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Chebyshev polynomials and G-distributed functions of F-distributed variables

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Abstract: We address a more general version of a classic question in probability theory. Suppose $\mathbf{X} \sim \mathbf{N_p}(\mu, \mathbf{\Sigma})$. What functions of $\mathbf{X}$ also have the $N_p(\mu, \Sigma)$ distribution? For $p = 1$, we give a general result on functions that cannot have this special property. On the other hand, for the $p = 2, 3$ cases, we give a family of new nonlinear and non-analytic functions with this property by using the Chebyshev polynomials of the first, second and the third kind. As a consequence, a family of rational functions of a Cauchy-distributed variable are seen to be also Cauchy distributed. Also, with three i.i.d. $N(0, 1)$ variables, we provide a family of functions of them each of which is distributed as the symmetric stable law with exponent $\frac{1}{2}$. The article starts with a result with astronomical origin on the reciprocal of the square root of an infinite sum of nonlinear functions of normal variables being also normally distributed; this result, aside from its astronomical interest, illustrates the complexity of functions of normal variables that can also be normally distributed.

1. Introduction

It is a pleasure for both of us to be writing to honor Herman. We have known and admired Herman for as long as we can remember. This particular topic is close to Herman's heart; he has given us many cute facts over the years. Here are some to him in reciprocation.

Suppose a real random variable $X \sim N(\mu, \sigma^2)$. What functions of X are also normally distributed? In the one dimensional case, an analytic map other than the linear ones cannot also be normally distributed; in higher dimensions, this is not true. Also, it is not possible for any one-to-one map other than the linear ones to be normally distributed. Textbook examples show that in the one dimensional case nonlinear functions $U(X)$, not analytic or one-to-one, can be normally distributed if X is normally distributed; for example, if $Z \sim N(0, 1)$ and Φ denotes the $N(0, 1)$ CDF, then, trivially, $U(Z) = \Phi^{-1}(2\Phi(|Z|) - 1)$ is also distributed as $N(0, 1)$. Note that this function $U(.)$ is not one-to-one; neither is it analytic.

One of the present authors pointed out the interesting fact that if X, Y are i.i.d. $N(0, 1)$, then the nonlinear functions $U(X, Y) = \frac{2XY}{\sqrt{X^2+Y^2}}$ and $V(X, Y) = \frac{X^2-Y^2}{\sqrt{X^2+Y^2}}$ are also i.i.d. $N(0, 1)$-distributed (see Shepp (1962), Feller (1966)). These are obviously nonlinear and not one-to-one functions of X, Y. We present a collection of new pairs of functions $U(X, Y), V(X, Y)$ that are i.i.d. $N(0, 1)$-distributed. The functions $U(X, Y), V(X, Y)$ are constructed by using the sequence of Chebyshev polynomials of the first, second and the third kind and the terrain corresponding to the plots of $U(X, Y), V(X, Y)$ gets increasingly more rugged, and yet with a visual regularity, as one progresses up the hierarchy. Certain other results about

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Cauchy-distributed functions of a Cauchy-distributed variable and solutions of certain Fredholm integral equations follow as corollaries to these functions U, V being i.i.d. $N(0,1)$ distributed, which we point out briefly as a matter of fact of some additional potential interest. Using the family of functions $U(X,Y), V(X,Y)$, we also provide a family of functions $f(X,Y,Z), g(X,Y,Z), h(X,Y,Z)$ such that f, g, h are i.i.d. $N(0,1)$ if X, Y, Z are i.i.d. $N(0,1)$. The article ends with a family of functions of three i.i.d. $N(0,1)$ variables, each distributed as a symmetric stable law with exponent $\frac{1}{2}$; the construction uses the Chebyshev polynomials once again.

We start with an interesting example with astronomical origin of the reciprocal of the square root of an infinite sum of dependent nonlinear functions of normally distributed variables being distributed as a normal again. The result also is relevant in the study of total signal received at a telephone base station when a fraction of the signal emitted by each wireless telephone gets lost due to various interferences. See Heath and Shepp (2003) for description of both the astronomical and the telephone signal problem. Besides the quite curious fact that it should be normally distributed at all, this result illustrates the complexity of functions of normal variables that can also be normally distributed.

