Games against a prophet for stochastic processes

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Abstract

Two players (the „prophet" and the gambler) observe a uniformly bounded stochastic process $(X_s)_{s\in S}$. The prophet's maximal expected gain $E(\sup_{s\in S} X_s)$ is compared with the maximal expected gain $\sup_\tau EX_\tau$ of the gambler who is restricted to use stopping rules τ. Games against a prophet are two-person zero-sum games where the prophet picks the distribution and the gambler chooses a stopping rule. To obtain minimax-theorems for these games one has to admit mixed or randomized stopping rules. It is shown that mixed threshold stopping rules can be used to construct saddle-points for several cases.

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1. Introduction

Prophet theory is concerned with problems of the following kind: Two players, the *prophet* and the gambler, observe a (uniformly bounded) stochastic process $X_S = (X_s)_{s\in S}$ where $S = \{1, \dots, n\}$ (*finite horizon*), $S = \mathbb{N}$ (*infinite horizon*) and $S = [a, b] \subset [0, \infty)$ (*continuous time*) are the most interesting special cases. The gambler may stop this process at any time $s \in S$. His decision, leading to the reward X_s, may take into account the previous observations $X_t, t \leq s$, but not the future ones, i.e. he is restricted to use non-anticipating stopping functions τ. The supremum over the expected

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rewards EX_τ is

$$V(X_S) := \sup\{E\ X_\tau : \tau \text{ stopping rule}\}.$$

The *prophet* has complete foresight; the supremum over his expected rewards is therefore

$$M(X_S) := E(\sup_{s\in S} X_s)$$

(in the continuous time case obviously some measurability conditions are needed). Prophet theory compares the functionals M and V; in particular, for a variety of classes of (discrete time) stochastic processes sharp inequalities for the difference $M(X_s) - V(X_S)$ and the ratio $M(X_S)/V(X_s)$ have been derived (for details we refer to the excellent review articles by Kertz (1986) and by Hill and Kertz (1992) and to the monograph by Harten, Meyerthole and Schmitz (1997)). According to the interpretation as returns of a gambler and a prophet „playing the same game" (see Kertz (1986)) it was shown (see Gdde (1991), Schmitz (1992)) that these bounds turn out to be the lower values of special zero-sum games, and game-theoretical solutions for mixed extensions of these games have been derived. Our goal is to consider games of this kind for stochastic processes:

Definition 1.1.

a) A *game against a prophet* is a zero-sum two-person game $\Gamma = (\mathcal{P}, \mathcal{T}, a)$ where $\mathcal{P}$ is a class of distributions on $(\mathbb{R}^S, \mathbb{B}^S)$, $\mathcal{T}$ ist a class of (generalized) stopping rules with respect to a filtration $(\mathcal{G}_s)_{s\in S}$, and a: $\mathcal{P} \times \mathcal{T} \to \overline{\mathbb{R}}$ is a pay-off function of the form $a(P, \tau) = \tilde{a}(E_P(\sup_{s\in S} X_s),\ E_P(X_\tau))$ where $\tilde{a}$ is increasing in the first and decreasing in the second component and $X_s, s \in S$, are $\mathcal{G}_s$-measurable random variables (r.v.) which are integrable for each $P \in \mathcal{P}$.

b) A *prophet inequality* is an upper bound for the lower value

$$W_\star(\Gamma) = \sup_{P\in\mathcal{P}} \inf_{\tau\in\mathcal{T}} a(P, \tau)$$

of a game against a prophet; a *sharp prophet inequality* is the lower value $W_\star(\Gamma)$ of a game against a prophet.

To illustrate this notion we mention two examples:

Examples 1.2.

a) The famous result, due to Krengel and Sucheston (1978), to Garling and to Hill and Kertz (1981 a)), that $M(X_1, \ldots, X_n) \leq 2 \cdot V(X_1, \ldots, X_n)$ holds true for all non-negative, independent random variables $X_1, \ldots, X_n$ and all $n \geq 2$, gives the lower values 2 for the games $\Gamma_n = (\mathcal{P}_n, \mathcal{T}_n, a)$ against a prophet where $\mathcal{P}_n = \{\prod_{i=1}^n P_i,\ P_i$ probability measure on $[0,\infty)\}$, $X_i = \pi_i$ (i-th projection), $\mathcal{G}_i = \sigma(X_1,\ldots,X_i)$, $\mathcal{T}_n = \{\tau : \tau$ stopping rules $\leq n$ w.r.t. $(\mathcal{G}_i)_{i\leq n}\}$, $a_R(P,\tau) = E_P(\max_{1\leq i\leq n} X_i)/E_P(X_\tau)$.

b) Similarly, the result by Hill and Kertz (1981 b)) that $M(X_1,\ldots,X_n) - V(X_1,\ldots,X_n) \leq (d-c)/4$ holds true for all independent, $[c,d]$-valued random variables $X_1,\ldots,X_n$ and all $n \geq 2$ gives the lower value $(d-c)/4$ for the corresponding games with the pay-off function $a_D(P,\tau) = E_P(\max_{1\leq i\leq n} X_i) - E_P(X_\tau)$. □

Moreover, a great variety of classes $\mathcal{P}$ (e.g. i.i.d. sequences, mixtures of i.i.d. sequences, discrete-time martingales, general sequences), of classes $\mathcal{T}$ (e.g. bounded stopping rules, generalized stopping rules, threshold stopping rules) and of pay-off functions a (e.g. discounted rewards, rewards with costs of observation) has been considered; see Hill and Kertz (1992) and Harten, Meyerthole and Schmitz (1997).

2. Reduction principles for the prophet

To analyze such games one will try to reduce/simplify the intricate set of strategies of the prophet. A first reduction principle is provided by a technique which is (in the special case of independent r.v.) known under different notions (*balayage* (see Boshuizen (1991)), *spreading* (see Kertz (1986)), *dilation* (see Jones (1990)), *r.v. with maximal variance* (see Badewitz (1989)):

Definition 2.1. Let Y be a r.v. on $(\Omega,\mathcal{F},P), \mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ and $a,b\in\mathbb{R}$, $a<b$. A r.v. Y_a^b s.t.

