

A note on bounds for VC dimensions

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Abstract: We provide bounds for the VC dimension of class of sets formed by unions, intersections, and products of VC classes of sets $\mathcal{C}_1, \dots, \mathcal{C}_m$.

1. Introduction and main results

Let \mathcal{C} be a class of subsets of a set \mathcal{X} . An arbitrary set of n points $\{x_1, \dots, x_n\}$ has 2^n subsets. We say that \mathcal{C} *picks out* a certain subset from $\{x_1, \dots, x_n\}$ if this can be formed as a set of the form $C \cap \{x_1, \dots, x_n\}$ for some $C \in \mathcal{C}$. The collection \mathcal{C} is said to *shatter* $\{x_1, \dots, x_n\}$ if each of its 2^n subsets can be picked out by \mathcal{C} . The *VC - dimension* $V(\mathcal{C})$ is the largest cardinality of a set shattered by \mathcal{C} (or $+\infty$ if arbitrarily large finite sets are shattered); more formally, if

$$\Delta_n(\mathcal{C}, x_1, \dots, x_n) = \#\{C \cap \{x_1, \dots, x_n\} : C \in \mathcal{C}\},$$

then

$$V(\mathcal{C}) = \sup \left\{ n : \max_{x_1, \dots, x_n} \Delta_n(\mathcal{C}, x_1, \dots, x_n) = 2^n \right\},$$

and $V(\mathcal{C}) = -1$ if \mathcal{C} is empty. (The VC-dimension $V(\mathcal{C})$ defined here corresponds to $S(\mathcal{C})$ as defined by [5] page 134. Dudley, and following him ourselves in [11], used the notation $V(\mathcal{C})$ for the *VC-index*, which is the dimension plus 1. We have switched to using $V(\mathcal{C})$ for the VC-dimension rather than the VC-index, because formulas are simpler in terms of dimension and because the machine learning literature uses dimension rather than index.)

Now suppose that $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m$ are VC-classes of subsets of a given set \mathcal{X} with VC dimensions V_1, \dots, V_m . It is known that the classes $\sqcup_{j=1}^m \mathcal{C}_j, \sqcap_{j=1}^m \mathcal{C}_j$ defined by

$$\begin{aligned} \sqcup_{j=1}^m \mathcal{C}_j &\equiv \{ \cup_{j=1}^m C_j : C_j \in \mathcal{C}_j, j = 1, \dots, m \}, \\ \sqcap_{j=1}^m \mathcal{C}_j &\equiv \{ \cap_{j=1}^m C_j : C_j \in \mathcal{C}_j, j = 1, \dots, m \}, \end{aligned}$$

are again VC: when $\mathcal{C}_1 = \dots = \mathcal{C}_m = \mathcal{C}$ and $m = k$, this is due to [2] (see also [3], Theorem 9.2.3, page 85, and [5], Theorem 4.2.4, page 141); for general $\mathcal{C}_1, \mathcal{C}_2$ and $m = 2$ it was shown by [3], Theorem 9.2.6, page 87, (see also [5], Theorem 4.5.3, page 153), and [9], Lemma 15, page 18. See also [8], Lemma 2.5, page 1032. For a summary of these types of VC preservation results, see e.g. [11], page 147. Similarly,

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if $\mathcal{D}_1, \dots, \mathcal{D}_m$ are VC-classes of subsets of sets $\mathcal{X}_1, \dots, \mathcal{X}_m$, then the class of product sets $\boxtimes_{j=1}^m \mathcal{D}_j$ defined by

$$\boxtimes_{j=1}^m \mathcal{D}_j \equiv \{D_1 \times \dots \times D_m : D_j \in \mathcal{D}_j, j = 1, \dots, m\}$$

is a VC-class of subsets of $\mathcal{X}_1 \times \dots \times \mathcal{X}_m$. This was proved in [1], Proposition 2.5, and in [3], Theorem 9.2.6, page 87 (see also [5], Theorem 4.2.4, page 141).

