CHAPTER 1

Integrals and the Haar Measure

In this and the next chapter, background material on integration, topological groups and group actions is presented. The concrete examples described below provide a direct connection between the rather abstract theory of Haar measure and its application to situations which are relevant in statistical applications.

1.1. Integrals and measures. Our purpose in this section is to describe the connection between two different approaches to integration theory. The measure theory approach is commonly used in probability theory and so is familiar to most of us. However, the algebraic or linear functional approach is not so familiar. A good reference for the details of both approaches is Segal and Kunze (1978). A brief description of the linear functional approach and its relationship to the measure theory approach follows.

Let **X** be a locally compact topological space (Hausdorff) for which the topology has a countable base. That is, **X** is a topological space such that:

(i) The topology is Hausdorff.
(ii) Each point of **X** has a compact neighborhood.
(iii) The topology has a countable base.

Throughout these notes, the word "space" refers to a topological space **X** *satisfying* (i), (ii) *and* (iii). In virtually all of our examples, **X** is an open or closed subset of a finite dimensional vector space (with the usual Euclidean topology) so that **X** with its inherited topology is a space.

In many discussions of integration on locally compact spaces, (iii) is not assumed. However, when (iii) does hold the σ-algebra generated by the open sets (the Borel σ-algebra) and the σ-algebra generated by the compact sets (the Baire σ-algebra) are identical. This circumstance simplifies the discussion somewhat and from our point of view is not a serious restriction of generality. The σ-algebra associated with the space **X** is the Borel σ-algebra ≡ Baire σ-algebra.

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Given a space $\mathbf{X}$, let $K(\mathbf{X})$ be the set of all continuous real valued functions f defined on $\mathbf{X}$ which have compact support. Thus if $f \in K(\mathbf{X})$, there is a compact set $C \subset \mathbf{X}$ (which may vary from one f to another) such that $f(x) = 0$ if $x \notin C$. For $f_1, f_2 \in K(\mathbf{X})$ and $a_1, a_2 \in R$, it is clear that $a_1 f_1 + a_2 f_2 \in K(\mathbf{X})$ so that $K(\mathbf{X})$ is a real vector space.

DEFINITION 1.1.   A real valued function J defined on $K(\mathbf{X})$ which satisfies

(i) $J(a_1 f_1 + a_2 f_2) = a_1 J(f_1) + a_2 J(f_2)$ for $f_1, f_2 \in K(\mathbf{X})$ and $a_1, a_2 \in R$,
(ii) $J(f) \geq 0$ whenever $f \in K(\mathbf{X})$ and f is nonnegative,
(iii) $J(f) > 0$ for some $f \in K(\mathbf{X})$

is called an *integral.*

An integral J is a linear function defined on the vector space $K(\mathbf{X})$ which maps nonnegative functions into nonnegative numbers. Assumption (iii) is to rule out the trivial case $J = 0$. Examples of integrals are easily constructed using Lebesgue measure when the space $\mathbf{X}$ is a subset of n dimensional Euclidean space R^n.

EXAMPLE 1.1.   Assume $\mathbf{X} \subseteq R^n$ with its inherited topology is a space and let dx denote Lebesgue measure restricted to $\mathbf{X}$. If

$$\int_{\mathbf{X}} dx > 0,$$

define J on $K(\mathbf{X})$ by

$$J(f) = \int_{\mathbf{X}} f(x)\, dx.$$

Because the Lebesgue measure of compact sets is finite, $J(f)$ is a well defined finite number for $f \in K(\mathbf{X})$. That J is an integral follows easily. However, care must be taken in some cases. For example, take $\mathbf{X} = (-1, 1)$ and define a measure μ on the Borel sets of $\mathbf{X}$ using the function

$$g_0(x) = \begin{cases} \dfrac{1}{|x|}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

via the equation

$$\mu(B) = \int_B g_0(x)\, dx.$$

Then μ is a well defined σ-finite measure on $\mathbf{X}$ but μ does not define an integral via

$$f \to \int f(x)\mu(dx)$$

because there are functions in $K(\mathbf{X})$ for which

$$\int f(x)\mu(dx) = +\infty.$$

Of course, the problem arises because μ assigns measure $+\infty$ to some of the compact sets in $\mathbf{X}$. $\square$

In order to describe the relationship between integrals (as they are defined above) and measures, Example 1.1 shows the same sort of restriction has to be made. Here is the appropriate definition which facilitates the identification of integrals with measures and vice versa.

DEFINITION 1.2.   On a space $\mathbf{X}$, a measure μ defined on the Borel subsets of $\mathbf{X}$ is a *Radon measure* if $\mu(C) < +\infty$ for all compact sets $C \subseteq \mathbf{X}$ and $\mu \not\equiv 0$.

THEOREM 1.1.   *Given an integral J defined on $K(\mathbf{X})$, there is a unique Radon measure μ such that*

$$(1.1) \qquad\qquad J(f) = \int f(x)\mu(dx), \qquad f \in K(\mathbf{X}).$$

Conversely, a Radon measure $\mu \not\equiv 0$ defines an integral via (1.1).

The representation (1.1) sets up a one-to-one correspondence between integrals and Radon measures. In these notes, both representations are used with the choice being dictated by notational convenience and taste. Equation (1.1) allows the definition of an integral J to be extended to the class of functions which are μ-integrable. That is, if f is Borel-measurable and if

$$\int |f(x)|\mu(dx) < +\infty,$$

then the right-hand side of (1.1) is well defined and so $J(f)$ is defined via (1.1). In what follows, all integrals are automatically extended to the class of Borel-measurable and μ-integrable functions.

