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Zeroes of infinitely differentiable characteristic functions

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Abstract: We characterize the sets where an n-dimensional, infinitely differentiable characteristic function can have its real part zero, positive, and negative, and where it can have its imaginary part zero, positive, and negative.

1. Introduction and summary

Let $f : \mathbb{R}^n \to \mathbb{C}$ be the characteristic function of a probability distribution on $\mathbb{R}^n$. Let $A^+ \subset \mathbb{R}^n$ be the set on which $\mathrm{Re}\{F(\cdot)\}$ is strictly positive, and let A^- be the set on which $\mathrm{Re}\{F(\cdot)\}$ is strictly negative. Let B^+ be the set on which $\mathrm{Im}\{f(\cdot)\}$ is strictly positive. What can we say about the sets A^+, A^- , and B^+? Since f is continuous, A^+, A^- , and B^+ are open sets. Since $f(t) = \overline{f(-t)}$ for all $t \in \mathbb{R}^n$, we have $A^+ = -A^+, A^- = -A^-$, and $B^+ \cap (-B^+) = \emptyset$. Clearly, $A^+ \cap A^- = \emptyset$. Finally, it follows from $f(0) = 1$ that $0 \in A^+$ and $0 \notin B^+$.

This paper will show that these obviously necessary conditions on the triple (A^+, A^-, B^+) are also sufficient to insure the existence of an n-dimensional characteristic function whose real part is positive precisely on A^+ and negative precisely on A^-, and whose imaginary part is positive precisely on B^+. Furthermore, this characteristic function may be taken to be infinitely differentiable.

Let $A^0 \subset \mathbb{R}^n$ be a closed set satisfying $0 \notin A^0$ and $A^0 = -A^0$. Let $B^0 \subset \mathbb{R}^n$ be a closed set containing 0 whose complement $(B^0)^c$ can be expressed as $(B^0)^c = B^+ \cup (-B^+)$, where B^+ is an open set satisfying $B^+ \cap (-B^+) = \emptyset$. It follows immediately from the main result that there exists an n-dimensional C^∞ characteristic function whose real part is zero precisely on A^0 and whose imaginary part is zero precisely on B^0. These sufficient conditions on A^0 and B^0 are obviously necessary.

Examples of one-dimensional characteristic functions with compact support are well known. However, the usual examples, and all those obtainable from the famous sufficient condition of Polya (see Theorem 6.5.3 of Chung (1974)) are not differentiable at zero, and the authors are not aware of any previously published examples of C^∞ characteristic functions with compact support.

2. Construction of the characteristic functions $g_{1,n}$ and $g_{2,n}$

For $x \in \mathbb{R}, x \neq 0$, define

$$r(x) = \frac{6}{x^2}\left(1 - \frac{\sin x}{x}\right).$$

Let $r(0) = 1$, so that r is continuous.

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Lemma 1. *The characteristic function of the probability density* $(3/2)\{(1-|t|)^+\}^2$ *is* r.

Proof. Direct calculation. $\square$

Lemma 2. *The function* r *is unimodal and positive.*

Proof. Since r is symmetric and since $r(0) = 1$ and $\lim_{x\to\infty} r(x) = 0$, it will suffice to prove that the first derivative $r'(\cdot)$ has no zeroes for $x \in (0, \infty)$. But

$$r'(x) = -\frac{6}{x^4}\big[(2+\cos x)x - 3\sin x\big],$$

so that it will suffice to prove that $w(\cdot)$ defined by

$$w(x) = (2+\cos x)x - 3\sin x$$

has no zeroes on $(0, \infty)$. It is easy to see that $w(x)$ is positive for $x \geq \pi$. To take care of $x \in (0, \pi)$, note that

$$\begin{aligned} w'(x) &= 2 - 2\cos x - x\sin x \\ w''(x) &= \sin x - x\cos x \\ w'''(x) &= x\sin x \end{aligned}$$

The third derivative $w'''(x)$ is positive for $x \in (0, \pi)$. Since $w''(0) = w'(0) = w(0) = 0$, it follows that $w(x)$ is positive for $x \in (0, \pi)$, and we are done. $\square$