2. Normal function of an infinite i.i.d. $N(0,1)$ sequence: An astronomy example

Proposition 1. *Suppose $\eta_0, \eta_1, \eta_2, \dots$ is a sequence of i.i.d. $N(0,1)$ variables. We show the following remarkable fact: let $S_n = \sum_{k=1}^{2n} \eta_k^2$. Then*

$$N = \frac{\operatorname{sgn}(\eta_0)}{\sqrt{\sum_{n=1}^{\infty} \frac{1}{S_n^2}}} \sim N\left(0, \frac{8}{\pi}\right).$$

The problem has an astronomical origin. Consider a fixed plane and suppose stars are distributed in the plane according to a homogeneous Poisson process with intensity λ; assume λ to be 1 for convenience. Suppose now that each star emits a constant amount of radiation, say a unit amount, and that an amount inversely proportional to some power k of the star's distance from a fixed point (say the origin) reaches the point.If $k = 4$, then the total amount of light reaching the origin would equal $L = \pi^2 \sum_{n=1}^{\infty} \frac{1}{(\gamma_1+\gamma_2+\cdots+\gamma_n)^2}$, where the γ_i are i.i.d. standard exponentials, because if the ordered distances of the stars from the origin are denoted by $R_1 < R_2 < R_3 < \dots$, then $R_n^2 \sim \frac{1}{\pi}(\gamma_1 + \gamma_2 + \cdots + \gamma_n)$, where the γ_i are i.i.d. standard exponentials. Since the sum of squares of two i.i.d. standard normals is an exponential with mean 2, it follows that L has the same distribution as $\frac{4\pi^2}{N^2}$, where N is as in Proposition 1 above. In particular, L does not have a finite mean. Earlier contributions to this problem are due to Chandrasekhar, Cox, and others; for detailed references, see Heath and Shepp (2003).

To prove the Proposition, we will show the following two facts:

(a) The Laplace transform of $\sum_{n=1}^{\infty} \frac{1}{R_n^4}$ equals $Ee^{-\lambda \sum_{n=1}^{\infty} \frac{1}{R_n^4}} = e^{-\pi^{\frac{3}{2}}\sqrt{\lambda}}$.

(b) If $\eta \sim N(0,1)$, then the Laplace transform of $\frac{1}{\eta^2}$ equals $e^{-\sqrt{2\lambda}}$.

To prove (a), consider the more general Laplace transform of the sum of the fourth powers of the reciprocals of $R \in S$ only for $0 < a < R < b$, where a, b are fixed, $\phi(\lambda, a, b) = Ee^{-\lambda \sum_{\{a<R<b\}} \frac{1}{R^4}}$. We want $\phi(\lambda, 0, \infty)$, but we can write the

"recurrence" relation:

$$\phi(\lambda, a, b) = e^{-\pi(b^2-a^2)} + \int_a^b e^{-\pi(r^2-a^2)}\phi(\lambda, r, b)e^{-\lambda r^{-4}} 2\pi r\, dr$$

where the first term considers the possibility that there are no points of S in the annulus $a < r < b$ and the integral is written by summing over the location of the point in the annulus with the smallest value of $R = r$ and then using the independence properties of the Poisson random set.

Now multiply both sides by $e^{-\pi a^2}$ and differentiate on a, regarding both b and λ as fixed constants, to get

$$\big(-2\pi a\phi(\lambda, a, b) + \phi'(\lambda, a, b)\big)e^{-\pi a^2} = -2\pi a e^{-\pi a^2}\phi(\lambda, a, b)e^{-\lambda a^{-4}}.$$

Dividing by $e^{-\pi a^2}$ and solving the simple differential equation for $\phi(\lambda, a, b)$, we get,

$$\phi(\lambda, a, b) = \phi(\lambda, 0, b)e^{2\pi \int_0^a (1-e^{-\lambda u^{-4}})u du}.$$

Since $\phi(\lambda, b, b) = 1$, we find that

$$\phi(\lambda, 0, b) = e^{-2\pi \int_0^b (1-e^{-\lambda u^{-4}})u\, du}.$$

Finally let $b \to \infty$ to obtain $\phi(\lambda, 0, \infty)$ as was desired. Evaluating the integral by changing $u = t^{-\frac{1}{4}}$ and integration by parts, gives the answer stated in (a).