1.2. Topological groups.   A group is a nonempty set G together with a binary operation $\circ$ such that the following conditions hold:

   (i) $g_1, g_2 \in G$ implies $g_1 \circ g_2 \in G$.
   (ii) $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$ for $g_1, g_2, g_3 \in G$.
   (iii) There exists an element $e \in G$ such that $e \circ g = g \circ e = g$ for $g \in G$.
   (iv) For each $g \in G$, there exists a unique element $g^{-1} \in G$ such that $g \circ g^{-1} = g^{-1} \circ g = e$.

The element $e \in G$ is called the *identity* and g^{-1} is called the *inverse* of g. It is customary to suppress the symbol $\circ$ when composing group elements. In this case, (ii) would be written $(g_1 g_2) g_3 = g_1(g_2 g_3)$. This custom is observed in what follows.

Now, assume that G is a group and that G is also a space as defined in Section 1.1. When the group operations fit together with the topology (i.e., are continuous), then G is a topological group. Here is the formal definition.

DEFINITION 1.3. Let the group G be a locally compact topological space (Hausdorff) whose topology has a countable base (i.e., G is a space). Assume that the functions

(i) $$(g_1, g_2) \to g_1 g_2$$

defined on $G \times G$ to G and

(ii) $$g \to g^{-1}$$

defined on G to G are both continuous. Then G is a *topological group*.

Again, the assumption that the topology for G has a countable base is one of convenience and is ordinarily not part of the definition of a topological group. In almost all the examples considered here, G is an open or closed subset of a finite dimensional vector space with the inherited topology. In the examples below, the verification that G is a topological group is not too hard and most of the details are left to the reader. Recall that $H \subset G$ is a *subgroup* of G if H, with its inherited operations from G, is a group.

EXAMPLE 1.2. Take $G = R^n$ with addition as the group operation and the usual topology on R^n. Obviously, $0 \in R^n$ is the identity and $-x$ is the inverse of $x \in R^n$. This group is *commutative*, that is, $x + y = y + x$, for $x, y \in R^n$. This example clearly extends to the case where G is a finite dimensional real vector space with the Euclidean topology. □

EXAMPLE 1.3. Let $G = \text{Gl}_n$, which is the group of all $n \times n$ nonsingular real matrices with matrix multiplication as the group operation—commonly called the *general linear group*. For $g \in \text{Gl}_n$, g^{-1} means the matrix inverse of g so that the identity in Gl_n is the $n \times n$ identity matrix. Let $\mathscr{L}_{n,n}$ be the real vector space of $n \times n$ real matrices so $\text{Gl}_n \subseteq \mathscr{L}_{n,n}$. The determinant function det is defined on $\mathscr{L}_{n,n}$ and is continuous. Since

$$\text{Gl}_n = \big\{ x | x \in \mathscr{L}_{n,n}, \det(x) \neq 0 \big\},$$

we see that Gl_n is an open subset of the Euclidean space $\mathscr{L}_{n,n}$. That is, Gl_n is the complement of the closed set $\{x | x \in \mathscr{L}_{n,n}, \det(x) = 0\}$. Thus, Gl_n is a space with the topology inherited from $\mathscr{L}_{n,n}$. That the group operations are continuous is not too hard to check. □

EXAMPLE 1.4. Let G be the group G_T^+ of $n \times n$ lower triangular matrices whose diagonal elements are positive. The group operation is matrix multiplication and G_T^+ is easily shown to be closed under this operation. That $g^{-1} \in G_T^+$ for $g \in G_T^+$ is most easily established by induction. For $g \in G_T^+$, partition g as

$$g = \begin{pmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{pmatrix},$$

where g_{11} is $(n-1) \times (n-1)$ and is lower triangular with positive diagonals,

g_{21} is $1 \times (n - 1)$ and $g_{22} \in (0, \infty)$. The inverse of g is

$$
\begin{pmatrix}
g_{11}^{-1} & 0 \\
-g_{22}^{-1} g_{21} g_{11}^{-1} & g_{22}^{-1}
\end{pmatrix},
$$

which shows that $g^{-1} \in G_T^+$. The topology for G_T^+ is that obtained by regarding G_T^+ as a subset of $n(n + 1)/2$ dimensional coordinate space and using the inherited topology for G_T^+. Clearly G_T^+ is an open subset of this $n(n + 1)/2$ dimensional vector space. The continuity of the group operations follows from the fact that G_T^+ is a (closed) subgroup of Gl_n and hence inherits the continuity of the group operations from Gl_n. $\square$

EXAMPLE 1.5. Let G be the group O_n of $n \times n$ real orthogonal matrices. Since O_n is a closed bounded set in $\mathscr{L}_{n, n'}$, O_n is a compact space with its inherited topology. Further, O_n is a closed subgroup of Gl_n and its topology is equal to that which it inherits from Gl_n. Thus the group operations in O_n are continuous. O_n is called the *orthogonal group*. $\square$

