned}$$

We suppose without loss of generality the former. In order for the lemma not to be true, clearly we must be able to couple so that X jumps to $X_0 \pm 1 \Leftrightarrow X'_0$ jumps to $X'_0 \mp 1$ so we must have $q_{x_0 x_0 \pm 1} = q_{x'_0 x'_0 \mp 1}$ and that $\psi(x_0+1) = \psi(x'_0-1)$, $\psi(x_0-1) = \psi(x'_0+1)$. Repeating this argument at (x_0-1, x'_0+1), or (x_0+1, x'_0-1) and continuing yields the result that the chain must be a reflection. This contradiction proves the lemma.

Corollary 2.1. *If (X, X') is an efficient Markov coupling of a nonreflection NN Markov chain on $C_n = \{0, 1, \ldots\ n-1\}$ with $X_0 \neq X'_0$ then $\exists C > 0$ so that*

$$P(\tau \geq t) > Ce^{-\lambda_2 t}$$

C does not depend on particular choice of coupling.

Theorem 2.1. *If (X, X') is an efficient coupling of a nonreflection Markov chain with $X_0 \neq X'_0$ then there is (at least one) "asymptotic monotone function "f and $(x, y), x \neq y$ so that with positive probability (X, X') reaches (x, y) and such that*

$$P^{x,y}(f(X_t) < f(X'_t), \forall t < \tau) = 1\,.$$

Proof. Consider (X, X') conditioned on $(\tau \geq t)$. We obtain $(\tilde{X}, \tilde{X}')$, $0 \leq s \leq t$, a time inhomogeneous Markov chain on $S \times S/$ Diag $(S \times S)$ such that if it is possible for the unconditioned process (X, X') to jump from (u,v) to (w,z) with $w \neq z$ at rate $q^{(X,X')}_{(u,v)(w,z)}$, then the jump rate for the process $(\tilde{X}, \tilde{X}')$ from (u,v) to (w,z) at time s is equal to

$$(*) \qquad q^{(X,X')}_{(u,v)(w,z)} \frac{P^{w,z}(\tau \geq t-s)}{P^{u,v}(\tau \geq t-s)} > c > 0$$

where c can be chosen independently of u, v, w or z. Let (x, y) be a point in an irreducible set for $(\tilde{X}, \tilde{X}')$. By Lemma 2.1 we can assume that $\psi_2(x) \neq \psi_2(y)$ for some second eigenvector ψ_2 and so that there exists z so that

$$P_t(x, z) - P_t(y, z) \geq ce^{-\lambda_2 t}\,.$$

We claim that $f(u) = \sum \psi_i(u)\psi_i(z)$ is monotone for our coupling starting from (x, y) where the summation (here and in subsequent uses) is over an orthonormal basis of second eigenvectors. If not let $T = \inf\{t : \sum \psi_i(X_t)\psi_i(z) \leq \sum \psi_i(X'_t)\psi_i(z)\}$. Then by (*), $\exists c, C$ and $0 < \delta < 1$ not depending on t so that

$$P^{x,y}(T \geq \delta t \mid \tau \geq t) \leq Ce^{-ct}\,.$$

Now

$$\begin{aligned}
&P_t(x, z) - P_t(y, z) \\
&\quad = P^{(x,y)}(X_t = z) - P^{x,y}(X'_t = z) \\
&\quad = E^{x,y}\left((\chi_{X_t = z} - \chi_{X'_t = z})I_{T \leq \delta t}\right) \\
&\quad + E^{x,y}\left((\chi_{X_t = z} - \chi_{X'_t = z})I_{T > \delta t}\right)
\end{aligned}$$

but the latter term is bounded by $P(T \geq \delta t, \tau > t) \leq ce^{-\lambda_2 t}e^{-ct} = o(e^{-\lambda_2 t})$. Equally

$$E\left[(\chi_{X_t=z} - \chi_{X'_t=z})) \mid T \leq \delta t\right]$$
$$= \left(\sum \psi_i(X_T)\psi_i(z) - \sum \psi_i(X'_T)\psi_i(z)\right) e^{-\lambda_2(t-T)} - o(e^{-\lambda_3(t-T)}).$$

where $-\lambda_3$ is the smallest eigenvalue strictly below $-\lambda_2$. Provided $\lambda_3(1-\delta) > \lambda_2$ the latter term is $o(e^{-\lambda_2 t})$ while the first term is ≤ 0 by definition of T. A contradiction results.