(b) can be proved by direct calculation, but a better way to see this is to use the fact that the hitting time, τ_1, of level one by a standard Brownian motion, $W(t), t \geq 0$, has the same distribution as η^{-2} using the reflection principle,

$$\begin{aligned} P(\tau_1 < t) &= P\big(\max W(u), u \in [0, t] > 1\big) = 2P\big(W(t) > 1\big) = P\big(\sqrt{t}|\eta| > 1\big) \\ &= P\big(\eta^{-2} < t\big). \end{aligned}$$

Finally, Wald's identity

$$Ee^{\lambda W(\tau_1) - \frac{\lambda^2}{2}\tau_1} = 1, \lambda > 0,$$

and the fact that $W(\tau_1) = 1$ gives the Laplace transform of τ_1 and hence also of η^{-2}, as

$$Ee^{-\lambda\eta^{-2}} = Ee^{-\lambda\tau_1} = e^{-\sqrt{2\lambda}}.$$

This completes the proof of Proposition 1 and illustrates the complexity of functions of normal variables that can also be normally distributed.

3. Chebyshev polynomials and normal functions

3.1. A general result

First we give a general result on large classes of functions of a random variable Z that *cannot* have the same distribution as that of Z. The result is much more general than the special case of Z being normal.

Proposition 2. *Let Z have a density that is symmetric, bounded, continuous, and everywhere strictly positive. If $f(Z) \neq \pm Z$ is either one-to-one, or has a zero derivative at some point and has a uniformly bounded derivative of some order $r \geq 2$, then $f(Z)$ cannot have the same distribution as Z.*

Proof. It is obvious that if $f(z)$ is one-to-one then Z and $f(Z)$ cannot have the same distribution under the stated conditions on the density of Z, unless $f(z) = \pm z$.

Consider now the case that $f(z)$ has a zero derivative at some point; let us take this point to be 0 for notational convenience. Let us also suppose that $|f^{(r)}(z)| \leq K$ for all z, for some $K < \infty$. Suppose such a function $f(Z)$ has the same distribution as Z.

Denote $f(0) = \alpha$; then $P(|f(Z) - \alpha| \leq \epsilon) = P(|Z - \alpha| \leq \epsilon) \leq c_1\epsilon$ for some $c_1 < \infty$, because of the boundedness assumption on the density of Z.

On the other hand, by a Taylor expansion around 0, $f(z) = \alpha + \frac{z^2}{2} f''(0) + \cdots + \frac{z^r}{r!} f^{(r)}(z^*)$, at some point between 0 and z. By the uniform boundedness condition on $f^{(r)}(z)$, from here, one has $P(|f(Z) - \alpha| \leq \epsilon) \geq P(a_2|Z|^2 + a_3|Z|^3 + \cdots a_r|Z|^r \leq \epsilon)$, for some fixed positive constants $a_2, a_3, \ldots, a_r$. For sufficiently small $\epsilon > 0$, this implies that $P(|f(Z) - \alpha| \leq \epsilon) \geq P(M|Z|^2 \leq \epsilon)$, for a suitable positive constant M.

However, $P(M|Z|^2 \leq \epsilon) \geq c_2\sqrt{\epsilon}$ for some $0 < c_2 < \infty$, due to the strict positivity and continuity of the density of Z. This will contradict the first bound $P(|f(Z) - \alpha| \leq \epsilon) \leq c_1\epsilon$ for small ϵ, hence completing the proof. □

3.2. Normal functions of two i.i.d. $N(0,1)$ *variables*

Following standard notation, let $T_n(x), U_n(x)$ and $V_n(x)$ denote the nth Chebyshev polynomial of the first, second and third kind. Then for all $n \geq 1$, the pairs of functions (Z_n, W_n) in the following result are i.i.d. $N(0,1)$ distributed.

Proposition 3. *Let* $X, Y \overset{i.i.d.}{\sim} N(0,1)$. *For* $n \geq 1$, *let*

$$Z_n = YU_{n-1}\left(\frac{X}{\sqrt{X^2+Y^2}}\right), \qquad \textit{and}$$
$$W_n = \sqrt{X^2+Y^2}\,T_n\left(\frac{X}{\sqrt{X^2+Y^2}}\right).$$

Then, $Z_n, W_n \overset{i.i.d.}{\sim} N(0,1)$.

There is nothing special about X, Y being i.i.d. By taking a bivariate normal vector, orthogonalizing it to a pair of i.i.d. normals, applying Proposition 3 to the i.i.d. pair, and then finally retransforming to the bivariate normal again, one similarly finds nonlinear functions of a bivariate normal that have exactly the same bivariate normal distribution as well. Here is a formal statement.