(i) $P(Y_a^b \in B|\mathcal{G}) = P(Y\in B|\mathcal{G}) \quad \forall B \in \mathbb{B}_{|[a,b]^c}$

(ii) $P(Y_a^b = a|\mathcal{G}) = E((b-Y)1_{\{Y\in[a,b]\}}|\mathcal{G})/(b-a)$

(iii) $P(Y_a^b = b|\mathcal{G}) = E((Y-a)1_{\{Y\in[a,b]\}}|\mathcal{G})/(b-a)$

is called a *balayage of Y under $\mathcal{G}$*, for $\mathcal{G} = \sigma(Z)$ also *balayage of Y under Z.*

For sufficiently large $(\Omega,\mathcal{F})$ (e.g. for $(\mathbb{R},\mathbb{B})$) there always exists a balayage. Conditions (ii), (iii) yield that the (conditional) expectations of Y and Y_a^b coincide; but for the comparison with $\mathcal{G}$-measurable r.v. the balayage Y_a^b is „better" than the original Y:

Remark 2.2. *Let X,Y be integrable r.v., $a<b, Y_a^b$ a balayage of Y under $\mathcal{G}$ and X $\mathcal{G}$-measurable. Then*

a) $E(Y_a^b|\mathcal{G}) = E(Y|\mathcal{G})\ P|\mathcal{G}$- *a.s. and, therefore,* $E(Y_a^b) = E(Y)$.

b) $E(X\vee Y|\mathcal{G}) \leq E(X\vee Y_a^b|\mathcal{G})\ P|\mathcal{G}$-*a.s., and, therefore,* $E(X\vee Y) \leq E(X\vee Y_a^b)$.

A second reduction principle is suggested by the optional sampling theorem: If arbitrary dependences are allowed one will expect that the prophet may restrict attention to (special) supermartingales.

Let $Y_i, i \in \mathbb{N}$, be the conditional value *ess sup*$\{E(X_\tau|\mathcal{F}_i) : \tau \in \mathcal{T}_i := \{\tau \geq i\}\}$ of $X_i, X_{i+1}, \ldots$ under $\mathcal{F}_i$ (see Chow/Robbins/Siegmund (1971), p. 62). Using Theorem 4.1 of Chow/Robbins/Siegmund one obtains that $(Y_i, \mathcal{F}_i)_{i\in\mathbb{N}}$ is a $[0,1]$-valued supermartingale and that $M((X_i)_{i\in\mathbb{N}}) \leq M((Y_i)_{i\in\mathbb{N}})$. Moreover, the optional sampling theorem (for supermartingales) yields $V((Y_i)_{i\in\mathbb{N}}) = E\,Y_1 = V((X_i)_{i\in\mathbb{N}})$. Applying the balayage technique leads to a further simplification:

Theorem 2.3. *Let $(X_i, \mathcal{F}_i)_{i\in\mathbb{N}}$ be a $[0,1]$-valued supermartingale (martingale). Then there exists a $[0,1]$-valued supermartingale (martingale) $(Y_i, \mathcal{G}_i)_{i\in\mathbb{N}}$ s.t.*

(i) $Y_1 = X_1$ *(hence* $V(X_i)_{i\in\mathbb{N}}) = V((Y_i)_{i\in\mathbb{N}}))$

(ii) $P(\{Y_{i+1} \geq Y_i\} \cup \{Y_{i+1} = 0\}) = 1 \quad \forall i \in \mathbb{N}$

(iii) $M((X_i)_{i\in\mathbb{N}}) \leq M((Y_i)_{i\in\mathbb{N}})$.

Proof: According to the convergence of (super-) martingales there exists, for $\mathcal{F}_\infty := \sigma(\bigcup_{i\in\mathbb{N}} \mathcal{F}_i)$, a limit r.v. X_∞ s.t. $(X_i, \mathcal{F}_i)_{i\in\mathbb{N}\cup\{\infty\}}$ is a (super-) martingale. Let $Z_\infty := (X_\infty)_0^1$ be a balayage of X_∞ under $\mathcal{F}_\infty$, $Z_i := X_i$, and define (generalized) stopping rules τ_i by $\tau_1 \equiv 1$, $\tau_{i+1} := \inf\{j > \tau_i : Z_j = 0$ or $Z_j \geq Z_{\tau_i}\}$ if $\tau_i < \infty, \infty$ else. Then $Y_i := Z_{\tau_i}, \mathcal{G}_i := \mathcal{F}_{\tau_i}$ yield a (super-) martingale with the properties (i)-(iii). □

3. Games against a prophet for martingales

a) Time-discrete martingales with finite horizon

First we consider the classes

$$\begin{aligned} \mathcal{P}_n^{\text{Mart}} &:= \{P^{X_1,\ldots,X_n} : X_1,\ldots,X_n \text{ is a } [0,1]\text{-valued martingale}\} \\ \mathcal{P}_n^{\text{SMart}} &:= \{P^{X_1,\ldots,X_n} : X_1,\ldots,X_n \text{ is a } [0,1]\text{-valued supermartingale}\} \end{aligned}$$

of (super-) martingales (with respect to the canonical filtration). The *prophet regions*

$$\Pi_n^{\text{Mart}} := \left\{ (x,y) \in \mathbb{R}^2 : \begin{array}{c} \exists\, P^{X_1,\ldots,X_n} \in \mathcal{P}_n^{\text{Mart}} s.t. \\ x = V(X_1,\ldots,X_n),\ y = M(X_1,\ldots,X_n) \end{array} \right\}$$

and the corresponding Π_n^{SMart} are known to be (see Hill and Kertz (1983), Kertz (1986))

$$\begin{aligned}\Pi_n^{\text{Mart}} &= \Pi_n^{\text{SMart}} \\ &= \{(x,y) \in [0,1]^2 : x \le y \le u_n(x) := x + (n-1)\, x(1 - x^{\frac{1}{n-1}})\}.\end{aligned}$$