EXAMPLE 1.6. In this example, we describe what is usually called the *affine group*, Al_n. Elements of Al_n are pairs (g, x) with $g \in \mathrm{Gl}_n$ and $x \in R^n$. The group operation is defined by

$$
(g_1, x_1)(g_2, x_2) = (g_1 g_2, g_1 x_2 + x_1),
$$

where $g_1 g_2$ is the composition of g_1 and g_2 in Gl_n and $g_1 x_2 + x_1$ means the matrix g_1 multiplied times x_2 with x_1 added to $g_1 x_2$. The identity in Al_n is $(e, 0)$ where e is the identity in Gl_n so

$$
(g, x)^{-1} = (g^{-1}, -g^{-1} x).
$$

The topology for Al_n is that of the product space $\mathrm{Gl}_n \times R^n$, which is an open subset of the coordinate space $\mathscr{L}_{n, n} \times R^n$. The continuity of the group operations is left to the reader to check. $\square$

Finally, we mention two discrete groups which arise in statistical applications. The first of these is $\mathbf{P}_n$, the group of *permutation matrices*. Elements of $\mathbf{P}_n$ are $n \times n$ matrices g such that in each row and each column of g, there is exactly one element which is equal to 1 and all the remaining elements are 0. Each $g \in \mathbf{P}_n$ is an orthogonal matrix because gx and x have the same length for all $x \in R^n$. Thus $g^{-1} = g'$ where g' is the transpose of g. That $\mathbf{P}_n$ is closed under matrix multiplication is easily checked. Of course, $\mathbf{P}_n$ is a topological group with the discrete topology. Obviously $\mathbf{P}_n$ has $n!$ elements. Elements of $\mathbf{P}_n$ are called permutation matrices because for each $x \in R^n$, gx is a vector whose coordinates are a permutation of the coordinates of x.

The group of *coordinate sign changes* $\mathbf{D}_n$ has elements g which are $n \times n$ matrices with diagonal elements equal to ± 1 and off diagonal elements 0. Obviously each g is an orthogonal matrix and $\mathbf{D}_n$ has 2^n elements.

1.3. Haar measure. One of the most important theorems regarding topological groups asserts the existence and uniqueness (up to a positive constant) of left- and right-invariant integrals (measures). This theorem together with the relationship between left- and right-invariant integrals is described here. A more extensive discussion of these results together with proofs can be found in Nachbin (1965) and Segal and Kunze (1978).

To state things precisely let G be a topological group as in Definition 1.3. The real vector space $K(G)$ is the set of all continuous functions with compact support defined on G. Given $g \in G$, define the transformation L_g on $K(G)$ to $K(G)$ by

$$\left(L_g f\right)(x) = f\left(g^{-1}x\right)$$

for $g, x \in G$ and $f \in K(G)$.

DEFINITION 1.4.   An integral J on $K(G)$ is *left-invariant* if for all $f \in K(G)$,

$$J\left(L_g f\right) = J(f) \quad \text{for } g \in G.$$

If μ is the Radon measure corresponding to a left-invariant integral, then μ satisfies

$$\int_G f\left(g^{-1}x\right)\mu(dx) = \int_G f(x)\mu(dx), \qquad g \in G,$$

for $f \in K(G)$ and hence for all μ-integrable f. Alternatively, this condition is sometimes written less formally as

$$\mu(d(gx)) = \mu(dx), \qquad g \in G,$$

by simply defining $\mu(d(gx))$ as

$$\int f(x)\mu(d(gx)) = \int f(g^{-1}x)\mu(dx).$$

Here is the basic theorem.

THEOREM 1.2.   *On a topological group G, there exists a left-invariant integral (measure). If J_1 and J_2 are left-invariant integrals, then there exists a positive constant c such that $J_1 = cJ_2$.*

In these notes, we move freely between integrals and the corresponding Radon measure. If J is a left-invariant integral on $K(G)$, the corresponding measure, denoted by ν_l, satisfies

$$J(f) = \int_G f(x)\nu_l(dx) = \int_G f(g^{-1}x)\nu_l(dx)$$

for $g \in G$. The measure ν_l is often called a *left Haar measure*—named after A. Haar who first established the existence of invariant measures on many topological groups. Since an invariant integral is unique up to a positive constant, so is ν_l.

To describe the relationship between left- and right-invariant integrals, it is convenient to introduce the so-called (right-hand) modulus of G. Let ν_l be a left Haar measure and define J_1 on $K(G)$ by

$$J_1(f) = \int f(xg^{-1})\nu_l(dx)$$

for a fixed $g \in G$. It is easily verified that J_1 is left-invariant as is

$$J(f) = \int f(x)\nu_l(dx).$$

By the uniqueness portion of Theorem 1.2, there is a positive constant which is denoted by $\Delta(g)$ such that $J_1 = \Delta(g)J$. Thus, Δ is defined by

$$(1.2) \qquad \int f(xg^{-1})\nu_l(dx) = \Delta(g)\int f(x)\nu_l(dx), \qquad f \in K(G).$$

THEOREM 1.3. *The function Δ defined on G to $(0, \infty)$, called the modulus of G, is continuous and satisfies*

$$\Delta(g_1g_2) = \Delta(g_1)\Delta(g_2).$$

In particular $\Delta(e) = 1$ and $\Delta(g^{-1}) = 1/\Delta(g)$.

PROOF. A proof of the continuity can be found in Nachbin [(1965), page 77]. To verify the functional equation for Δ, compute

$$\Delta(g_1g_2)\int f(x)\nu_l(dx) = \int f\big(x(g_1g_2)^{-1}\big)\nu_l(dx)$$

$$= \int f\big((xg_2^{-1})g_1^{-1}\big)\nu_l(dx)$$

$$= \Delta(g_1)\int f\big(xg_2^{-1}\big)\nu_l(dx)$$

$$= \Delta(g_1)\Delta(g_2)\int f(x)\nu_l(dx). \qquad \square$$

To define right-invariant integrals, for $g \in G$ define R_g on $K(G)$ by

$$(fR_g)(x) = f(xg^{-1}).$$

Note that R_g is written on the right of f so that the equation

$$R_{g_1}R_{g_2} = R_{g_1g_2}$$

is valid. (More is said about this when group actions are discussed.)

