LECTURE 8

Examples of Long-Range Dependence

An early example of a process that might be termed long-range dependent was given in Rosenblatt (1961). The process $\{X_k\}$ is a quadratic function

$$X_k = Y_k^2 - 1$$

of a Gaussian stationary sequence Y_k with $EY_k \equiv 0$ and covariance sequence

$$r_k = (1 + k^2)^{-\gamma}, \qquad \gamma > 0.$$

The spectral density $g(\lambda)$ of the process Y_k is continuous and bounded away from zero if $|\lambda| > \varepsilon > 0$. Further, if $\gamma < \frac{1}{2}$, g has a singularity of the form $|\lambda|^{2\gamma-1}$ in the neighborhood of $\lambda = 0$. The covariance sequence of X_k is $r_k = 2(1 + k^2)^{-2\gamma}$. Let $\gamma < \frac{1}{4}$. Then the spectral density $f(\lambda)$ of $\{X_k\}$ has a singularity of the form $|\lambda|^{4\gamma-1}$ in the vicinity of $\lambda = 0$. Actually this family of processes was constructed to give simple examples of stationary sequences that are not strongly mixing. One can show that

$$n^{-1+2\gamma} \sum_{k=1}^{n} X_k$$

has a non-Gaussian limiting distribution as $n \to \infty$. This implies that the sequence $\{X_k\}$ is not strongly mixing for if it were, by the theorem of Rosenblatt (1956b), the limiting distribution would have to be normal since all the assumptions other than strong mixing are satisfied. The process $\{X_k\}$ would today be called long-range dependent because the partial sums have a non-Gaussian (and nonstable) limiting distribution. The non-Gaussian limiting behavior is easy to exhibit. The characteristic function of the normalized partial sum is

$$|I - 2itn^{-1+2\gamma}R|^{-1/2} \exp\{-in^{2\gamma}t\} = \exp\left\{\frac{1}{2} \sum_{k=2}^{\infty} (2itn^{-1+2\gamma})^k \operatorname{sp}(R^k)/k\right\}$$

and

$$\begin{aligned}(n^{-1+2\gamma})^k \operatorname{sp}(R^k) &= (n^{-1+2\gamma})^k \sum_{i_j=1}^{n} r_{i_1-i_2} r_{i_2-i_3} \cdots r_{i_k-i_1} \\ &\to \underbrace{\int \overset{1}{\underset{0}{\cdots}} \int}_{k} |x_1-x_2|^{-2\gamma} |x_2-x_3|^{-2\gamma} \cdots |x_k-x_1|^{-2\gamma}\, dx_1 \cdots dx_k\end{aligned}$$

as $n \to \infty$. Here $\operatorname{sp}(R)$ is the trace of the covariance matrix R of the process $\{Y_k\}$. Clearly the limiting distribution is non-Gaussian (and nonstable). It is an infinite weighted sum of independent χ^2 random variables.

Taqqu (1979) and Dobrushin and Major (1979) identified a much larger collection of processes obtained as nonlinear functionals of Gaussian sequences and their shifts as long-range dependent and obtained corresponding limit theorems. It will be convenient to follow the discussion of Dobrushin and Major. Assume that one is initially given a normal stationary sequence $\{Y_n\}$ with $EY_n \equiv 0$, $EY_n^2 = 1$ and covariance function $r(n)$ satisfying

$$r(n) = n^{-\alpha} L(n), \qquad 0 < \alpha < 1,$$

with $L(t)$, $0 < t$, a slowly varying function. The derived process X_n is given by an instantaneous function $H(\cdot)$ so that

$$X_n = H(Y_n), \qquad n = \dots, -1, 0, 1, \dots,$$

with

$$EX_n = EH(Y_n) = \int_{-\infty}^{\infty} H(y) \exp\left(-\frac{y^2}{2}\right) \frac{dy}{\sqrt{2\pi}} = 0$$

and

$$EX_n^2 = \sigma^2(X_n) = \int_{-\infty}^{\infty} H(y)^2 \exp\left(-\frac{y^2}{2}\right) \frac{dy}{\sqrt{2\pi}} < \infty.$$