Corollary 1. *Suppose* $(X_1, X_2) \sim N(0,0,1,1,\rho)$. *Then, for all* $n \geq 1$, *the pairs of functions* (Y_{1n}, Y_{2n}) *defined as*

$$\begin{aligned} Y_{1n} &= X_2 U_{n-1}\left(\frac{X_1 - \rho X_2}{\sqrt{X_1^2 + (1+\rho^2)X_2^2 - 2\rho X_1 X_2}}\right), \\ Y_{2n} &= \rho Y_{1n} + \sqrt{1-\rho^2}\sqrt{X_1^2 + (1+\rho^2)X_2^2 - 2\rho X_1 X_2}\,T_n \\ &\quad \times \left(\frac{X_1 - \rho X_2}{\sqrt{X_1^2 + (1+\rho^2)X_2^2 - 2\rho X_1 X_2}}\right), \end{aligned}$$

are also distributed as $N(0,0,1,1,\rho)$.

The first few members of the polynomials $T_n(x), U_n(x)$ are $T_1(x) = x, T_2(x) = 2x^2 - 1, T_3(x) = 4x^3 - 3x, T_4(x) = 8x^4 - 8x^2 + 1, T_5(x) = 16x^5 - 20x^3 + 5x, T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$, and $U_0(x) = 1, U_1(x) = 2x, U_2(x) = 4x^2 - 1, U_3(x) = 8x^3 - 4x, U_4(x) = 16x^4 - 12x^2 + 1, U_5(x) = 32x^5 - 32x^3 + 6x$; see, e.g, Mason and Handscomb (2003). Plugging these into the formulae for Z_n and W_n in Proposition 3, the following illustrative pairs of i.i.d. $N(0,1)$ functions of i.i.d. $N(0,1)$ variables X, Y are obtained.

Example 1. Pairs of i.i.d. $N(0,1)$ Distributed Functions when $X, Y \overset{i.i.d.}{\sim} N(0,1)$.

$$\frac{2XY}{\sqrt{X^2+Y^2}} \quad \text{and} \quad \frac{X^2-Y^2}{\sqrt{X^2+Y^2}} \quad \text{(Shepp's example)}$$

$$\left(3X^2-Y^2\right)\frac{Y}{X^2+Y^2} \quad \text{and} \quad \left(X^2-3Y^2\right)\frac{X}{X^2+Y^2}$$

$$\frac{X^4-6X^2Y^2+Y^4}{(X^2+Y^2)^{\frac{3}{2}}} \quad \text{and} \quad \frac{4XY(X^2-Y^2)}{(X^2+Y^2)^{\frac{3}{2}}}$$

$$\left(5X^4-10X^2Y^2+Y^4\right)\frac{Y}{(X^2+Y^2)^2} \quad \text{and} \quad \left(5Y^4-10X^2Y^2+X^4\right)\frac{X}{(X^2+Y^2)^2}$$

$$\frac{6X^5Y-20X^3Y^3+6XY^5}{(X^2+Y^2)^{\frac{5}{2}}} \quad \text{and} \quad \frac{X^6-15X^4Y^2+15X^2Y^4-Y^6}{(X^2+Y^2)^{\frac{5}{2}}}.$$

Remark 1. Since $Z_n(X,Y)$ and $W_n(X,Y)$ are i.i.d. $N(0,1)$ whenever X, Y are i.i.d. $N(0,1)$, one would get an i.i.d. pair of standard normals by considering the functions $Z_m(Z_n(X,Y), W_n(X,Y))$ and $W_m(Z_n(X,Y), W_n(X,Y))$. It is interesting that $Z_m(Z_n(X,Y), W_n(X,Y)) = Z_{mn}(X,Y)$ and $W_m(Z_n(X,Y), W_n(X,Y)) = W_{mn}(X,Y)$. Thus, iterations of the functions in Proposition 3 produce members of the same sequence.

Remark 2. Consider the second pair of functions in Example 1. One notices that but for a sign, the second function is obtained by plugging Y for X and X for Y in the first function. It is of course obvious that because X, Y are i.i.d., by writing Y for X and X for Y, we cannot change the distribution of the function. What is interesting is that this operation produces a function independent of the first function. This in fact occurs for all the even numbered pairs, as is formally stated in the following proposition.

Proposition 4. *For every $n \geq 0, W_{2n+1}(X,Y) = (-1)^n Z_{2n+1}(Y,X)$, and hence, for every $n \geq 0, Z_{2n+1}(X,Y)$ and $Z_{2n+1}(Y,X)$ are independently distributed.*

Progressively more rugged plots are obtained by plotting the functions $Z_n(x,y)$ and $W_n(x,y)$ as n increases; despite the greater ruggedness, the plots also get visually more appealing. A few of the plots are presented next. The plots labeled as V correspond to the functions W of Proposition 3.