DEFINITION 1.5. An integral J is *right-invariant* if $J(f) = J(fR_g)$ for $f \in K(G)$ and $g \in G$.

THEOREM 1.4. *Let J be a left-invariant integral given by*

$$J(f) = \int f(x)\nu_l(dx)$$

and let Δ be the modulus of G. Then the integral J_1 defined by

$$J_1(f) = \int f(x)\Delta(x^{-1})\nu_l(dx)$$

is right-invariant and satisfies

(1.3) $$\int f(x)\Delta(x^{-1})\nu_l(dx) = \int f(x^{-1})\nu_l(dx)$$

for $f \in K(G)$.

PROOF.   That J_1 is right-invariant follows from the definition of Δ and the calculation

$$\begin{aligned}
J_1(fR_g) &= \int f(xg^{-1})\Delta(x^{-1})\nu_l(dx) \\
&= \int f(xg^{-1})\frac{1}{\Delta(xg^{-1}g)}\nu_l(dx) \\
&= \frac{1}{\Delta(g)}\int \frac{f(xg^{-1})}{\Delta(xg^{-1})}\nu_l(dx) \\
&= \frac{\Delta(g)}{\Delta(g)}\int \frac{f(x)}{\Delta(x)}\nu_l(dx) = J_1(f).
\end{aligned}$$

A proof of the second equation can be found in Nachbin [(1965), page 78]. □

Equation (1.3) establishes a relationship between left and right Haar measure. Namely, if ν_l is a left Haar measure, then

(1.4) $$\nu_r(dx) = \Delta(x^{-1})\nu_l(dx)$$

is a right Haar measure. Further if ν_r is a right Haar measure, a similar construction shows that

(1.5) $$\nu_l(dx) = \Delta(x)\nu_r(dx)$$

is a left Haar measure.

Before turning to a few examples, we discuss the important special case of compact groups. Here is a characterization of compact groups in terms of Haar measure.

THEOREM 1.5.   *In order that G be compact, it is necessary and sufficient that there exist a finite Haar measure [that is, $\mu(G) < +\infty$].*

PROOF.   See Nachbin [(1965), page 75]. □

Let Δ be the modulus of a compact group G. We claim that $\Delta \equiv 1$. To see this, recall that Δ is a continuous function from G into $(0, \infty)$ which satisfies

$$\Delta(g_1g_2) = \Delta(g_1)\Delta(g_2),$$

that is, Δ is a group homomorphism. Thus $H = \{\Delta(g)|g \in G\}$ is a subgroup of the multiplicative group $(0, \infty)$. However, H is compact as it is the continuous image of the compact set G. If $H \neq \{1\}$, then there exists $a \in H$ such that $a > 1$. Hence $a^n \in H$ for $n = 1, 2, \dots$. But $\{a^n|n = 1, 2, \dots\}$ does not contain a convergent subsequence which contradicts the compactness of H. Thus $\Delta \equiv 1$ for any compact group G. Therefore all right Haar measures are left Haar measures and conversely. In these notes, the Haar measure on a compact group G is always normalized so that $\mu(G) = 1$, that is, the Haar measure is taken to be a probability measure.

Here are some examples of left and right Haar measures.

EXAMPLE 1.7. Take G to be the vector space R^n with addition as the group operation. Then ordinary Lebesgue measure on R^n is both left- and right-invariant (the group here is commutative) and so the modulus is $\Delta = 1$. $\square$

EXAMPLE 1.8. If G is any finite or denumerable discrete group, then counting measure is both left- and right-invariant and so the modulus is 1. $\square$

EXAMPLE 1.9. For this example, take $G = \mathrm{Gl}_n$ as in Example 1.3. Let dx denote Lebesgue measure restricted to Gl_n with Gl_n regarded as a subset of $\mathscr{L}_{n,n}$. Since G is open in $\mathscr{L}_{n,n}$, Gl_n has positive Lebesgue measure. Define an integral J by

$$J(f) = \int f(x) \frac{dx}{|\det(x)|^n}.$$

The claim is that J is both left- and right-invariant. To see this, first consider

$$J(L_g f) = \int f(g^{-1}x) \frac{dx}{|\det(x)|^n}$$

and make the change of variables $y = g^{-1}x$ so $x = gy$. To calculate the Jacobian of this transformation (defined by g), write x as a column vector

$$\begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix} \in R^{n^2},$$

where x^i is the ith column of x and do the same with y. In this notation, the equation $x = gy$ becomes

$$\begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix} = \begin{pmatrix} g & & & \\ & g & & \\ & & \ddots & \\ & & & g \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^n \end{pmatrix},$$

where the elements not indicated in the $n^2 \times n^2$ matrix are 0. Obviously this

linear transformation relating x to y has determinant equal to $(\det(g))^n$. Thus $x = gy$ implies

$$dx = \left|(\det(g))^n\right| dy = |\det(g)|^n \, dy.$$

Substituting this into the expression for $J(L_g f)$ yields

$$J(L_g f) = \int f(g^{-1}x) \frac{dx}{|\det(x)|^n} = \int f(y) \frac{|\det(g)|^n \, dy}{|\det(gy)|^n}$$

$$= \int f(y) \frac{dy}{|\det(y)|^n} = J(f).$$

Thus J is a left-invariant integral. A similar calculation (using rows of x and y instead of columns) shows that J is also right-invariant. Thus, the modulus of Gl_n is identically 1. $\square$

Groups for which the modulus is 1 are called *unimodular*. Clearly all commutative groups are unimodular as are finite or denumerable discrete groups and compact groups. In contrast to Gl_n, the following example shows that Al_n is not unimodular.