In the case of $m = 2$, consider the maximal VC dimensions $\max V(\mathcal{C}_1 \sqcup \mathcal{C}_2)$, $\max V(\mathcal{C}_1 \cap \mathcal{C}_2)$, and $\max V(\mathcal{D}_1 \boxtimes \mathcal{D}_2)$, where the maxima are over all classes $\mathcal{C}_1, \mathcal{C}_2$ (or $\mathcal{D}_1, \mathcal{D}_2$ in the last case) with $V(\mathcal{C}_1) = V_1$, $V(\mathcal{C}_2) = V_2$ for fixed V_1, V_2 . As shown in [3], Theorem 9.2.7, these are all equal:

$$\max V(\mathcal{C}_1 \sqcup \mathcal{C}_2) = \max V(\mathcal{C}_1 \cap \mathcal{C}_2) = \max V(\mathcal{D}_1 \boxtimes \mathcal{D}_2) \equiv S(V_1, V_2).$$

[3] provided the following bound for this common value:

Proposition 1.1. $S(V_1, V_2) \leq T(V_1, V_2)$ where, with $rC_{\leq v} \equiv \sum_{j=0}^v \binom{r}{j}$,

$$(1.1) \quad T(V_1, V_2) \equiv \sup\{r \in \mathbb{N} : rC_{\leq V_1} rC_{\leq V_2} \geq 2^r\}.$$

Because of the somewhat inexplicit nature of the bound in (1.1), this proposition seems not to have been greatly used so far.

Furthermore, [4] (Theorem 4.27, page 63; Proposition 4.38, page 64) showed that $S(1, k) \leq 2k + 1$ for all $k \geq 1$ with equality for $k = 1, 2, 3$.

Here we give a further more explicit bound for $T(V_1, V_2)$ and extend the bounds to the case of general $m \geq 2$. Our main result is the following proposition.

Theorem 1.1. *Let $V \equiv \sum_{j=1}^m V_j$. Then the following bounds hold:*

$$(1.2) \quad \left\{ \begin{array}{l} V(\sqcup_{j=1}^m \mathcal{C}_j) \\ V(\cap_{j=1}^m \mathcal{C}_j) \\ V(\boxtimes_1^m \mathcal{D}_j) \end{array} \right\} \leq c_1 V \log \left(\frac{c_2 m}{e^{Ent(\underline{V})/\bar{V}}} \right) \leq c_1 V \log(c_2 m),$$

where $\underline{V} \equiv (V_1, \dots, V_m)$, $c_1 \equiv \frac{e}{(e-1)\log(2)} \doteq 2.28231\dots$, $c_2 \equiv \frac{e}{\log 2} \doteq 3.92165\dots$,

$$Ent(\underline{V}) \equiv m^{-1} \sum_{j=1}^m V_j \log V_j - \bar{V} \log \bar{V}$$

is the ‘‘entropy’’ of the V_j ’s under the discrete uniform distribution with weights $1/m$ and $\bar{V} = m^{-1} \sum_{j=1}^m V_j$.

Corollary 1.1. *For $m = 2$ the following bounds hold:*

$$S(V_1, V_2) \leq T(V_1, V_2) \leq \left\lfloor c_1 (V_1 + V_2) \log \left(\frac{2c_2}{\exp(Ent(\underline{V})/\bar{V})} \right) \right\rfloor \equiv R(V_1, V_2)$$

where $c_1, c_2, Ent(\underline{V})$, and \bar{V} are as in Theorem 1.

Proof. The subsets picked out by $\cap_i \mathcal{C}_i$ from a given set of points $\{x_1, \dots, x_n\}$ in \mathcal{X} are the sets $C_1 \cap \dots \cap C_m \cap \{x_1, \dots, x_n\}$. They can be formed by first forming all *different* sets of the form $C_1 \cap \{x_1, \dots, x_n\}$ for $C_1 \in \mathcal{C}_1$, next intersecting each of these sets by sets in \mathcal{C}_2 giving all sets of the form $C_1 \cap C_2 \cap \{x_1, \dots, x_n\}$, etc. If $\Delta_n(\mathcal{C}, y_1, \dots, y_n) \equiv \#\{C \cap \{y_1, \dots, y_n\} : C \in \mathcal{C}\}$ and $\Delta_n(\mathcal{C}) = \max_{y_1, \dots, y_n} \Delta_n(\mathcal{C}, y_1, \dots, y_n)$ for every collection of sets \mathcal{C} and points y_1, \dots, y_n (as