It can be shown that the upper bounds of these regions are attained for special martingales which have been introduced by Dubins and Pitman (1980):

Definition 3.1. $X = (X_1, \ldots, X_n)$ is called *Dubins-Pitman-martingale for n and x* where $n \ge 2$ and $x \in (0,1)$ iff

(i) $P(X_1 = x) = 1$

(ii) $P(X_j \in \{0, x^{\frac{n-j}{n-1}}\}) = 1$

(iii) $P(X_j = x^{\frac{n-j}{n-1}} | X_{j-1} = x^{\frac{n-j+1}{n-1}}) = x^{\frac{1}{n-1}}$
$= 1 - P(X_j = 0 | X_{j-1} = x^{\frac{n-j+1}{n-1}}), 2 \le j \le n$

(iv) $P(X_j = 0 | X_{j-1} = 0) = 1,\ 2 \le j \le n.$

These Dubins-Pitman-martingales yield extremal distributions; indeed, no other extremal distributions exist:

Theorem 3.2. *(see Harten, Meyerthole and Schmitz (1997), Theorem (3.6))*
Let $n \in \mathbb{N}$ and $P^{X_1,\ldots,X_n} \in \mathcal{P}_n^{\text{SMart}}$ s.t. $EX_1 = x \in (0,1)$. Then $M(X_1,\ldots,X_n) = u_n(x)$ iff $X_1,\ldots,X_n$ is a Dubins-Pitman-martingale for n and x.

By considering tangents of the upper boundary of the prophet region one obtains sharp prophet inequalities:

Corollary 3.3. *Let $n \ge 2$, $P^{X_1,\ldots,X_n} \in \mathcal{P}_n^{\text{Mart}}$ and $\gamma \in [0,1]$. Then the sharp inequality*

$$M(X_1,\ldots,X_n) - (1-\gamma)^n \le n\,\gamma\, V(X_1,\ldots,X_n).$$

holds. In particular (for $\gamma = 1$ and $\gamma = 1/n$ resp.)

(i) $M(X_1,\ldots,X_n) \le n\, V(X_1,\ldots,X_n)$

(ii) $M(X_1,\ldots,X_n) - V(X_1,\ldots,X_n) \le \left(\frac{n-1}{n}\right)^n$,

where the equality in (i) holds only for the 0-martingale, in (ii) only for the Dubins-Pitman-martingale for n and $\left(\frac{n-1}{n}\right)^{n-1}$.

From these results we obtain some game-theoretical results:

Theorem 3.4.

a) *For $n \geq 2$ and $\gamma \in [0,1]$ the game $\Gamma_{1,\gamma} = (\mathcal{P}_n^{\text{Mart}}, T^n, a_\gamma)$ against a prophet where*

$$a_\gamma(P,\tau) := (E_P(\max_{1\leq i\leq n} X_i) - \gamma)/E_P\, X_\tau$$

is strictly determined with value $n(1-\gamma^{1/n})$. Each strategy of the gambler is a minimax-strategy; for $\gamma \neq 0$ the prophet has exactly one minimax-strategy, namely the distribution of the Dubins-Pitman-martingale for n and $\gamma^{(n-1)/n}$; for $\gamma = 0$ (ratio case) the value is n and there does not exist any minimax-strategy of the prophet.

b) *For $n \geq 2$ and $\gamma \in (0,n]$ the game $\Gamma_{2,\gamma} := (\mathcal{P}_n^{\text{Mart}}, T^n, a_\gamma)$ against a prophet where*

$$a_\gamma(P,\tau) := E_P(\max_{1\leq i\leq n} X_i) - \gamma\, E_P(X_\tau)$$

is strictly determined with value $(1-\gamma/n)^n$. Each strategy of the gambler is a minimax-strategy. The prophet has exactly one minimax-strategy, namely the distribution of the Dubins-Pitman-martingale for n and $(1-\gamma/n)^{n-1}$.

Proof (for part a); part b) is shown analogously): According to 3.3 we obtain

$$\begin{aligned} W_\star(\Gamma_{1,\gamma}) &= \sup_{P\in\mathcal{P}_n^{\text{Mart}}} \inf_{\tau\in T^n} a_\gamma(P,\tau) \\ &= \sup_{P\in\mathcal{P}_n^{\text{Mart}}} E_P(\max_{1\leq i\leq n} X_i - \gamma)/V(X_1,\ldots,X_n) = n(1-\gamma^{1/n}). \end{aligned}$$

Since the equality is attained by the Dubins-Pitman-martingale for n and $\gamma^{\frac{n-1}{n}}$, and since for each $\tau \in T^n$

$$\sup_{P\in\mathcal{P}_n^{\text{Mart}}} a_\gamma(P,\tau) = \sup_{P\in\mathcal{P}_n^{\text{Mart}}} E_P(\max_{1\leq i\leq n} X_i - \gamma)/E_P(X_1) = n(1-\gamma^{1/n}),$$

it follows $W^\star(\Gamma_{1,\gamma}) = n(1-\gamma^{1/n})$ and, therefore, the assertion. □

b) Time-discrete martingales with infinite horizon

In some aspects the infinite horizon case

$$\begin{aligned} \mathcal{P}_\infty^{\text{Mart}} &:= \{P^{(X_i)_{i\in\mathbb{N}}} : (X_i)_{i\in\mathbb{N}} \text{ is a [0,1]-valued martingale}\} \\ \mathcal{P}_\infty^{\text{SMart}} &:= \{P^{(X_i)_{i\in\mathbb{N}}} : (X_i)_{i\in\mathbb{N}} \text{ is a [0,1]-valued supermartingale}\} \end{aligned}$$

is the limiting case of the finite horizon cases – in particular the function $u(x) := x - x \ln x = \lim_{n\to\infty} u_n(x)$ plays an important role –; but there are some differences. An important fact is that the upper bound of the prophet region $\prod_\infty^{\text{Mart}}$ and $\prod_\infty^{\text{SMart}}$ is no longer attained (see Hill and Kertz (1983), Theorem 4.2):

$$\Pi_\infty^{\text{Mart}} = \Pi_\infty^{\text{SMart}} = \{(x,y) \in (0,1)^2 : x \leq y < x - x \ln x\} \cup \{(0,0),(1,1)\}.$$