Let the Fourier–Hermite expansion of H be

$$H(y) = \sum_{j=1}^{\infty} c_j H_j(y)$$

with $H_j(y)$ the Hermite polynomial of degree j and

$$\sum_{j=1}^{\infty} c_j^2 j! < \infty.$$

The object is to consider the sequences

$$X_n^N = \frac{1}{A_N} \sum_{j=N(n-1)}^{Nn-1} H(Y_j), \qquad n = \dots, -1, 0, 1, \dots, \qquad N = 1, 2, \dots,$$

with A_n a sequence of norming constants. The object is to determine circumstances under which there is a nontrivial limit process X_n^* in distribution. Since we will have $A_N \to \infty$, the limit process will be self-similar, that is, the probability structure of the process X_n^{*N} will be the same as that of X_n^*.

THEOREM. *Let k be the smallest index for which $c_k \neq 0$ and let $k\alpha < 1$,*

$$A_N = N^{1-(k\alpha)/2} L(N)^{k/2}.$$

Then the finite dimensional distributions of X_n^N, $n = \ldots, -1, 0, 1, \ldots$, tend to those of the sequence X_n^* given by

$$X_n^* = D^{-k/2} c_k \int \exp(in\{y_1 + \cdots + y_k\}) \frac{\exp(i(y_1 + \cdots + y_k)) - 1}{i(y_1 + \cdots + y_k)} \times |y_1|^{(\alpha-1)/2} \cdots |y_k|^{(\alpha-1)/2} \, dW(y_1) \cdots dW(y_k)$$

with

$$D = \int_{-\infty}^{\infty} \exp(iy)|y|^{\alpha-1}\, dy = 2\Gamma(\alpha)\cos\left(\frac{\alpha\pi}{2}\right).$$

In the integrals above it is understood that there is no contribution from $x_j = \pm x_l$ for $j \neq l$. Also W is the random spectral function of the Gaussian white noise process. Here W is Gaussian with mean zero,

$$W(\Delta) = \overline{W(-\Delta)},$$

$$E|W(\Delta)|^2 = \frac{1}{2\pi}|\Delta|$$

for any interval Δ. If $\Delta_1, \ldots, \Delta_j$ are disjoint intervals on the positive real axis, $W(\Delta_1), \ldots, W(\Delta_j)$ are independent. Also $\operatorname{Re} W(\Delta)$, $\operatorname{Im} W(\Delta)$ are independent variables with the same distribution if Δ is an interval of positive reals of the Gaussian white noise process.

First of all, note that if ξ and η are jointly normal variables with $E\xi = E\eta = 0$, $E\xi^2 = E\eta^2 = 1$, $E\xi\eta = r$, Mehler's formula implies that

$$EH_k(\xi)H_j(\eta) = \delta_{j,k} r^k k!.$$

One can see that if $k\alpha < 1$,

$$\sigma^2\left(\sum_{j-1}^{N} H(Y_j)\right) \asymp N^{2-k\alpha} L(N)^k$$

for

$$\sigma^2\left(\sum_{j-1}^{N} H(Y_j)\right) = \sum_{s=-N}^{N} (N - |s|) E(H(Y_0)H(Y_s))$$

$$\simeq \sum_{s--N}^{N} (N - |s|) c_k^2 (1 + |s|)^{-k\alpha} L(s)^k.$$

$A \asymp B$ if the ratios A/B, B/A are bounded. It is also clear that if $\alpha k > 1$, then $\sigma^2(\sum_{j=1}^{N} H(Y_j)) \asymp N$. This suggests that if this is the case, there is short-range dependence for the X_j's and the central limit theorem should hold (asymptotic normality) with the usual normalization.