EXAMPLE 1.10.   Take $G = \mathrm{Al}_n$ so elements of Al_n are (x, a) with $x \in \mathrm{Gl}_n$ and $a \in R^n$. That Al_n can be regarded as an open subset of $\mathscr{L}_{n,n} \times R^n$ was discussed in Example 1.6. Let $dx \, da$ denote Lebesgue measure on $\mathscr{L}_{n,n} \times R^n$ restricted to Al_n. Define an integral J by

$$J(f) = \int f[(x, a)] \frac{dx \, da}{|\det(x)|^{n+1}}.$$

We now show that J is left-invariant. With $g \in \mathrm{Gl}_n$ and $\alpha \in R^n$, $h = (g, \alpha) \in \mathrm{Al}^n$ and

$$J(L_h f) = \int f\left[h^{-1}(x, a)\right] \frac{dx \, da}{|\det(x)|^{n+1}}.$$

Set $(y, b) = h^{-1}(x, a)$ so that

$$(x, a) = h(y, b) = (gy, gb + \alpha).$$

Thus $x = gy$ and $a = gb + \alpha$. From the last example, the Jacobian of the transformation $x = gy$ is $|\det(g)|^n$ and it is obvious that the Jacobian of the transformation $a = gb + \alpha$ is $|\det(g)|$. Making this change of variables, we have

$$J(L_h f) = \int f[(y, b)] \frac{|\det(g)|^{n+1}}{|\det(gy)|^{n+1}} \, dy \, db$$

$$= \int f[(y, b)] \frac{dy \, db}{|\det(y)|^{n+1}} = J(f).$$

A similar (but slightly different) calculation shows that

$$J_1(f) = \int f[(x, a)] \frac{dx\, da}{|\det(x)|^n}$$

is a right-invariant integral. From Equation (1.4), it follows that the modulus of Al_n is

$$\Delta(x, a) = \frac{1}{|\det(x)|}. \qquad \Box$$

Other examples of invariant integrals are given in the next section after the discussion of multipliers and relatively invariant integrals.

1.4. Multipliers and relatively invariant integrals. The modulus Δ of a topological group G provides an example of a multiplier, namely a continuous homomorphism of G into the multiplicative group of $(0, \infty)$. Multipliers play the role of Jacobians for some integrals on $K(G)$.

DEFINITION 1.6.   A function χ defined on G to $(0, \infty)$ is a *multiplier* if χ is continuous and $\chi(g_1 g_2) = \chi(g_1)\chi(g_2)$ for $g_1, g_2 \in G$.

It is clear that $\chi(e) = 1$ and $\chi(g_1 g_2) = \chi(g_2 g_1)$ for any multiplier χ, so that $\chi(g^{-1}) = 1/\chi(g)$. Further, $\chi(G) = \{\chi(g) | g \in G\}$ is a subgroup of $(0, \infty)$. Thus, if G is compact, $\chi(G)$ must be a compact subgroup of $(0, \infty)$ and thus $\chi(G) = \{1\}$. Hence the trivial multiplier, $\chi = 1$, is the only multiplier defined on a compact group.

DEFINITION 1.7.   An integral J on $K(G)$ is *relatively* (left) *invariant* with multiplier χ if for each $g \in G_1$,

$$J(L_g f) = \int f(g^{-1}x)m(dx) = \chi(g)\int f(x)m(dx) = \chi(g)J(f).$$

The relationship between invariant and relatively invariant integrals is given in the following result.

THEOREM 1.6.   *Let χ be a multiplier on G.*

(i) *If $J_1(f) = \int f(x)m(dx)$ is relatively invariant with multiplier χ, then*

$$J(f) = \int f(x)\chi(x^{-1})m(dx)$$

is left-invariant.

(ii) *If $J(f) = \int f(x)\nu_l(dx)$ is left-invariant, then*

$$J_1(f) = \int f(x)\chi(x)\nu_l(dx)$$

is relatively invariant with multiplier χ.

PROOF.   Only the proof of (i) is given, as the proof of (ii) is similar. For $g \in G$,

$$J(L_g f) = \int f(g^{-1}x)\chi(x^{-1})m(dx) = \int f(g^{-1}x)\chi\big((g^{-1}x)^{-1}g^{-1}\big)m(dx)$$

$$= \chi(g^{-1})\chi(g)\int f(x)\chi(x^{-1})m(dx) = J(f).$$

Thus, J is left invariant. $\square$

Theorem 1.6 shows how to construct a relatively invariant integral with a given multiplier from a left-invariant measure, and conversely. In practice, it is the converse which is somewhat more useful since it provides a very useful method of constructing a left-invariant integral in some cases. Here is a rather informal description of the method which is used below to construct a left-invariant measure on the group G_T^+ of Example 1.4. Suppose the topological group G is subset of a Euclidean space R^m such that G has positive Lebesgue measure. Define a integral on $K(G)$ by

$$J_1(f) = \int_G f(x)\,dx,$$

where dx denotes Lebesgue measure restricted to G. If we can show that

(1.6) $$\int_G f(g^{-1}x)\,dx = \chi(g)\int_G f(x)\,dx \quad \text{for } g \in G,$$

then

$$J(f) = \int_G f(x)\chi(x^{-1})\,dx$$

is left-invariant. The verification of (1.6) is ordinarily carried out by changing variables and computing a Jacobian.