By considering tangents of the boundary of $\prod_\infty^{\text{Mart}}$ one obtains:

Lemma 3.5. *Let $P^{(X_i)_{i\in\mathbb{N}}} \in \mathcal{P}_\infty^{\text{Mart}}$ and $\gamma > 0$. Then*

$$M((X_i)_{i\in\mathbb{N}}) - e^{-\gamma} < \gamma\, V((X_i)_{i\in\mathbb{N}});$$

this inequality is sharp. In particular:

(i) *For each $C \in \mathbb{R}$ there exists a [0,1]-valued martingale $(X_i)_{i\in\mathbb{N}}$ s.t. $M((X_i)_{i\in\mathbb{N}}) \geq C \cdot V((X_i)_{i\in\mathbb{N}})$*

(ii) *For $P^{(X_i)_{i\in\mathbb{N}}} \in \mathcal{P}_\infty^{\text{Mart}}$ holds the (sharp) inequality*

$$M((X_i)_{i\in\mathbb{N}}) - V((X_i)_{i\in\mathbb{N}}) < 1/e.$$

For our game-theoretical considerations we have to take into account that the a.s. finiteness of a stopping rule heavily depends on the underlying probability measure. Hence we consider generalized stopping rules τ where

$$X_\tau := \limsup_{n\to\infty} X_n \quad \text{on} \quad \{\tau = \infty\};$$

the class of these stopping rules will be denoted by $\overline{T}$. According to Theorem 4.7 of Chow/Robbins/Siegmund (1971)/Theorem 11 of Shiryayev (1978) this generalization yields no advantage for the gambler, and for uniformly integrable martingales holds $E(X_\tau) = E(X_1)\ \forall \tau \in \overline{T}$.

Whereas the ratio case (i.e. $\gamma = 0$) is, due to 3.5(i), of no game-theoretical interest we obtain for $\gamma > 0$, analogously to Theorem 3.4, a complete solution for the corresponding games against a prophet:

Theorem 3.6. *For $\gamma > 0$ the game $\Gamma_{3,\gamma} := (\mathcal{P}_\infty^{\text{Mart}}, \overline{T},\ a_\gamma)$ against a prophet where*

$$a_\gamma(P,\tau) := E_P(\sup_{i\in\mathbb{N}} X_i) - \gamma\, E_P(X_\tau)$$

is strictly determined with value $e^{-\gamma}$. Each strategy of the gambler is a minimax-strategy; there does not exist any minimax-strategy of the prophet.

c) Time-continuous martingales

In the time-continuous case ($S = [0, b]$ *horizon* b or $S = [0, \infty)$ *infinite horizon*) some additional measure-theoretical problems arise. To ensure that $\sup_{s \in S} X_S$ as well as X_τ is measurable we assume (see Meyer (1966), Ch. IV.3) that $(X_t, \mathcal{F}_t)_{t \in S}$, where $(\mathcal{F}_t)_{t \in S}$ denotes the canonical filtration, is a [0,1]-valued martingale with right-continuous paths (then the functionals $M(X_S)$ and $V(X_S)$ are defined and the optional sampling theorem yields $V(X_S) = E(X_0)$). For $\mathcal{P}^{\text{Mart}}_{[0,\infty)} = \{P^{(X_t)_{t \geq 0}} : (X_t)_{t \geq 0}$ is a $[0,1]$-valued martingale with right-continuous paths$\}$ we then obtain

Theorem 3.7. *(see Meyerthole (1995), Harten, Meyerthole and Schmitz (1997), Theorem (3.21))*
Let $P^{(X_t)_{t \geq 0}} \in \mathcal{P}^{\text{Mart}}_{[0,\infty)}$ *and* $EX_0 \in (0,1)$. *Then* $M((X_t)_{t \geq 0}) \leq EX_0 - EX_0 \ln EX_0$.

Moreover, one can show that the upper bound in 3.7 is really attained:

Theorem 3.8. *(see Harten, Meyerthole and Schmitz (1997), Lemma (3.27))*
For each $x \in (0,1)$ *there exists a* $P^{(X_t)_{t \geq 0}} \in \mathcal{P}^{\text{Mart}}_{[0,\infty)}$ *s.t.*

$$EX_0 = x \quad \textit{and} \quad E(\sup_{t \geq 0} X_t) = x - x \ln x.$$

A special stochastic process of this kind is $(B^x_{t \wedge \tau})_{t \geq 0}$ *where* $(B^x_t)_{t \geq 0}$ *is a (one-dimensional) Brownian motion with continuous paths starting in* x *and* $\tau := \inf\{t : B^x_t \in \{0,1\}\}$.