We first discuss the case $H(y) = H_k(y)$. One can show that

$$H_k(Y_n) = H_k\left(\int e^{in\lambda} Z_G(d\lambda)\right)$$
$$= \int \exp(in(\lambda_1 + \cdots + \lambda_k)) Z_G(\lambda_1) \cdots Z_G(d\lambda_k)$$

with Z_G the random spectral measure of the Gaussian process (Y_n). Then

$$X_n^N = \frac{1}{N} \sum_{j=N(n-1)}^{Nn-1} \int \exp\left\{i \frac{1}{N} j(\lambda_1 + \cdots + \lambda_k)\right\} Z_{G_N}(d\lambda_1) \cdots Z_{G_N}(d\lambda_k)$$
$$= \int \exp\{in(\lambda_1 + \cdots + \lambda_k)\} K_N(\lambda_1, \dots, \lambda_k) Z_{G_N}(d\lambda_1) \cdots Z_{G_N}(d\lambda_k)$$

with

$$G_N(A) = \frac{N^\alpha}{L(N)} G(N^{-1}A)$$

and Z_{G_N} is the random measure corresponding to G_N. Also

$$K_N(\lambda_1, \dots, \lambda_k) = \sum_{0 \le j < N-1} \frac{1}{N} \exp\left\{i \frac{j}{N}(\lambda_1 + \cdots + \lambda_k)\right\}$$
$$= \frac{\exp\{i(\lambda_1 + \cdots + \lambda_k)\} - 1}{[\exp\{i(1/N)(\lambda_1 + \cdots + \lambda_k)\} - 1] N}.$$

Let

$$\varphi_N(t_1, \dots, t_k) = \int \exp\left\{\frac{i}{N}(j_1\lambda_1 + \cdots + j_k\lambda_k)\right\} \times |K_N(\lambda_1, \dots, \lambda_k)|^2 G_N(d\lambda_1) \cdots G_N(d\lambda_k) \tag{8.1}$$

where $j_s = [t_s N]$, $s = 1, \dots, k$. By using (8.1) one can see that

$$\varphi_N(t_1, \dots, t_k) = \frac{1}{N^{2-k\alpha} L(N)^k} \sum_{p,q \in B^N} r(p - q + j_1) \cdots r(p - q + j_k)$$
$$= \frac{1}{N^{2-k\alpha} L(N)^k} \sum_{p \in \tilde{B}^N} (N - |p|) r(p + j_1) \cdots r(p + j_k), \tag{8.2}$$

where

$$B^N = \{p | 0 \le p < N - 1\}, \qquad \tilde{B}^N = \{p | -N < p < N\}.$$

To prove the theorem one needs the following lemmas.

LEMMA 1. *$\lim_{N\to\infty} \varphi_N(t_1,\ldots,t_k) = g(t_1,\ldots,t_k)$ uniformly in every bounded cube where*

$$g(t_1,\ldots,t_k) = \int_{[-1,1]} (1-|x|) \prod_{j=1}^{k} |x+t_j|^{-\alpha}\, dx.$$

It is clear that $g(t_1,\ldots,t_k)$ is a continuous function.
The claim is that if $\alpha k < 1$, then

$$\int \left| \frac{\exp(i(\lambda_1 + \cdots + \lambda_k)) - 1}{i(\lambda_1 + \cdots + \lambda_k)} \right|^2 |\lambda_1|^{\alpha-1} \cdots |\lambda_k|^{\alpha-1}\, d\lambda_1 \cdots d\lambda_k < \infty.$$

This lemma follows immediately from (8.2).

LEMMA 2. *Let $\mu_1, \mu_2, \ldots$ be a sequence of measures on R^k such that $\mu_N(R^k - [-c_N\pi, c_N\pi]^k) = 0$ with a sequence $c_N \to \infty$. Let*

$$\varphi_N(t) = \int_{R^k} \exp(i((j/c_N)\cdot\lambda))\mu_N(d\lambda)$$

with $j \in Z^k$ and $j = [tc_n]$. If $\varphi_n(t)$ tends to a function $\varphi(t)$ continuous at the origin, then μ_N tends weakly to a finite measure μ_0 and $\varphi(t)$ is the Fourier transform of μ_0.

This result is an analogue of the classical continuity theorem for characteristic functions.