Analogous to the Chebyshev polynomials of the first and second kind, those of the third kind also produce standard normal variables. However, this time there is no independent mate.

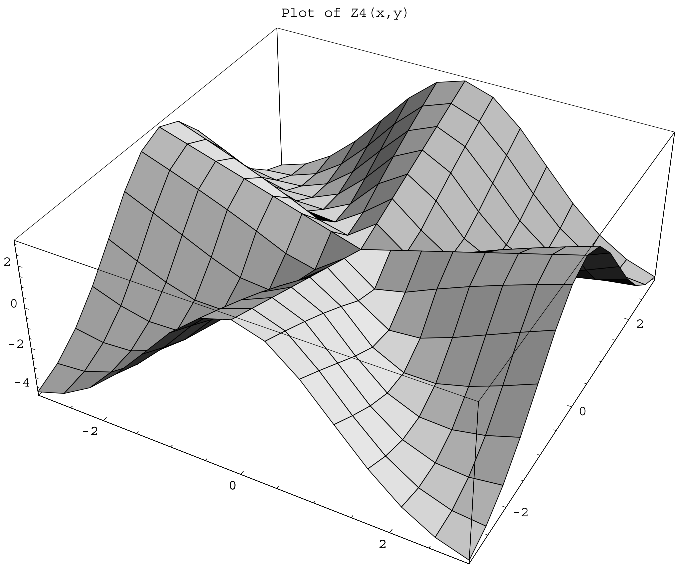
Proposition 5. *Let $X, Y \overset{i.i.d.}{\sim} N(0,1)$. For $n \geq 1$, let*

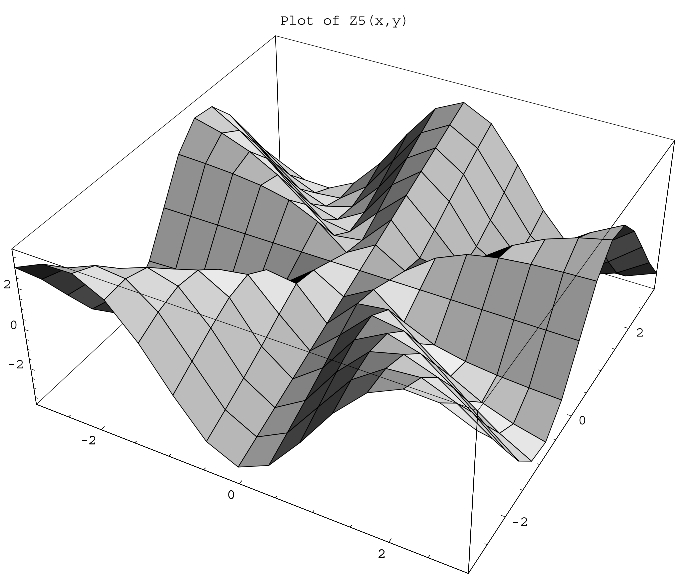
$$Q_n = \frac{\text{sgn}(Y)}{\sqrt{2}}\sqrt{X^2+Y^2+X\sqrt{X^2+Y^2}}\,V_n\left(\frac{X}{\sqrt{X^2+Y^2}}\right).$$

Then $Q_n \sim N(0,1)$.

The first few polynomials $V_n(x)$ are $V_1(x) = 2x-1, V_2(x) = 4x^2-2x-1, V_3(x) = 8x^3-4x^2-4x+1, V_4(x) = 16x^4-8x^3-12x^2+4x+1$. Plugging these into the formula for Q_n, a sequence of increasingly complex standard normal functions of X, Y are obtained.

For example, using $n = 1$, if X, Y are i.i.d. $N(0,1)$, then $\frac{\text{sgn}(Y)}{\sqrt{2}}(2X - \sqrt{X^2+Y^2})\sqrt{1+\frac{X}{\sqrt{X^2+Y^2}}}$ is distributed as $N(0,1)$. In comparison to the $N(0,1)$ functions Z_2, W_2 in Section 3.2, this is a more complex function with a $N(0,1)$ distribution.