A further difference to the time-discrete case is that there exists a great variety of extremal distributions: The trivial modifications

$$\hat{B}^0_t := \begin{cases} B^0_t & t \leq t_1 \\ B^0_{t_1} \quad \text{for} & t_1 < t < t_2 \\ B^0_{t-t_2+t_1} & t \geq t_2 \end{cases}, \quad \text{where } 0 \leq t_1 < t_2,$$

of the Brownian motion $(B^0_t)_{t \geq 0}$ yield further (closely related) extremal distributions. But there are essentially different extremal distributions (due to a personal comment by F. Boshuizen): Let Z be an Exp(1)-distributed r.v., $x \in (0,1)$, and define for $t \in [0, \infty)$

$$X_t := \begin{cases} x e^t \, 1_{\{Z > t\}} & t \leq -\ln x \\ \quad\quad\quad \text{for} & \\ X_{-\ln x} & t > -\ln x. \end{cases}$$

Then $(X_t)_{t \geq 0}$ is a $[0,1]$-valued martingale with right-continuous paths and

$$EX_0 = x, \; E(\sup_{t \geq 0} X_t) = x - x \ln x;$$

the ,,discretizations" $X_0, X_{-\ln x/(n-1)}, X_{-2\ln x/(n-1)}, \dots, X_{-\ln x}$ yield the Dubins-Pitman martingales for n and x.

Combining the previous results one obtains:

Theorem 3.9. *a) For all* $b > 0$

$$\Pi^{\text{Mart}}_{[0,b]} = \Pi^{\text{Mart}}_{[0,\infty)} = \{(x,y) \in (0,1)^2 : x \le y \le x - x\ln x\} \cup \{(0,0),(1,1)\};$$

the extremal distributions are not uniquely determined.

b) For each $P^{(X_t)_{t\ge 0}} \in \mathcal{P}^{\text{Mart}}_{[0,\infty)}$ *and* $\gamma > 0$

$$M((X_t)_{t\ge 0}) - \gamma\, V((X_t)_{t\ge 0}) \le e^{-\gamma}.$$

The inequality is sharp; equality is attained.

The game-theoretical analysis of the situation is additionally complicated by the fact that even the first entrance times

$$\tau_A := \inf\{t \ge 0 : X_t \in A\}, \quad \text{where} \quad \inf\emptyset := \infty,$$

may fail to be stopping rules for *all* strategies of the prophet (see e.g. Bauer (1991), §49). On the other hand, each interesting set of strategies of the gambler should include the class

$$\mathcal{T}_C := \{\tau \equiv c : c \ge 0\}$$

of constant stopping rules. But this class is already rich enough to lead to a saddle-point theorem:

Theorem 3.10. *For each* $\gamma > 0$ *the game* $\Gamma_{4,\gamma} := (\mathcal{P}^{\text{Mart}}_{[0,\infty)}, \mathcal{T}_C,\ a_\gamma)$ *against a prophet*

where $a_\gamma(P,\tau) := E_P(\sup_{t\ge 0} X_t) - \gamma\ E_P(X_\tau)$ *is strictly determined with value* $e^{-\gamma}$. *Both players have minimax-strategies (in fact, each* $\tau \in \mathcal{T}_C$ *is a minimax-strategy of the gambler).*

Proof: The results of 3.7-3.9 yield $W_\star(\Gamma_{4,\gamma}) = e^{-\gamma}$ and the existence of minimax-strategies of the prophet. On the other hand, it follows for each $\tau \in \mathcal{T}_C$ that

$$\sup_{P \in \mathcal{P}^{\text{Mart}}_{[0,\infty)}} \big(E(\sup_{t\ge 0} X_t) - \gamma\, EX_\tau\big) = \sup_{P \in \mathcal{P}^{\text{Mart}}_{[0,\infty)}} E(\sup_{t\ge 0} X_t) - \gamma\, EX_0 = W^\star(\Gamma_{4,\gamma}).$$

This yields the minimax-property of τ and the strict determinedness of $\Gamma_{4,\gamma}$. □

It is obvious that all the results of this chapter hold true for [0,1]-valued supermartingales (with right-continuous paths) and that they may be generalized (by linear transforms) to $[c, d\,]$-valued (super) martingales.

4. Games against a prophet for general stochastic processes

The results of the chapters 2 and 3 represent the essential tools to solve the general case.

a) Time-discrete processes with finite horizon

For the classes $\mathcal{P}_n := \{P^{(X_1,\dots,X_n)} : X_1,\dots,X_n$ are [0,1]-valued r.v.$\}$ and the corresponding prophet regions Π_n one obtains (see Hill and Kertz (1983), Theorem 3.2) for each $n \in \mathbb{N}$

$$\Pi_n = \Pi_n^{\mathrm{Mart}} = \{(x,y) \in [0,1]^2 : x \le y \le x + (n-1)x(1 - x^{\frac{1}{n-1}})\}.$$

It turns out that again the Dubins-Pitman-martingales represent the only extremal distributions:

Lemma 4.1. *Let $n \in \mathbb{N}$ and $P^{X_1,\dots,X_n} \in \mathcal{P}_n$ s.t. $EX_1 = x \in (0,1)$. Then $M(X_1,\dots,X_n) = u_n(x)$ iff $X_1,\dots,X_n$ is a Dubins-Pitman-martingale for n and x.*

Proof: For the remaining part we assume that $Y_1,\dots,Y_n$ is the supermartingale which is constructed according to Chapter 2. Due to Theorem 3.2 $Y_1,\dots,Y_n$ must be a Dubins-Pitman-martingale for n and x. By the construction of $Y_1,\dots,Y_n$ we obtain $\{X_i = 0\} \supset \{Y_i = 0\}$. To show that $\{X_i = x^{\frac{n-i}{n-1}}\} \supset \{Y_i = x^{\frac{n-i}{n-1}}\}$ one may use backward induction. By the stopping rule $\tau = i$, if $X_i = x^{\frac{n-i}{n-1}}$, $i+1$ elsewhere, one obtains $P(X_i < x^{\frac{n-i}{n-1}}, Y_i = x^{\frac{n-i}{n-1}}, X_{i+1} = Y_{i+1} = x^{\frac{n-i-1}{n-1}}) = 0$. □

By considering tangents of the upper boundary one obtains, in the same way as before, sharp prophet inequalities.