Let $X_1, X_2, \ldots$ be $\ldots$ random variables with density $(3/2)\{(1-|t|)^+\}^2$. Define

$$S_1 = \sum_{k=1}^{\infty} X_k/k^2 \qquad \text{and} \qquad S_2 = \sum_{k=1}^{\infty} X_k/k^4.$$

Let h_1 be the density of S_1, and let h_2 be the density of S_2. Since $\sum_{k=1}^{\infty} k^{-2} = \pi^2/6$, the density h_1 is positive precisely on the interval $(-\pi^2/6, \pi^2/6)$. Likewise, since $\sum_{k=1}^{\infty} k^{-4} = \pi^4/90$, h_2 is positive precisely on $(-\pi^4/90, \pi^4/90)$.

It follows from Lemma 1 that the characteristic functions of S_1 and S_2 are given by

$$q_1(x) = \prod_{k=1}^{\infty} r\big(x/k^2\big) \qquad \text{and} \qquad q_2(x) = \prod_{k=1}^{\infty} r\big(x/k^4\big),$$

respectively.

By the Fourier inversion theorem (see the corollary on p. 155 of Chung (1974)),

$$h_j(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt} q_j(x)\,dx,$$

for $j = 1, 2$. Setting $t = 0$ yields

$$2\pi h_j(0) = \int_{-\infty}^{\infty} q_j(x)\,dx.$$

Thus, $\tilde{p}_j(\cdot)$ defined by

$$\tilde{p}_j(\cdot) = \frac{q_j(x)}{2\pi h_j(0)}$$

is a probability density with characteristic function given by

$$\tilde{g}_j(t) = h_j(t)/h_j(0), \qquad j = 1, 2.$$

Obviously, $\tilde{g}_1$ and $\tilde{g}_2$ are positive precisely on $(-\pi^2/6, \pi^2/6)$ and $(-\pi^4/90, \pi^4/90)$, respectively. Since $r(\cdot)$ is symmetric about 0 and unimodal, $\tilde{p}_1$ and $\tilde{p}_2$ are also symmetric and unimodal. From the definitions of $r(\cdot)$ and $q_j(\cdot)$ above, it is easy to see that

$$\lim_{x\to\infty} x^m \tilde{p}_j(x) = 0$$

for $j = 1, 2$ and for all $m > 0$. Thus, the densities $\tilde{p}_1$ and $\tilde{p}_2$ have all moments. It follows that $\tilde{g}_1$ and $\tilde{g}_2$ are C^∞. (See Theorem 6.4.1 of Chung (1974)). Finally, we need to show that the tails of $\tilde{p}_2$ are fatter than those of $\tilde{p}_1$ in the sense that, for each real $a > 0$,

$$\lim_{x\to\infty} \frac{\tilde{p}_1(ax)}{\tilde{p}_2(x)} = 0. \tag{2.1}$$

To do this, it will suffice to show that

$$\lim_{x\to\infty} \frac{q_1(ax)}{q_2(x)} = 0. \tag{2.2}$$

If $b, c > 0$, then obviously $\frac{r(bx)}{r(cx)} \to \frac{c^2}{b^2}$ as $x \to \infty$. Also, if $b > c > 0$, then $0 < \frac{r(bx)}{r(cx)} \leq 1$ for all $x \in \mathbb{R}$, by Lemma 2. But

$$\frac{q_1(ax)}{q_2(x)} = \prod_{k=1}^{\infty} \frac{r(ax/k^2)}{r(x/k^4)},$$

and the kth factor converges to $(a^2k^4)^{-1}$. There are only finitely many k's for which $(a^2k^4)^{-1} \geq 1$. If $(a^2k^4)^{-1} < 1$, then $0 < \frac{r(ax/k^2)}{r(x/k^4)} \leq 1$ for all x, and the limiting value $(a^2k^4)^{-1}$ can be made arbitrarily small by choosing k sufficiently large. This suffices to prove (2.2) and hence (2.1).