One can also define relatively right-invariant integrals, but their relationship to relatively left integrals follows from (1.5) and the equation

(1.7) $$m(dx) = \chi(x)\nu_l(dx),$$

which relates the left Haar measure ν_l and the measure m which defines a relatively invariant integral with multiplier χ. For this reason, relatively right-invariant integrals are not mentioned explicitly.

It follows from Theorem 1.6 that the problem of describing all the relatively invariant integrals on $K(G)$ can be solved by first exhibiting one relatively invariant integral and then describing all the multipliers on G. This program is carried out in some examples that follow. Before turning to these examples, it is useful to describe a few general facts concerning multipliers.

THEOREM 1.7.   *Suppose the topological group G is the direct product $G_1 \times G_2$ of two topological groups [that is, each $g \in G$ has the form $g = (g_1, g_2)$ with $g_i \in G_i$ and the group operation is $g \circ h = (g_1 h_1, g_2 h_2)$ where $g = (g_1, g_2)$ and*

$h = (h_1, h_2)$]. *The topology on G is assumed to be the product topology. Then χ is a multiplier on G iff*

(1.8)
$$\chi[(g_1, g_2)] = \chi_1(g_1)\chi_2(g_2),$$

where χ_i is a multiplier on G_i, $i = 1, 2$.

PROOF. If χ has the form (1.8), it is clear that χ is a multiplier. Conversely, suppose that χ is a multiplier. For $g = (g_1, g_2) \in G$, write

$$(g_1, g_2) = (g_1, e_2)(e_1, g_2),$$

where e_i is the identity in G_i, $i = 1, 2$. Then consider

$$\chi_1(g_1) = \chi[(g_1, e_2)], \qquad g_1 \in G_1,$$
$$\chi_2(g_2) = \chi[(e_1, g_2)], \qquad g_2 \in G_2.$$

The continuity of χ_i on G_i follows since χ is continuous and χ_i is χ restricted to a closed subset of G. That (1.8) holds follows since χ is a homomorphism. □

Also observe that if χ is a multiplier on G and H is a closed subgroup of G, then the restriction of χ to H is a multiplier on H. In particular, if H is a compact subgroup of G, then $\chi(h) = 1$ for all $h \in H$ since compact groups have only trivial multipliers.

EXAMPLE 1.11. In this example, all the relatively invariant integrals on R^n are computed. Because R^n is an n-fold product group, Theorem 1.7 shows that it is sufficient to treat the case $n = 1$. Thus, suppose that χ_1 is a continuous function from R^1 to $(0, \infty)$ which satisfies the equation

$$\chi_1(x + y) = \chi_1(x)\chi_1(y).$$

Taking logs of both sides of this leads to

$$\log \chi_1(x + y) = f(x + y) = f(x) + f(y)$$

for $x, y \in R^1$. This is the well known Cauchy functional equation which has the solution (because f is continuous)

$$f(x) = cx,$$

where c is some real number. Thus,

$$\chi_1(x) = \exp[cx], \qquad x \in R^1,$$

where c is a fixed real number. Applying Theorem 1.7 shows that χ is a multiplier on R^n iff χ has the form

$$\chi(u) = \exp\left[\sum_1^n c_i u_i\right],$$

where $u \in R^n$ has coordinates $u_1, \ldots, u_n$ and $c_1, \ldots, c_n$ are fixed real numbers. Thus, a Radon measure m is relatively invariant with multiplier χ given above iff

$$m(du) = c_0\chi(u)\,du, \qquad u \in R^n,$$

where c_0 is a fixed constant and du is Lebesgue measure on R^n.

When G is the product multiplicative group $(0, \infty)^n$, the continuous isomorphism $\phi\colon G \to R^n$ given by

$$\phi(x) = \begin{pmatrix} \log x_1 \\ \log x_2 \\ \vdots \\ \log x_n \end{pmatrix} \in R^n, \qquad x \in (0, \infty)^n,$$

can be used to show that the measure

$$\nu_l(dx) = \frac{dx_1 \cdots dx_n}{\prod_{i=1}^n x_i}$$

is right- and left-invariant on G. Further, χ is a multiplier on G iff χ has the form

$$\chi(x) = \prod_{i=1}^n (x_i)^{c_i},$$

where $c_1, \ldots, c_n$ are fixed real numbers. Thus, all the relatively invariant integrals on $K(G)$ are known. $\square$

EXAMPLE 1.12. Consider $G = \mathrm{Gl}_n$ as in Example 1.3 so

$$\nu_l(dx) = \frac{dx}{|\det(x)|^n}$$

is left- and right-invariant. To describe all the multipliers on Gl_n, the functional equation