Remark. If the Markov chain is a reflection so that two points are preserved then it is easy to see that in an efficient coupling there must be an "asymptotic monotone function" but if for the reflection θ a point i is adjacent to $\theta(i)$ then it is possible that there is no asymptotic function.

Section Three

Before presenting the examples we present a notion, the transposition property, introduced in [BK]. This version of the transposition property is slightly weaker than the one in [BK]. However, an examination of their proof that the transposition property implies the coupling is efficient shows that this implication remains valid under the weaker version. A coupling (X, X') is said to have the transposition property with respect to (x, x'), $x \neq x'$, if whenever (a, b) with $a \neq b$ is accessible for (X, X') from (x, x') then both (x, x') and (x', x) are accessible from (a, b). Also, at least one (a, b) not on the diagonal is accessible from (x, x'). Existence of the transposition property is a useful device for proving inefficiency of a coupling. [BK] used it to demonstrate the impossibility of efficient couplings for certain Markov processes. The transposition property in [BK] is the transposition property with respect to (x, x') for every $x \neq x'$. We use it in the following example which demonstrates that even on a very simple state space, no efficient coupling may exist. We noted previously that on C_3 or a line, efficient couplings always exist for nearest neighbor reversible Markov processes. We now exhibit a Markov process on C_4 for which there is no efficient coupling. This answers a question of [BK].

Example 1. We consider a reversible NN Markov chain on C_4. The jump rates are $q_{01} = 1$, $q_{12} = m$, $q_{23} = m^2$, $q_{30} = m^3$, $q_{03} = cm^6$, $q_{32} = c^{-1/3}$, $q_{21} = c^{-1/3}$, $q_{10} = c^{-1/3}$. We take $m >> c > 1$. We first list cycles which will have positive probability under any coupling, (we suggest the reader draw the square $C_4 \times C_4$ indicating these cycles)

$$A_0 = \{(1,2) \to (1,3) \to (1,0) \to (1,3)\}$$
$$A_1 = \{(2,1) \to (3,1) \to (0,1) \to (3,1)\}$$
$$A_2 = \{(0,2) \to (3,2) \to (0,2)\}$$
$$A_3 = \{(2,0) \to (2,3) \to (2,0)\}.$$

Now we observe

$$P(\{(0,3) \to (1,3), (0,3) \to (1,0)\}) > 0$$

(meaning that since $q_{30} >> q_{01}$ it may be the case that whenever X tries to jump from 0 to 1, X' may jump from 3 to 0, but that since $q_{01} >> q_{32}$ it must be possible for X to jump from 0 to 1 without X' jumping from 3 to 2) and

$$P(\{(3,0) \to (3,1), (3,0) \to (0,1)\}) > 0$$

which implies A_0 is accessible from $(0,3)$ and A_2 is accessible from $(3,0)$ under any coupling. Moreover, for any coupling,

$$P(\{(0,2) \to (1,2), (0,2) \to (1,3)\}) > 0$$
$$P(\{(2,0) \to (2,1), (2,0) \to (3,1)\}) > 0$$

which implies A_0 is accessible from A_2 and A_1 is accessible from A_3. Also, under any coupling

$$P(\{(2,3) \to (1,2), (2,3) \to (1,3), (2,3) \to (1,0)\}) > 0$$
$$P(\{(3,2) \to (2,1), (3,2) \to (3,1), (3,2) \to (0,1)\}) > 0$$

which implies A_0 is accessible from A_3 and A_1 is accessible from A_2. Finally, under any coupling

$$P(\{(1,3) \to (2,3), (1,3) \to (2,0)\}) > 0$$
$$>P(\{(3,1) \to (3,2), (3,1) \to (0,2)\})0$$

which implies A_3 is accessible from A_0 and A_2 is accessible from A_1.

Note that the above imply that both (1, 0) and (0, 1) are accessible from any point since they can be reached from any point on A_0 and A_1, respectively. This implies that any coupling has the transposition property with respect to (1, 0). Thus, no Markov coupling for this Markov process can be efficient.