Define $g_1, g_2, p_1,$ and p_2 by rescaling $\tilde{g}_1, \tilde{g}_2, \tilde{p}_1,$ and $\tilde{p}_2$ as follows.

$$\begin{aligned} g_1(t) &= \tilde{g}_1\big(\pi^2 t/6\big) & g_2(t) &= \tilde{g}_2\big(\pi^4 t/90\big) \\ p_1(x) &= \big(6/\pi^2\big)\tilde{p}_1\big(6x/\pi^2\big) & p_2(x) &= \big(90/\pi^2\big)\tilde{p}_2\big(90x/\pi^4\big). \end{aligned}$$

Our results for $\tilde{g}_1, \tilde{g}_2, \tilde{p}_1,$ and $\tilde{p}_2$ imply the results for $g_1, g_2, p_1,$ and p_2 given in the following lemma.

Lemma 3. *The functions g_1 and g_2 defined above are real-valued, nonnegative, C^∞ characteristic functions which are positive precisely on $(-1, 1)$. The corresponding probability densities p_1 and p_2 are unimodal, and the tails of p_2 are fatter then those of p_1 in the sense that, for every $a > 0$, $\underset{x\to\infty}{lim} \frac{p_1(ax)}{p_2(x)} = 0$.*

In order to prove our main theorem, we will need an n-dimensional analog of Lemma 3. For the remainder of this paper, t and x will denote points in $\mathbb{R}^n$ with respective coordinates t_i and $x_i, i = 1, \ldots, n$.

For j=1 and 2, let $\mathbf{Y}_j$ be a random vector in $\mathbb{R}^n$ whose coordinates are i.i.d. random variables with density p_j. Then $\mathbf{Y}_j$ has density

$$\hat{p}_{j,n}(x) = \prod_{i=1}^{n} p_j(x_i)$$

and characteristic function

$$\hat{g}_{j,n}(t) = \prod_{i=1}^{n} g_j(t_i).$$

Let M be a random $n \times n$ orthogonal matrix (with the normalized Haar measure on the group of $n \times n$ orthogonal matrices as its probability distribution), and suppose M is independent of $\mathbf{Y}_j$. Then $\mathbf{Z}_j = M\mathbf{Y}_j$ is a spherically symmetric random vector in $\mathbb{R}^n$ with density

$$\tilde{p}_{j,n}(x) = \int_{S^{n-1}} \hat{p}_{j,n}\big(\|x\|u\big) d\upsilon(u),$$

where $S^{n-1} = \{t \in \mathbb{R}^n : \|t\| = 1\}$ is the unit sphere in $\mathbb{R}^n$, and υ is the rotation invariant probability measure on S^{n-1}. The characteristic function of $\mathbf{Z}_j$ is

$$\tilde{g}_{j,n}(t) = \int_{S^{n-1}} \hat{g}_{j,n}\big(\|t\|u\big) d\upsilon(u),$$

which is C^∞ and is positive precisely on $\{t \in \mathbb{R}^n : \|t\| < \sqrt{n}\}$. For j=1 and 2, let

$$g_{j,n^{(t)}} = \tilde{g}_{j,n}\big(\sqrt{n}t\big) \tag{2.3}$$

and

$$p_{j,n}(x) = n^{-1/2}\tilde{p}_{j,n}\big(n^{-1/2}x\big). \tag{2.4}$$

The following lemma gives us the results we need to prove the main theorem.

Lemma 4. *The functions $g_{1,n}$ and $g_{2,n}$ defined above are real-valued, nonnegative, C^∞ characteristic functions which are positive precisely on $\{t \in R^n : \|t\| < 1\}$. For each $a > 0$, there is a constant $L(a)$ such that the corresponding densities functions $p_{1,n}$ and $p_{2,n}$ satisfy*

$$p_{1,n}(ax) < L(a)p_{2,n}(x)$$

for all $x \in \mathbb{R}^n$.