But under the game-theoretical point of view there are considerable differences to the martingale case. In particular, it was shown (see Schmitz (1992)) that for the difference case a_D (see Example 1.2 b)) as well as for the ratio case a_R the games $(\mathcal{P}_n, T^n, a)$ fail to be strictly determined. According to general game theory one is, therefore, led to admit mixed or randomized stopping rules (see Irle (1990)). Indeed, it was shown by Gdde (1991) for the difference case and by Schmitz (1992) for the (modified) ratio case that this leads to strictly determined games.

b) Time-discrete processes with infinite horizon

In the same way as in part a) we obtain for

$$\mathcal{P}_\infty := \{P^{(X_i)_{i\in\mathbb{N}}} : X_i, i \in \mathbb{N}, \text{ are [0,1]-valued r.v.}\}$$

the prophet region

$$\prod\nolimits_\infty = \prod\nolimits_\infty^{\text{Mart}} = \{(x,y) \in (0,1)^2 : x \leq y < x - x\ln x\} \cup \{(0,0),(1,1)\}.$$

(see Hill and Kertz (1983), Theorem 4.2) and sharp prophet inequalities.

Under the game-theoretical point of view the same difficulties arise as for the infinite horizon martingale-case; hence we again consider generalized stopping rules. Due to (3.5)(i) the ratio case is of no game-theoretical interest. For the (generalized) difference case the situation considerably differs from the martingale case:

Theorem 4.2. *For each $\gamma > 0$ the game $\Gamma_{5,\gamma} := (\mathcal{P}_\infty, \overline{T}, a_\gamma)$ against a prophet where $a_\gamma(P,\tau) := E_P(\sup_{i\in\mathbb{N}} X_i) - \gamma\, E_P(X_\tau)$ fails to be strictly determined; the lower value is $W_\star(\Gamma_{5,\gamma}) = e^{-\gamma}$, the upper value is $W^\star(\Gamma_{5,\gamma}) = 1/(1+\gamma)$. A minimax-strategy $\tau^\star$ of the gambler is given by*

$$\tau^\star(x_1, x_2, \ldots) := \inf\{i \in \mathbb{N} : x_i \geq 1/(1+\gamma)\},\ \inf\emptyset := \infty;$$

there does not exist any minimax-strategy of the prophet.

Proof: The statements on the lower value and on the non-existence of minimax-strategies of the prophet follow from Lemma 3.5. On the other hand, one obtains for each $\tau \in \overline{T}$

$$\begin{aligned}
&\sup_{x_i\in[0,1]} (\sup_{i\in\mathbb{N}} x_i - \gamma \cdot x_\tau) \\
\geq\ &\sup_{P\in\mathcal{P}_\infty} \int_{\mathbb{R}^{\mathbb{N}}} (\sup_{i\in\mathbb{N}} x_i - \gamma\, x_\tau)\, dP^{(X_i)_{i\in\mathbb{N}}} \\
=\ &\sup_{P\in\mathcal{P}_\infty} (E_P(\sup_{i\in\mathbb{N}} X_i) - \gamma\, E_P(X_\tau)) \geq \sup_{x_i\in[0,1]} (\sup_{i\in\mathbb{N}} x_i - \gamma\, x_\tau).
\end{aligned}$$

Defining $B_\tau := \{x \in \mathbb{R} : \tau(x,\ldots) = 1\}$ and

$$(x_1^\star,\ldots) := \begin{cases} (1/(1+\gamma),1,0,\ldots) & \text{if } 1/(1+\gamma) \in B_\tau \\ (1/(1+\gamma),0,0,\ldots) & \text{if } 1/(1+\gamma) \notin B_\tau \end{cases}$$

leads to

$$\sup_{x_i\in[0,1]} (\sup_{i\in\mathbb{N}} x_i - \gamma\, x_\tau) \geq 1/(1+\gamma)$$

and, therefore, $W^\star(\Gamma_{5,\gamma}) \geq 1/(1+\gamma)$. But for each $(z_i)_{i\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$

$$\sup_{i\in\mathbb{N}} z_i - \gamma\, z_{\tau^\star} = \begin{cases} \sup_{i\in\mathbb{N}} z_i - \gamma \overline{\lim} z_i \leq 1/(1+\gamma) \text{ if all } z_i \leq 1/(1+\gamma) \\ \sup_{i\in\mathbb{N}} z_i - \gamma\, z_{k_0} \leq 1/(1+\gamma) \text{ if } k_0 := \inf\{j : z_j \geq \frac{1}{1+\gamma}\} < \infty. \end{cases}$$

Hence $W^\star(\Gamma_{5,\gamma}) = 1/(1+\gamma)$ and $\tau^\star$ is a minimax-strategy of the gambler.□

Therefore, one is again led to consider randomized stopping rules. In the difference case ($\gamma = 0$), according to Gdde (1991) a canonical candidate for an optimal strategy of the gambler seems to be $\varphi^\star = (\varphi_n^\star)_{n\in\mathbb{N}}$ where

$$\begin{aligned} \varphi_1^\star((x_i)_{i\in\mathbb{N}}) &:= \begin{cases} (x_1 - 1/e)/x_1 & \text{if } x_1 \geq 1/e \\ 0 & \text{elsewhere,} \end{cases} \\ \varphi_n^\star((x_i)_{i\in\mathbb{N}}) &:= (x_n - \max\{x_1, \ldots, x_{n-1}, 1/e\})^+/x_n \end{aligned}$$

where $0/0 := 0$ and $z^+ := \max\{0, z\}$. To show that $\varphi^\star$ is really a generalized randomized stopping rule, we mention that for each sequence $(y_i)_{i\in\mathbb{N}_0}$ s.t. $y_i \in [0,1]$ and $y_{i-1} < y_i\ \forall i \in \mathbb{N}$ holds

$$\sum_{i=1}^{\infty} \frac{y_i - y_{i-1}}{y_i} \leq \int_{y_0}^{1} \frac{1}{y} dy = -\ln y_0.$$

Applying this to the indices n_j where $\varphi_{n_j}((x_i)_{i\in\mathbb{N}}) \neq 0$, i.e. $1/e =: x_{n_0} < x_{n_1} < \ldots$, yields

$$\sum_{n=1}^{\infty} \varphi_n^\star((x_i)_{i\in\mathbb{N}}) \leq 1 \qquad \forall (x_i)_{i\in\mathbb{N}} \in [0,1]^{\mathbb{N}};$$

hence $\varphi^\star$ is a generalized randomized stopping rule.