We say that a sequence of locally finite measures μ_N $[\mu_N(B) < \infty$ for every bounded $B]$ tends to a locally finite measure μ_0 locally weakly if $\int h(x)\mu_N(dx) \to \int h(x)\mu_0(dx)$ for each continuous function h of bounded support. In the case of the sequence of measures $G_N(\cdot)$, one can show that they converge locally weakly to the measure

$$G_0(dx) = D^{-1}|x|^{\alpha-1}\, dx. \tag{8.3}$$

Further the measure G_0 has the self-similarity property

$$G_0(A) = t^{-\alpha} G_0(tA). \tag{8.4}$$

The locally finite measure G_0 is determined by the relation

$$2\int e^{itx} \frac{1-\cos x}{x^2} G_0(dx) = \int_{-1}^{1} (1-|x|) \frac{1}{|x+t|^{\alpha}}\, dx.$$

If we apply Lemmas 1 and 2 in the case $k = 1$, the sequence of measures

$$\mu_N(B) = \int_B \overline{K}_N(x) G_N(dx)$$

with $\overline{K}_N(x) = |K_N(x)|^2$ converge to the finite measure μ_0 determined by its Fourier transform $g(t)$. But then for any bounded open interval B,

$$\lim_{N\to\infty} G_N(B) = \int [\overline{K}_0(x)]^{-1} \mu_0(dx) = G_0(B)$$

with

$$\overline{K}_0(x) = 2\frac{1-\cos x}{x^2} = \lim_{N\to\infty} \frac{1-\cos x}{N^2(1-\cos(2/N))} = \lim_{N\to\infty} \overline{K}_N(x).$$

Thus G_N converges locally weakly to G_0. Let $M = [N/t]$. We then have

$$G_N(tB) \simeq \left(\frac{N}{M}\right)^{\alpha} \frac{L(M)}{L(N)} G_M\left(t\frac{M}{N}B\right)$$

for B an open interval. Since $(N/M)^{\alpha}(L(M)/L(N)) \to 1$, $G_0(tB) = t^{\alpha}G_0(B)$. It is clear that G_0 must be characterized by (8.4) and so we have determined G_0 as given by (8.3).

LEMMA 3. *Let G_N be a sequence of nonatomic measures in R^k tending locally weakly to a nonatomic measure G_0. Suppose $\hat{K}(x_1, \ldots, x_k)$ is a sequence of measurable functions tending to a limit $\hat{K}_0(x_1, \ldots, x_k)$ uniformly continuous on any finite rectangle $[-A, A]^k$. Also let the $\hat{K}$ satisfy*

$$\lim_{A\to\infty} \int_{B^k-[-A,A]^k} |\hat{K}_N(x_1, \ldots, x_k)|^2 G_N(dx_1) \cdots G_N(dx_k) = 0$$

uniformly for $N = 0, 1, \ldots$. The Wiener–Itô integrals

$$\int \hat{K}_0(x_1, \ldots, x_k) Z_{G_0}(dx_1) \cdots Z_{G_0}(dx_k) \tag{8.5}$$

exist and the sequence of Wiener–Itô integrals

$$\int \hat{K}_N(x_1, \ldots, x_k) Z_{G_N}(dx_1) \cdots Z_{G_N}(dx_k) \tag{8.6}$$

converges in distribution to (8.5) as $N \to \infty$.

The following limiting relation can be seen to hold for functions of finite support $h(x_1, \ldots, x_k)$ taking on only a finite number of values and each value on a product set:

$$\int h(x_1, \ldots, x_k) Z_{G_N}(dx_1) \cdots Z_{G_N}(dx_k)$$
$$\to \int h(x_1, \ldots, x_k) Z_{G_0}(dx_1) \cdots Z_{G_0}(dx_k)$$

in distribution as $N \to \infty$. This follows because the expressions can be seen to be just polynomials in variables $Z_{G_N}(B)$ and $Z_{G_0}(B)$. The general case of Lemma 3 follows by a standard approximation argument.