Proof. Only the second assertion remains to be proved. Fix $a > 0$. It follows from Lemma 3 that there exists a number $K(a) > 0$ such that $p_1(ax_1) < K(a)p_2(x_1)$ for all $x_1 \in \mathbb{R}$. Thus

$$\hat{p}_{1,n}(ax) = \prod_{i=1}^{n} p_1(ax_i) < K^n(a)\prod_{i=1}^{n} p_2(x_i) = K^n(a)\hat{p}_{2,n}(x)$$

Furthermore,

$$\begin{aligned} \tilde{p}_{1,n}(ax) &= \int_{S^{n-1}} \hat{p}_{1,n}(a\|x\|u)\, d\upsilon(u) < K^n(a)\int_{S^{n-1}} \hat{p}_{2,n}(\|x\|u)\, d\upsilon(u) \\ &= K^n(a)\tilde{p}_{2,n}(x). \end{aligned}$$

Let $L(a) = K^n(a)$. Then it follows from (2.4) that $p_{1,n}(ax) < L(a)p_{2,n}(x)$ for all $x \in \mathbb{R}^n$. □

Remark. It is not hard to show that the spherically symmetric densities $p_{1,n}$ and $p_{2,n}$ are unimodal, and that, for each $a > 0$, they satisfy

$$\lim_{\|x\|\to\infty} \frac{p_{1,n}(ax)}{p_{2,n}(x)} = 0.$$

We will only need the facts given in Lemma 4, however.

3. The main theorem

Theorem. *Let A^+, A^-, and B^+ be open subsets of R^n satisfying $A^+ = -A^+, A^- = -A^-, B^+ \bigcap (-B^+) = \emptyset, A^+ \bigcap A^- = \emptyset, 0 \in A^+$, and $0 \notin B^+$. Then there exists an infinitely differentiable characteristic function f on $\mathbb{R}^n$ satisfying*

$$\begin{aligned} A^+ &= \{t \in \mathbb{R}^n : \mathrm{Re}(f(t)) > 0\} \\ A^- &= \{t \in \mathbb{R}^n : \mathrm{Re}(f(t)) < 0\} \\ B^+ &= \{t \in \mathbb{R}^n : \mathrm{Im}(f(t)) > 0\}. \end{aligned}$$

Proof. For $c \in \mathbb{R}^n$ and r a positive constant, let

$$B_r(c) = \{t \in \mathbb{R}^n : \|t - c\| < r\}$$

be the open ball in $\mathbb{R}^n$ with center c and radius r. We may assume without loss of generality that $B_1(0) \subset A^+$. Define

$$\tilde{A}^+ = A^+ \bigcap \{t \in \mathbb{R}^n : \|t\| > 1/2\}.$$

Since $\tilde{A}^+$ is open, it is the union of a countable set $\{B_{r_i}(c_i)\}_{i=1}^{\infty}$ of open balls. Since $\tilde{A}^+ = -\tilde{A}^+$, we have $B_{r_i}(-c_i) \subset \tilde{A}^+$ for all i. Define

$$f_i^+(t) = g_{1,n}\{(t - c_i)/r_i\} + g_{1,n}\{(t + c_i)/r_i\}.$$

By Lemma 4, f_i^+ is positive precisely on $B_{r_i}(c_i) \bigcup B_{r_i}(-c_i)$. Taking a Fourier transform yields

$$\begin{aligned} (2\pi)^{-n} \int_{\mathbb{R}^\ltimes} e^{-i(x \cdot t)} f_i^+(t)\, dt &= \{e^{-i(x \cdot c_i)} + e^{i(x \cdot c_i)}\} r_i p_{1,n}(r_i x) \\ &= 2 r_i \cos(x \cdot c_i) p_{1,n}(r_i x) \end{aligned}$$

(See Theorem 7.7(c) of Rudin (1973)).

Let $\{\alpha_i\}_{i=1}^{\infty}$ be a sequence of positive constants satisfying $\alpha_i < 2^{-i-2} \times \{2 r_i L(r_i)\}^{-1}$. Then

$$\left| (2\pi)^{-n} \int_{R^n} e^{-i(x \cdot t)} \sum_{i=1}^{\infty} \alpha_i f_i^+(t)\, dt \right| < \sum_{i=1}^{\infty} 2^{-i-2} \{L(r_i)\}^{-1} p_{1,n}(r_i x) < \frac{1}{4} p_{2,n}(x).$$

Furthermore, by choosing the α_i's to converge to zero sufficiently fast, we can insure that $f^+(\cdot)$ defined by

$$f^+(t) = \sum_{i=1}^{\infty} \alpha_i f_i^+(t)$$

is C^∞ and in $L^1(\mathbb{R}^n)$. Note that the real-valued, nonnegative function $f^+(\cdot)$ is nonzero precisely on A^+.