Theorem 4.3. *The game $\Gamma_6 := (\mathcal{P}_\infty, \Phi_\infty, A)$ against a prophet where Φ_∞ denotes the class of all generalized randomized stopping rules and*

$$A(P,\varphi) := E_P(\sup_{i\in\mathbb{N}} X_i) - E_P(X_\varphi)$$

is strictly determined with value $1/e$. $\varphi^\star$ is a minimax-strategy of the gambler whereas the prophet has no minimax-strategy.

Proof: The statements on the lower value $W_\star(\Gamma_t)$ and on the non-existence of minimax-strategies of the prophet follow from Theorem 4.2. On the other hand,

$$\begin{aligned} A(P,\varphi) &= \int \sum_{j\in\overline{\mathbb{N}}} (\sup_{i\in\mathbb{N}} x_i - x_j)\ \varphi_j((x_i)_{i\in\mathbb{N}}) dP^{(X_i)_{i\in\mathbb{N}}} \\ &\leq \sup_{x_i\in[0,1]} \sum_{j\in\overline{\mathbb{N}}} (\sup_{i\in\mathbb{N}} x_i - x_j)\ \varphi_j((x_i)_{i\in\mathbb{N}}) \end{aligned}$$

where $x_\infty := \limsup_i x_i$ on $\{\varphi_\infty := 1 - \sum_{j=1}^\infty \varphi_j > 0\}$, and therefore

$$\sup_{P\in\mathcal{P}_\infty} A(P,\varphi) = \sup_{(x_i)_{i\in\mathbb{N}}} A(\delta_{(x_i)_{i\in\mathbb{N}}},\varphi).$$

It is sufficient to consider the case $\sup_{i\in\mathbb{N}} x_i > 1/e$. If $\sup_{i\in\mathbb{N}} x_i = x_k$ then

$$\begin{aligned}
A(\delta_{(x_i)_{i\in\mathbb{N}}},\varphi^\star) &\leq x_k - \sum_{j=1}^{k} x_j\, \varphi_j^\star((x_i)_{i\in\mathbb{N}}) \\
&= x_k - \sum_{r=1}^{s} x_{j_r} \frac{x_{j_r} - x_{j_{r-1}}}{x_{j_r}} \\
\text{where } 1 \leq j_1 < \ldots < j_s &\leq k \text{ are those indices} \leq k \text{ for which } \varphi_{j_r}^\star \neq 0 \\
&\quad \text{and } x_{j_0} := 1/e \\
&= x_k - x_{j_s} + x_{j_0} = 1/e.
\end{aligned}$$

If $x_n \neq \sup_{i\in\mathbb{N}} x_i = x_\infty\ \forall n \in \mathbb{N}$ we obtain in a similar way

$$\begin{aligned}
A(\delta_{(x_i)_{i\in\mathbb{N}}},\varphi^\star) &= \sum_{j=1}^{\infty}(x_\infty - x_j)\, \varphi_j^\star((x_i)_{i\in\mathbb{N}}) \\
&= \sum_{r=1}^{\infty}(x_\infty - x_{j_r})\frac{x_{j_r} - x_{j_{r-1}}}{x_{j_r}} \\
&\quad \text{where } (i_r)_{r\in\mathbb{N}} \text{ are the indices s.t. } \varphi_{i_r}^\star \neq 0 \\
&\leq x_\infty - \lim_{r\to\infty} x_{i_r} + x_{i_0} = 1/e. \qquad \square
\end{aligned}$$

As for the finite horizon case (see Gdde (1991)) there exist further minimax-strategies of the gambler.

Besides randomized stopping rules one may consider also mixed stopping rules (see Irle (1990)). Since these turn out to be important also for the time-continuous case we prove a minimax-result for this class, too. In particular, we consider mixtures of threshold stopping rules

$$\tau(c) := \{n \in \mathbb{N} : X_n \geq c\},\ \inf\emptyset := \infty$$

and identify these with their thresholds $c \in [0,1]$. Mixed threshold stopping rules may then be identified with probability measures on $([0,1], \mathbb{B}_{|[0,1]})$; let $\mathcal{G}$ denote the class of all these stopping rules.

Theorem 4.4. *For each $\gamma > 0$ the game $\Gamma_{7,\gamma} := \{\mathcal{P}_\infty, \overline{T} \cup \mathcal{G}, A_\gamma)$ against a prophet where*

$$\begin{aligned}
A_\gamma(P,\tau) &:= E_P(\sup_{i\in\mathbb{N}} X_i) - \gamma\, E_P(X_\tau) \qquad \textit{for } \tau \in \overline{T} \\
A_\gamma(P,Q) &:= E_P(\sup_{i\in\mathbb{N}} X_i) - \gamma\, E_P\Big(\int_{[0,1]} X_{\tau(c)} dQ(c)\Big) \ \textit{for } Q \in \mathcal{G},
\end{aligned}$$

is strictly determined with value $e^{-\gamma}$. The mixed threshold stopping rule $Q^\star$ defined by the Lebesgue-density $1_{[e^{-\gamma},1]}(c)/\gamma c$ is a minimax-strategy of the gambler whereas the prophet has no minimax-strategy.