Example 2. We now exhibit a coupling for a reversible, nearest neighbor Markov process on C_6 which is efficient but which does not possess an order function for all time starting from some states. The jump rates are $q_{01} = q_{10} = q_{34} = q_{43} = \varepsilon$, $q_{21} = q_{32} = q_{50} = q_{45} = 1$, $q_{12} = q_{05} = 1 + \delta$, $q_{23} = q_{54} = 1 + 2\delta$. Again, the reader will find it useful to draw a diagram of the state space C_6 with jump rates on the bonds. The coupling starts at $(x, x') = (1, 2)$. Initially, the components of (X, X') will be run as independent versions of the Markov process. Define a reflection R by $R0 = 1$, $R5 = 2$, $R4 = 3$ and $R^2 = I$ and set $y = RX'$, set $U = \inf\{t > 0 : X_t = X'_t \quad \text{or} \quad X_t = y_t\}$. If $U = T \equiv \inf\{t > 0 : X_t = X'_t\}$ then we consider them as coupled and put $X_t = X'_t$ for $t > T$. If $U < T$, then X and X' will no longer run as independent processes. If (X, X') occupies (2, 5) (or (5, 2)) they will jump to (3, 4) (or (4, 3)) or (1, 0) (or (0, 1)) synchronously. If (X, X') occupies (1, 0) (or (0, 1)) take independent, exponential random variables $\tau_{1+\delta}, \tau_\varepsilon, \tau'_\varepsilon$ with parameters $1 + \delta$, ε, ε respectively. When $\tau_{1+\delta} < \tau_\varepsilon \wedge \tau'_\varepsilon$, (X, X') jumps to (2, 5) (or (5, 2)). If $\tau_\varepsilon < \tau_{1+\delta} \wedge \tau'_\varepsilon$, X jumps onto X' and when $\tau'_\varepsilon < \tau_{1+\delta} \wedge \tau_\varepsilon$, X' jumps onto

X. In the last two cases X and X' are coupled at the jump time. The coupling behaves in an analogous way when (X, X') occupies (3, 4) (or (4, 3)).

Starting from $A = \{(0,1),(1,0),(2,5),(5,2),(3,4) \text{ or } (4,3)\}$ the above coupling is efficient since $f : C_6 \to \mathbb{R}$ defined by $f(1) = f(2) = f(3) = 1$, $f(0) = f(5) = f(4) = 0$ is an order function. Thus, if $\lambda_2 = \lambda_2(\varepsilon)$ is the second eigenvalue for this process on C_6 and $(a,b) \in A$,

$$P^{a,b}(T > t) = O(e^{\lambda_2 t}).$$

Notice that $\lambda_2(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$. Also, U can be viewed as the coupling time of an independent pair of Markov processes on $\{1,2,3\}$ with rates $q_{11} = q_{33} = \varepsilon$, $q_{12} = 1+\delta$, $q_{23} = 1+2\delta$, $q_{21} = q_{32} = 1$. Since $f(1) = 0$, $f(2) = 1$, $f(3) = 2$ provides an order function for this coupling, it must be efficient. Thus, if $\tilde{\lambda}_2 = \tilde{\lambda}_2(\varepsilon)$ is the second eigenvalue for this process

$$P^{1,2}(U > t) = O(e^{-\tilde{\lambda}_2 t}).$$

Notice that $\lim_{\varepsilon \to 0} \tilde{\lambda}_2(\varepsilon) = \tilde{\lambda}_2(0) > 0$. This all implies that

$$\begin{aligned}
P^{1,2}(T > t) &= P^{1,2}(T > t, U < T) + P^{1,2}(T > t, U = T) \\
&\leq P^{1,2}(T > t | U < T) P^{1,2}(U < T) + P^{1,2}(U > t) \\
&= E^{1,2}(P^{(X_U, X'_U)}(T \cdot \theta_U > (t - U)^+) P^{1,2}(U < T) \\
&\quad + P^{1,2}(U > t)
\end{aligned}$$

The latter term is $O(e^{-\tilde{\lambda}_2 t})$, while the former term is less than $\sum_{n \geq 1} E[P^{X_U, X'_U}(T > (t-n)(,+,).\ I_{U \in [n-1,n]}]$ which is $O(e^{-\tilde{\lambda}_2 t}) + O(e^{-\lambda_2 t})$ (assuming that λ_2 and $\tilde{\lambda}_2$ are not equal).

Now notice $P(U < T) > 0$ for all $\varepsilon > 0$ and we can select ε so small that $\tilde{\lambda}_2 > \lambda_2$. Thus, $P^{1,2}(T > t) = O(e^{-\lambda_2 t})$ and the coupling is efficient. Finally, no order function can exist for the coupling started at (1, 2). Suppose otherwise and that $f(1) < f(2)$. Since $(1,2) \to (0,2) \to (5,2) \to (4,3)$ has positive probability we must have $f(4) > f(3)$. But, the path $(1,2) \to (1,3) \to (1,4) \to (2,4) \to (3,4)$ has positive probability so we must also have $f(3) > f(4)$ which contradicts the existence of an order function for the coupling.

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