The proof of the main result for the case $H(x) = H_k(x)$ follows by applying Lemma 3 with

$$\hat{K}_N(x_1,\dots,x_k) = \sum_{j=1}^{l} \alpha_j \exp(inj\{x_1 + \cdots + x_k\}) K_N(x_1,\cdots,x_k)$$

and

$$\hat{K}_0(x_1,\dots,x_k) = \sum_{j=1}^{l} \alpha_j \exp(inj\{x_1 + \cdots + x_k\}) K_0(x_1,\dots,x_k).$$

In the case of a general $H(x)$, let

$$Z^N = \sum_{s\in B^N} \sum_{j=k+1}^{\infty} c_j H(Y_s), \qquad N = 1, 2, \dots .$$

Then

$$E(Z^N)^2 = \sum_{j=k+1} c_j^2 j! \sum_{s,t\in B^N} [r(s-t)]^j$$

and this implies that

$$E(Z^N)^2 = O\big(N^{2-(k+1)\alpha} L(N)^{k+1}\big) + O(N)$$

as $N \to \infty$. Therefore $A_N^{-1} Z^N \to 0$ in probability as $N \to \infty$. In the theorem $H(x)$ can be replaced by $c_k H_k(x)$.

The results just obtained were for an instantaneous function of Y_0 and its shifts Dobrushin and Major obtain a result for noninstantaneous functions of the Y process. Suppose ξ is a random variable with finite second moment, $E\xi = 0$, that is defined on the space of the Y process. The random variable ξ can then be written

$$\xi = \sum_{j=1}^{\infty} \frac{1}{j!} \int \alpha_j(y_1,\dots,y_k) Z_G(dy_1) \cdots Z_G(dy_k),$$

where

$$\alpha_j(y_1,\dots,y_j) = \overline{\alpha_j(-y_1,\dots,-y_j)}$$

and

$$\sum_{j=1}^{\infty} \frac{1}{j!} \int |\alpha_j(y_1,\dots,y_j)|^2 G(dy_1) \cdots G(dy_j) < \infty.$$

Let us assume the process $\{X_n\}$ is obtained by setting X_n equal to the nth time shift of ξ. Then

$$X_n = \sum_{k=1}^{\infty} \int \exp[in(y_1 + \cdots + y_k)] \alpha_k(y_1,\dots,y_k) Z_G(dy_1) \cdots Z_G(dy_k).$$

One sometimes refers to (X_n) as a process subordinate to (Y_n). Let

$$X_n^N = A_N^{-1} \sum_{j=N(n-1)}^{Nn-1} X_j.$$

THEOREM. *Let k be the largest integer such that for every $j < k$, $\alpha_j = 0$. Assume that $\alpha_k(y_1, \dots, y_k)$ is bounded, continuous at the origin and such that*

$$\alpha_k(0, \dots, 0) \neq 0.$$

Let $\alpha k < 1$ and

$$\sum_{j=k+1}^{\infty} \frac{1}{j!} \frac{N^{-(j-k)\alpha}}{L(N)^{j-k}} \int \left| \alpha_j\left(\frac{y_1}{N}, \dots, \frac{y_j}{N}\right) \right|^2 |K_N(y_1, \dots, y_j)|^2$$
$$\times G_N(dy_1) \cdots G_N(dy_j) \to 0.$$

Then if $A_N = N^{(1-k\alpha)/2} L(N)^{k/2}$, the finite-dimensional distributions of X_n^N tend to those of $(1/k!)\alpha_k(0, \dots, 0)X_n^*$, with

$$X_n^* = \int \exp(in\{y_1 + \cdots + y_k\}) K_0(y_1, \dots, y_k) Z_{G_0}(dy_1) \cdots Z_{G_0}(dy_k).$$

A discussion of what may happen if $\alpha_k(0, \dots, 0) = 0$ is given in Rosenblatt (1979) and Major (1981). References and a discussion of similar limit theorems for processes derived from one with a non-Gaussian distribution can be found in Giraitis and Surgailis (1986).