Proof: Since

$$\begin{aligned} A(P,Q) &= \int_{[0,1]} (E_P(\sup_{i\in\mathbb{N}} X_i) - \gamma\, E_P(X_{\tau(c)})\, dQ(c) \\ &\geq \inf_{\tau\in\overline{T}} (E_P(\sup_{i\in\mathbb{N}} X_i) - \gamma\, E_P(X_\tau)) \end{aligned}$$

the statements on the lower value and the non-existence of a minimax-strategy of the prophet follow as before. On the other hand, we obtain for $\sup_{i\in\mathbb{N}} x_i \leq e^{-\gamma}$ that $A_\gamma(\delta_{(x_i)_{i\in\mathbb{N}}}, Q^\star) \leq e^{-\gamma}$, and for $x_\infty := \sup_{i\in\mathbb{N}} x_i > e^{-\gamma}$ that

$$A_\gamma(\delta_{(x_i)_{i\in\mathbb{N}}}, Q^\star) = x_\infty - \gamma \int_{[0,1]} x_{\tau(c)} dQ^\star(c) \leq x_\infty - \gamma \int_{e^{-\gamma}}^{x_\infty} c\, dQ^\star(c) = e^{-\gamma}.$$

This yields $W^\star(\Gamma_{7,\gamma}) \leq e^{-\gamma}$ and, therefore, the remaining part of Theorem 4.4. □

c) Time-continuous processes

Due to the same reasons as for the time-continuous martingale case we restrict attention to stochastic processes with right-continuous paths, i.e. we consider the classes

$$\mathcal{P}_{[0,\infty)} := \{P^{(X_t)_{t\geq 0}} : \begin{array}{l}(X_t)_{t\geq 0} \text{ is a [0,1]-valued stochastic} \\ \text{process with right-continuous paths}\}\end{array}$$

and (for $b > 0$)

$$\mathcal{P}_{[0,b]} := \left\{ P^{(X_t)_{t\in[0,b]}} : \begin{array}{l}(X_t)_{t\in[0,b]} \text{ is a [0,1]-valued stochastic} \\ \text{process with right-continuous paths}\end{array} \right\}$$

and the corresponding prophet region $\Pi_{[0,\infty)}$ and $\Pi_{[0,b]}$.

Theorem 4.5.

$$\begin{aligned} \Pi_{[0,\infty)} &= \Pi_{[0,b]} = \Pi^{\text{Mart}}_{[0,\infty)} \\ &= \{(x,y) \in (0,1)^2 : x \leq y \leq x - x\ln x\} \cup \{(0,0),(1,1)\} \end{aligned}$$

Proof: Obviously $\Pi^{\text{Mart}}_{[0,\infty)} \subset \Pi_{[0,\infty)}$. For the remaining part we assume that there exists a $(x,y) \in \Pi_{[0,\infty)} \backslash \Pi^{\text{Mart}}_{[0,\infty)}$, i.e. there exists a $P^{(X_t)_{t\geq 0}} \in \mathcal{P}_{[0,\infty)}$ s.t. $0 < V((X_t)_{t\geq 0}) < 1$ and

$$(\star) \qquad E(\sup_{t\geq 0} X_t) > V((X_t)_{t\geq 0}) - V((X_t)_{t\geq 0}) \ \ln\ V((X_t)_{t\geq 0}).$$

Since $\mathbb{Q}^+$ is a separating set for $(X_t)_{t\geq 0}$ we have $E(\sup_{t \in \mathbb{Q}^+_t} X_t) = E(\sup_{t\geq 0} X_t)$. Considering the ordered initial parts of a counting of $\mathbb{Q}^+$ (and using monotone convergence) we obtain from $(\star)$ a contradiction to the results of parts a), b). The case $\Pi_{[0,b]}$ is treated in an analogous way. □

A special consequence of this result is that the reduction to martingales is also possible in the time-continuous case. Moreover, one immediately obtains sharp prophet inequalities:

Corollary 4.6. *Let $P^{(X_t)_{t\geq 0}} \in \mathcal{P}_{[0,\infty)}$ and $\gamma > 0$. Then*

$$M((X_t)_{t\geq 0}) - e^{-\gamma} \leq \gamma\, V((X_t)_{t\geq 0});$$

equality is attained e.g. by a stopped Brownian motion $(B_{t\wedge\tau})_{t\geq 0}$ with start in $e^{-\gamma}$ and stopping rule $\tau := \inf\{t \geq 0 : B_t \in \{0,1\}\}$. For each $C > 0$ there exists a $P^{(X_t)_{t\geq 0}} \in \mathcal{P}_{[0,\infty)}$ s.t.

$$M((X_t)_{t\geq 0}) \geq C \ \ V((X_t)_{t\geq 0}).$$

For a game-theoretical analysis of the situation one has to ensure that each strategy of the gambler is, for each strategy of the prophet, a stopping rule. Hence we restrict attention, on the one hand, on the class $\mathcal{P}^c_{[0,\infty)}$ of stochastic processes with continuous paths and, on the other hand, to threshold stopping rules (and their mixtures); then according to Bauer (1991), Theorem 49.5, the desired property is fulfilled.

Theorem 4.7. *For $\gamma > 0$ the game $\Gamma_7 = (\mathcal{P}^c_{[0,\infty)}, \mathcal{G}, A_\gamma)$ against a prophet where*

$$A_\gamma(P,Q) := E_P(\sup_{t\geq 0} X_t) - \gamma\ E_P\Big(\int_{[0,1]} X_{\tau(c)}\, dQ(c)\Big)$$

(and hence each larger mixed extension) is strictly determined with value $e^{-\gamma}$. The stopped Brownian motion B_t with start in $e^{-\gamma}$ and stopping rule $\tau := \inf\{t \geq 0 : B_t \in \{0,1\}\}$ is a minimax-strategy of the prophet; the distribution $Q^\star$ with Lebesgue-density $1_{[e^{-\gamma},1]}(c)/\gamma\, c$ is a minimax-strategy of the gambler.

Proof: Using similar arguments as in the proof of 4.4 one can show that the stopped Brownian motion and the distribution $Q^\star$ build a saddle-point. □

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