(1.9) $$\chi(xy) = \chi(x)\chi(y), \qquad x, y \in \mathrm{Gl}_n,$$

must be solved. To this end, recall the singular value decomposition of an $n \times n$ matrix $x \in \mathrm{Gl}_n$; it is

$$x = \alpha D \beta,$$

where $\alpha, \beta \in O_n$ and D is an $n \times n$ diagonal matrix with positive diagonal elements [for example, see Eaton (1983), page 58]. Thus if χ is continuous and satisfies (1.9), then

$$\chi(x) = \chi(\alpha D \beta) = \chi(\alpha)\chi(D)\chi(\beta) = \chi(D)$$

since α and β are both elements of the compact group O_n so $\chi(\alpha) = \chi(\beta) = 1$. Now, write D as

$$D = \prod_{i=1}^n D_i(d_i),$$

where d_i is the ith diagonal of D and $D_i(d_i)$ has ith diagonal equal to d_i and all other diagonal elements equal to 1. Then, because $D_i(d_i) \in \mathrm{Gl}_n$,

$$\chi(D) = \chi\left(\prod_{i=1}^n D_i(d_i)\right) = \prod_{i=1}^n \chi(D_i(d_i)).$$

A bit of reflection shows that there exists a permutation matrix $P \in \mathbf{P}_n \subset O_n$ such that

$$D_i(d_i) = PD_1(d_i)P', \qquad i = 1, \ldots, n,$$

so that

$$\chi(D_i(d_i)) = \chi(D_1(d_i)), \qquad i = 1, \ldots, n,$$

because $\chi(P) = \chi(P') = 1$. Thus,

$$\chi(D) = \prod_{i=1}^{n} \chi(D_i(d_i)) = \prod_{i=1}^{n} \chi(D_1(d_i))$$

$$= \chi\left(\prod_{i=1}^{n} D_1(d_i)\right) = \chi\left(D_1\left(\prod_{i=1}^{n} d_i\right)\right).$$

This shows that χ is determined by its values on matrices of the form

$$D_1(\lambda) = \begin{pmatrix} \lambda & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \in \mathrm{Gl}_n, \qquad \lambda > 0.$$

Since these matrices form a subgroup of Gl_n which is isomorphic to the multiplicative group $(0, \infty)$, it follows from Example 1.11 that

$$\chi(D_1(\lambda)) = \lambda^c$$

for some fixed real number c. Now, retracing the above steps, we have

$$\chi(x) = \chi(D) = \left(\prod_{i=1}^{n} d_i\right)^c$$

$$= |\det(x)|^c.$$

Hence, all multipliers on Gl_n are some power of $|\det(x)|$. The converse is clear. Thus every relatively invariant measure has the form

$$c_0 |\det(x)|^c \frac{dx}{|\det(x)|^n}$$

for some fixed number c and some $c_0 > 0$. $\square$

EXAMPLE 1.13.   With $G = \mathrm{Al}_n$ of Example 1.6, recall that the group operation in Al_n is

$$(g_1, x_1)(g_2, x_2) = (g_1 g_2, g_1 x_2 + x_1)$$

for $g_i \in \mathrm{Gl}_n$ and $x_i \in R_n$, $i = 1, 2$. To describe the multipliers on Al_n, consider χ on Al_n and write $(g, x) \in \mathrm{Al}_n$ as

$$(g, x) = (g, 0)(e, g^{-1}x) = (e, x)(g, 0),$$

where e is the identity in Gl_n. Thus, a multiplier χ must satisfy

$$\chi(g, x) = \chi(g, 0)\chi(e, g^{-1}x) = \chi(g, 0)\chi(e, x)$$

so

$$\chi(e, g^{-1}x) = \chi(e, x)$$

for all $g \in \mathrm{Gl}_n$. Letting g^{-1} converge to 0, the continuity of χ implies that

$$1 = \chi(e, 0) = \chi(e, x), \qquad x \in R^n.$$

Hence

$$\chi(g, x) = \chi(g, 0), \qquad g \in \mathrm{Gl}_n.$$

But $g \to \chi(g, 0)$ defines a multiplier on Gl_n so by the previous example,

$$\chi(g, x) = |\det(g)|^c$$

for some fixed real number c. Conversely, it is clear that such a function on Al_n is a multiplier. Thus, all the multipliers on Al_n have been given. $\square$

EXAMPLE 1.14. The final example of this chapter concerns $G_T^+ \subset \mathrm{Gl}_n$. The left and right Haar measures, the modular function and all the multipliers for G_T^+ are derived here. We are going to use the method described after the proof of Theorem 1.6. For $x \in G_T^+$, x has the form

$$x = \begin{pmatrix} x_{11} & 0 & \cdots & 0 \\ x_{21} & x_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

with $x_{ii} > 0$ and $x_{ij} \in R^n$, $i > j$. Let dx denote Lebesgue measure restricted to the set of such x's in $n(n+1)/2$ dimensional space. Define an integral J on $K(G_T^+)$ by

$$J(f) = \int f(x)\, dx$$

and for $g \in G_T^+$, consider

$$J(L_g f) = \int f(g^{-1}x)\, dx.$$

With $y = g^{-1}x$ so $x = gy$, the Jacobian of this transformation on $n(n+1)/2$ coordinate space is

$$\chi_0(g) = \prod_{i=1}^{n} g_{ii}^i,$$

where $g_{11}, \ldots, g_{nn}$ are the diagonal elements of $g \in G_T^+$ [see Eaton (1983), page 171]. Hence $dx = \chi_0(g)\, dy$ so that

$$J(L_g f) = \chi_0(g) J(f).$$

From Theorem 1.6, we have that

(1.10)
$$\nu_l(dx) = \frac{dx}{\chi_0(x)} = \frac{dx}{\prod_{i=1}^{n} x_{ii}^i}$$

is a left Haar measure on $G_T^+ \subset \mathrm{Gl}_n$. The modulus of G_T^+ is computed using the definition (1.2). Thus, consider