Let $\{B_{r_i'}(c_i')\}_{i=1}^{\infty}$ be a sequence of open balls whose union is A^-, and let

$$f_i^-(t) = -g_{1,n}\{(t - c_i')/r_i'\} - g_{1,n}\{(t + c_i')/r_i'\}.$$

The same argument used above shows that we can choose a sequence of positive constants $\{\beta_i\}_{i=1}^{\infty}$ such that $f^-(\cdot)$ defined by

$$f^-(t) = \sum_{i=1}^{\infty} \beta_i f_i^-(t)$$

is C^∞, in $L^1(\mathbb{R}^n)$, and satisfies

$$\left|(2\pi)^{-n}\int_{\mathbb{R}^{\ltimes}} e^{-(x\cdot t)} f^-(t)\,dt\right| < \frac{1}{4}p_{2,n}(x).$$

Note that the real-valued, nonpositive function $f^-(\cdot)$ is nonzero precisely on A^-.

Let $\{B_{r_i''}(c_i'')\}_{i=1}^\infty$ be a sequence of open balls whose union is B^+. Let

$$f_i^{im}(t) = i\big[g_{i,n}\big\{\big(t-c_i''\big)/r_i''\big\} - g_{1,n}\big\{\big(t+c_i''\big)/r_i''\big\}\big]$$

Then

$$\begin{aligned}(2\pi)^{-n}\int_{\mathbb{R}^{\ltimes}} e^{-i(x\cdot t)} f_i^{im}(t)\,dt &= \big\{e^{-i(x\cdot c_i'')} - e^{i(x\cdot c_i'')}\big\} r_i'' p_{1,n}\big(r_i'' x\big)\\ &= -2r_i''\sin\big(x\cdot c_i''\big)p_{1,n}\big(r_i'' x\big)\end{aligned}$$

Again, we can choose a sequence of positive constants $\{\gamma_i\}_{i=1}$ so that $f^{im}(\cdot)$ defined by

$$f^{im}(t) = \sum_{i=1}^{\infty}\gamma_i f_i^{im}(t)$$

is C^∞, in $L^1(\mathbb{R}^n)$, and satisfies

$$\left|(2\pi)^{-n}\int_{\mathbb{R}^{\ltimes}} e^{-i(x\cdot t)} f^{im}(t)\,dt\right| < \frac{1}{4}p_{2,n}(x).$$

Note that the function $f^{im}(\cdot)$ is pure imaginary, and that its imaginary part is positive precisely on B^+.

Now let

$$f(t) = g_{2,n}(t) + f^+(t) + f^-(t) + f^{im}(t).$$

Clearly the real and imaginary parts of f are positive and negative on the proper sets. The function f is C^∞, and in $L^1(\mathbb{R}^n)$.

Define

$$p(x) = (2\pi)^{-n}\int_{\mathbb{R}^{\ltimes}} e^{-i(x\cdot t)} f(t)\,dt.$$

Since

$$\left|(2\pi)^{-n}\int_{\mathbb{R}^{\ltimes}} e^{-i(x\cdot t)}\big(f^+(t)+f^-(t)+f^{im}(t)\big)\,dt\right| < \frac{3}{4}p_{2,n}(x),$$

and

$$(2\pi)^{-n}\int_{\mathbb{R}^{\ltimes}} e^{-i(x\cdot t)} g_{2,n}(t)\,dt = p_{2,n}(x),$$

we have

$$\frac{1}{4}p_{2,n}(x) < p(x) < 2p_{2,n}(x).$$

By the Fourier inversion theorem (again, see Theorem 7.7(c) of Rudin (1973)),

$$f(t) = \int_{\mathbb{R}^{\ltimes}} e^{i(x\cdot t)} p(x)\,dx.$$

Also, since $f(0) = g_{2,n}(0) = 1$, we have

$$\int_{\mathbb{R}^{\ltimes}} p(x)\,dx = f(0) = 1.$$

Thus, $f$is the characteristic function of the probability density p, and f satisfies all the requirements of the theorem. □

Addendum

Except for slight corrections, the present paper was completed in 1984. Results very similar to the one-dimensional version of our main theorem appear in Sasvári (1985).

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