Plot of Z4(x,y)
2
0
-2
-4
-2
0
2
2
0
-2


Plot of Z5(x,y)
2
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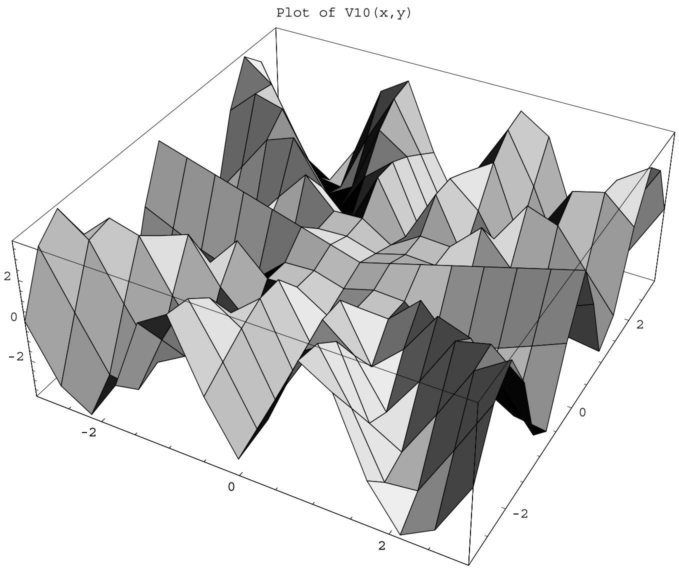
-2
-2
0
2
2
0
-2


Plot of V10(x,y)
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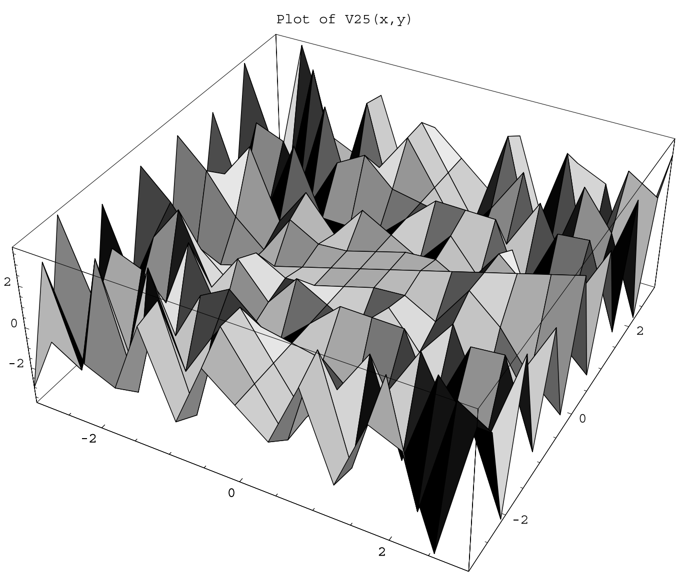
2
2
0
-2


Plot of V25(x,y)
2
0
-2
-2
0
2
2
0
-2

3.3. The case of three

It is interesting to construct explicitly three i.i.d. $N(0,1)$ functions $f(X,Y,Z)$, $g(X,Y,Z), h(X,Y,Z)$ of three i.i.d. $N(0,1)$ variables X,Y,Z. In this section, we present a method to explicitly construct such triplets of functions $f(X,Y,Z)$, $g(X,Y,Z), h(X,Y,Z)$ by using Chebyshev polynomials, as in the case with two of them. The functions f,g,h we construct are described below.

Proposition 6. *Let $X,Y,Z \overset{i.i.d.}{\sim} N(0,1)$. If $U(X,Y), V(X,Y)$ are i.i.d. $N(0,1)$, then $f(X,Y,Z), g(X,Y,Z), h(X,Y,Z)$ defined as*

$$\begin{aligned} f(X,Y,Z) &= U\big(V(X,Y), V\big(U(X,Y),Z\big)\big), \\ g(X,Y,Z) &= V\big(V(X,Y), V\big(U(X,Y),Z\big)\big), \\ h(X,Y,Z) &= U\big(U(X,Y),Z\big) \end{aligned}$$

are also distributed as i.i.d. $N(0,1)$.

Example 2. For $U(X,Y), V(X,Y)$, we can use the pair of i.i.d. $N(0,1)$ functions of Proposition 3. This will give a family of i.i.d. $N(0,1)$ functions f,g,h of X,Y,Z. The first two functions f,g of Proposition 6 are too complicated even when we use $U = Z_2$ and $V = W_2$ of Proposition 3. But the third function h is reasonably tidy. For example, using $U = Z_n$, and $V = W_n$ with $n = 2$, one gets the following distributed as $N(0,1)$:

$$h(X,Y,Z) = \frac{4XYZ}{\sqrt{4X^2Y^2 + Z^2(X^2+Y^2)}}.$$

4. Cauchy distributed functions, Fredholm integral equations and the stable law of exponent $\frac{1}{2}$

4.1. Cauchy distributed functions of Cauchy distributed variables

It follows from the result in Proposition 3 that if C has a *Cauchy*$(0,1)$ distribution, then appropriate sequences of rational functions $C\lambda_n(C)$ also have a *Cauchy*$(0,1)$ distribution. These results generalize the observations in Pitman and Williams (1967). This results, by consideration of characteristic functions, in the *Cauchy*$(0,1)$ density being solutions to a certain Fredholm integral equation of the first kind. This connection seems to be worth pointing out. First the functions $f_n(C)$ attributed to above are explicitly identified in the next result.