$$\int f(xg^{-1})\nu_l(dx) = \int f(xg^{-1})\frac{dx}{\chi_0(x)}$$

and let $y = xg^{-1}$ so $x = yg$. The Jacobian of this transform is

$$\chi_1(g) = \prod_{i=1}^{n} g_{ii}^{n-i+1},$$

where $g_{11}, \ldots, g_{nn}$ are the diagonal elements of g. Making this change of variables in the above integral yields

$$\int f(xg^{-1})\nu_l(dx) = \int f(y)\chi_1(g)\frac{dy}{\chi_0(yg)}$$

$$= \frac{\chi_1(g)}{\chi_0(g)}\int f(y)\frac{dy}{\chi_0(y)}$$

$$= \frac{\chi_1(g)}{\chi_0(g)}\int f(x)\nu_l(dx).$$

Thus, by definition, the modulus of G_T^+ is

$$(1.11) \qquad \Delta(g) = \frac{\chi_1(g)}{\chi_0(g)} = \prod_{i=1}^{n} g_{ii}^{n-2i+1}$$

so a right Haar measure is

$$(1.12) \qquad \nu_r(dx) = \frac{1}{\Delta(x)}\nu_l(dx) = \frac{dx}{\prod_{i=1}^{n}x_{ii}^{n-i+1}}. \qquad \square$$

Here is a characterization of the multipliers on $G_T^+ \subset \mathrm{Gl}_n$.

THEOREM 1.8. *All multipliers on $G_T^+ \subset \mathrm{Gl}_n$ have the form*

$$\chi(g) = \prod_{i=1}^{n}(g_{ii})^{c_i},$$

where $c_1, \ldots, c_n$ are fixed real numbers. Conversely, any such function is a multiplier.

PROOF. That such a function is a multiplier is easily checked. To show that all multipliers have the claimed form, we argue by induction. For $n = 1$, $G_T^+ = (0, \infty)$ and the assertion follows from Example 1.11. Assume the result is true for $G_T^+ \subset \mathrm{Gl}_n$ and consider $G_T^+ \subset \mathrm{Gl}_{n+1}$. For $g \in G_T^+ \subset \mathrm{Gl}_{n+1}$, partition g as

$$g = \begin{pmatrix} x_{11} & 0 \\ x_{21} & x_{22} \end{pmatrix},$$

where x_{11} is $n \times n$, x_{21} is $1 \times n$ and x_{22} is in $(0, \infty)$. With e denoting the $n \times n$

identity matrix, observe that

$$\begin{pmatrix} x_{11} & 0 \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{11} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e & 0 \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} e & 0 \\ x_{21}x_{11}^{-1} & x_{22} \end{pmatrix} \begin{pmatrix} x_{11} & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, if χ is a multiplier,

$$\chi \begin{pmatrix} x_{11} & 0 \\ 0 & 1 \end{pmatrix} \chi \begin{pmatrix} e & 0 \\ x_{21} & x_{22} \end{pmatrix} = \chi \begin{pmatrix} e & 0 \\ x_{21}x_{11}^{-1} & x_{12} \end{pmatrix} \chi \begin{pmatrix} x_{11} & 0 \\ 0 & 1 \end{pmatrix}$$

so that

$$\chi \begin{pmatrix} e & 0 \\ x_{21} & x_{22} \end{pmatrix} = \chi \begin{pmatrix} e & 0 \\ x_{21}x_{11}^{-1} & x_{22} \end{pmatrix}.$$

Letting x_{11}^{-1} converge to 0, the continuity of χ implies that

$$\chi \begin{pmatrix} e & 0 \\ x_{21} & x_{22} \end{pmatrix} = \chi \begin{pmatrix} e & 0 \\ 0 & x_{22} \end{pmatrix}.$$

But, the function

$$x_{22} \rightarrow \chi \begin{pmatrix} e & 0 \\ 0 & x_{22} \end{pmatrix}$$

defines a multiplier on $(0, \infty)$, so

$$\chi \begin{pmatrix} e & 0 \\ 0 & x_{22} \end{pmatrix} = (x_{22})^{c_{n+1}}$$

for some real c_{n+1}. Also, the function

$$x_{11} \rightarrow \chi \begin{pmatrix} x_{11} & 0 \\ 0 & 1 \end{pmatrix}$$

defines a multiplier on $G_T^+ \subset \mathrm{Gl}_n$. Since the diagonal elements of x_{11} are $g_{11}, \ldots, g_{n,n}$ and $x_{22} = g_{n+1,n+1}$, using the induction hypothesis, we have

$$\chi(g) = \chi \begin{pmatrix} x_{11} & 0 \\ x_{21} & x_{22} \end{pmatrix} = \chi \begin{pmatrix} x_{11} & 0 \\ 0 & 1 \end{pmatrix} \chi \begin{pmatrix} e & 0 \\ 0 & x_{22} \end{pmatrix}$$

$$= \left(\prod_{i=1}^{n} g_{ii}^{c_i} \right) g_{n+1}^{c_{n+1}} = \prod_{i=1}^{n+1} g_{ii}^{c_i}.$$

This completes the proof. $\square$

That the compact group O_n has a unique left- and right-invariant probability measure follows from the general theory of Haar measure. However, the explicit construction of this probability, in terms of a "coordinate system" for O_n is a bit technical and requires some familiarity with differential forms. This topic is not discussed here, but in subsequent material, it is shown how to construct a random orthogonal matrix in O_n whose distribution is exactly the Haar measure on O_n. This construction involves the multivariate normal distribution.