Proposition 7. *Let $C \sim$ Cauchy$(0,1)$. Let $R = \frac{1}{\sqrt{1+C^2}}$ and for $k \geq 1$,*

$$\begin{aligned} f_k(C) &= \frac{1 + 2T_2(R) + 2T_4(R) + \cdots + T_{2k}(R)}{T_{2k}(R)}, \qquad \textit{and} \\ g_k(C) &= \frac{2T_1(R) + 2T_3(R) + \cdots + T_{2k+1}(R)}{T_{2k+1}(R)}. \end{aligned}$$

Then $Cf_k(C)$ and $Cg_k(C)$ are also $\sim$ Cauchy$(0,1)$.

Example 3. The functions f_k, g_k for small values of k are as follows:

$$\begin{aligned} f_1(C) &= \frac{2}{1-C^2}; \qquad g_1(C) = \frac{C^2-3}{3C^2-1}; \\ f_2(C) &= \frac{4-4C^2}{C^4-6C^2+1}; \qquad g_2(C) = \frac{C^4-10C^2+5}{5C^4-10C^2+1}; \\ f_3(C) &= \frac{6C^4-20C^2+6}{C^6-15C^4+15C^2-1}; \qquad g_3(C) = \frac{C^6-21C^4+35C^2-7}{7C^6-35C^4+21C^2-1}. \end{aligned}$$

Note that f_k, g_k are rational functions of C. Proposition 7 thus gives an infinite collection of rational functions, say $\lambda_n(C)$, such that $C\lambda_n(C) \sim C \forall n$. This implies the following result on Fredholm integral equations.

Proposition 8. *Consider the Fredholm integral equation* $\int_{-\infty}^{\infty} K(t,y)p(y)dy = g(t)$, *where* $K(t,y) = cos(ty\lambda(y))$ *and* $g(t) = e^{-|t|}$. *Then for any of the rational functions* $\lambda(y) = f_k(y), g_k(y)$ *in Proposition 7, the Cauchy*$(0,1)$ *density* $p(y) = \frac{1}{\pi(1+y^2)}$ *is a solution of the above Fredholm equation.*

4.2. The stable law with exponent $\frac{1}{2}$

Starting with three i.i.d. standard normal variables, one can construct an infinite collection of functions of them, each having a symmetric stable distribution with exponent $\frac{1}{2}$. The construction uses, as in the previous sections, the Chebyshev polynomials. It is described in the final result.

Proposition 9. *Let* X, Y, N *be i.i.d.* $N(0,1)$. *Then, for each* $n \geq 1$, $S_{1,n} = \frac{N}{Z_n(X,Y)W_n^2(X,Y)}$, *as well as* $S_{2,n} = \frac{N}{W_n(X,Y)Z_n^2(X,Y)}$ *have a symmetric stable distribution with exponent* $\frac{1}{2}$.

Example 4. Using $n = 2, 3$, the following are distributed as a symmetric stable law of exponent $\frac{1}{2}$:

$$N\frac{(X^2+Y^2)^{\frac{3}{2}}}{4X^2Y^2(X^2-Y^2)} \quad \text{and} \quad N\frac{(X^2+Y^2)^{\frac{3}{2}}}{2XY(X^2-Y^2)^2};$$
$$N\frac{(X^2+Y^2)^3}{XY^2(X^2-3Y^2)(3X^2-Y^2)^2} \quad \text{and} \quad N\frac{(X^2+Y^2)^3}{X^2Y(3X^2-Y^2)(X^2-3Y^2)^2}.$$

5. Appendix

Proof of Proposition 3. Proposition 3 is a restatement of the well known fact that if X, Y are i.i.d. $N(0,1)$, and if r, θ denote their polar coordinates, then for all $n \geq 1, r\cos n\theta$ and $r \sin n\theta$ are i.i.d. $N(0,1)$, and that the Chebyshev polynomials $T_n(x), U_n(x)$ are defined by $T_n(x) = \cos n\theta, U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}$ with $x = \cos\theta$.

Proof of Proposition 4. We need to prove that for all x, y,

$$\begin{aligned} xU_{2n}\left(\frac{y}{\sqrt{x^2+y^2}}\right) &= (-1)^n\sqrt{x^2+y^2}\,T_{2n+1}\left(\frac{x}{\sqrt{x^2+y^2}}\right) \\ &\Leftrightarrow \forall w, \sqrt{1-w^2}\,U_{2n}(w) \\ &= (-1)^n T_{2n+1}\big(\sqrt{1-w^2}\,\big). \end{aligned}$$

Note now that

$$\frac{d}{dw}(-1)^n T_{2n+1}\big(\sqrt{1-w^2}\,\big) = (-1)^{n+1}\frac{w}{\sqrt{1-w^2}}(2n+1)U_{2n}\big(\sqrt{1-w^2}\,\big),$$

by using the identity

$$\frac{d}{dw}T_k(w) = kU_{k-1}(w).$$

On the other hand,

$$\begin{aligned} &\frac{d}{dw}\sqrt{1-w^2}\,U_{2n}(w) \\ &\quad = -\frac{w}{\sqrt{1-w^2}}U_{2n}(w) + \sqrt{1-w^2}\,\frac{(n+1)U_{2n-1}(w) - nU_{2n+1}(w)}{1-w^2}, \end{aligned}$$

by using the identity

$$\frac{d}{dw}U_k(w) = \frac{(k+2)U_{k-1}(w) - kU_{k+1}(w)}{2(1-w^2)};$$

see Mason and Handscomb (2003) for these derivative identities.

It is enough to show that the derivatives coincide. On some algebra, it is seen that the derivatives coincide iff $U_{2n-1}(w) - wU_{2n}(w) = (-1)^{n+1}wU_{2n}(\sqrt{1-w^2})$, which follows by induction and the three term recursion for the sequence U_n.

Proof of Proposition 5. Proposition 5, on some algebra, is a restatement of the definition of the Chebyshev polynomials of the third kind as $V_n(x) = \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{\theta}{2}}$. We omit the algebra.

Proof of Proposition 6. If X, Y, Z are i.i.d. $N(0,1)$, and $U(X,Y), V(X,Y)$ are also i.i.d. $N(0,1)$, then, obviously, $U(X,Y), V(X,Y), Z$ are i.i.d. $N(0,1)$. At the next step, use this fact with X, Y, Z replaced respectively by $U(X,Y), Z, V(X,Y)$. This results in $U(U(X,Y),Z), V(U(X,Y),Z), V(X,Y)$ being i.i.d. $N(0,1)$. Then as a final step, use this fact one more time with X, Y, Z replaced respectively by $V(X,Y), V(U(X,Y),Z), U(U(X,Y),Z)$. This completes the proof.

Proof of Proposition 7. From Proposition 3, $\frac{Z_n(X,Y)}{W_n(X,Y)} \sim \mathit{Cauchy}(0,1)$ for all $n \geq 1$. Thus, we need to reduce the ratio $\frac{Z_n(X,Y)}{W_n(X,Y)}$ to $Cf_k(C)$ when $n = 2k$ and to $Cg_k(C)$ when $n = 2k+1$, with C standing for the Cauchy-distributed variable $\frac{Y}{X}$.

The reduction for the two cases $n = 2k$ and $n = 2k+1$ follow, again on some algebra, on using the following three identities:

(i) $wU_{n-1}(w) = U_n(w) - T_n(w)$;

(ii) $U_{2k}(w) = T_0(w) + 2T_2(w) + \cdots + 2T_{2k}(w)$;

(iii) $U_{2k+1}(w) = 2T_1(w) + 2T_3(w) + \cdots + 2T_{2k+1}(w)$;

see Mason and Handscomb (2003) for the identities (i)–(iii). Again, we omit the algebra.

Proof of Proposition 8. Proposition 8 follows from Proposition 7 on using the facts that each f_k, g_k are even functions of C, and hence the characteristic function of $Cf_k(C)$ and $Cg_k(C)$ is the same as its Fourier cosine transform, and on using also the fact that the characteristic function of a *Cauchy*$(0,1)$ distributed variable is $e^{-|t|}$.

Proof of Proposition 9. Proposition 9 follows from Proposition 3 and the well known fact that for three i.i.d. standard normal variables X, Y, N, $\frac{N}{XY^2}$ is symmetric stable with exponent $\frac{1}{2}$; see, e.g., Kendall, Stuart and Ord (1987).

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