in [11], page 135), then in the first step we obtain at most $\Delta_n(\mathcal{C}_1)$ different sets, each with n or fewer points. In the second step each of these sets gives rise to at most $\Delta_n(\mathcal{C}_2)$ different sets, etc. We conclude that

$$\Delta_n(\prod_i \mathcal{C}_i) \leq \prod_i \Delta_n(\mathcal{C}_i) \leq \prod_i \left(\frac{en}{V_i}\right)^{V_i},$$

by [11], Corollary 2.6.3, page 136, and the bound $(en/s)^s$ for the number of subsets of size smaller than s for $n \geq s$. By definition the left side of the display is 2^n for n equal to the VC-dimension of $\prod_i \mathcal{C}_i$. We conclude that

$$2^n \leq \prod_{i=1}^m \left(\frac{en}{V_i}\right)^{V_i},$$

or

$$n \log 2 \leq \sum_{i=1}^m V_i \log(e/V_i) + \left(\sum_{i=1}^m V_i\right) \log n.$$

With $V \equiv \sum_i V_i$, define $r = en/V$. Then the last display implies that

$$rV \frac{\log 2}{e} \leq \sum_i V_i \log(e/V_i) + V \log(rV/e),$$

or

$$\begin{aligned} r \frac{\log 2}{e} &\leq \log r + \log V - \frac{\sum_i V_i \log V_i}{V} \\ &= \log r + \log m - \frac{Ent(\underline{V})}{\bar{V}} = \log \left(\frac{mr}{e^{Ent(\underline{V})/\bar{V}}} \right), \end{aligned}$$

and this inequality can in turn be rewritten as

$$\frac{x}{\log x} \equiv \frac{mr/e^{Ent(\underline{V})/\bar{V}}}{\log \left(mr/e^{Ent(\underline{V})/\bar{V}} \right)} \leq \frac{m}{e^{Ent(\underline{V})/\bar{V}}} \cdot \frac{e}{\log 2} \equiv y.$$

Now note that $g(x) \equiv x/\log x \leq y$ for $x \geq e$ implies that $x \leq (e/(e-1))y \log y$; g is minimized by $x = e$ and is increasing; furthermore $y \geq g(x)$ for $x \geq e$ implies that

$$\log y \geq \log x - \log \log x = \log x \left(1 - \frac{\log \log x}{\log x} \right) \geq \log x \left(1 - \frac{1}{e} \right)$$

so that

$$x \leq y \log x \leq y \left(1 - \frac{1}{e} \right)^{-1} \log y = \frac{e}{e-1} y \log y.$$

Thus we conclude that

$$\frac{mr}{e^{Ent(\underline{V})/\bar{V}}} \leq \frac{e}{e-1} \frac{me}{e^{Ent(\underline{V})/\bar{V}} \log 2} \log \left(\frac{m}{e^{Ent(\underline{V})/\bar{V}}} \cdot \frac{e}{\log 2} \right),$$

which implies that

$$r \leq \frac{e^2}{(e-1) \log 2} \log \left(\frac{m}{\exp(Ent(\underline{V})/\bar{V})} \cdot \frac{e}{\log 2} \right).$$

Expressing this in terms of n yields the first inequality (1.2). The second inequality holds since $Ent(\underline{V}) \geq 0$ implies $\exp(Ent(\underline{V})/\bar{V}) \geq 1$.

The corresponding statement for the unions follows because a class \mathcal{C} of sets and the class \mathcal{C}^c of their complements possess the same VC-dimension, and $\cup_i C_i = (\cap_i C_i^c)^c$.

In the case of products, note that

$$\Delta_n(\boxtimes_1^m \mathcal{D}_j) \leq \prod_1^m \Delta_n(\mathcal{D}_j) \leq \prod_{j=1}^m \left(\frac{en}{V_j} \right)^{V_j},$$

and then the rest of the proof proceeds as in the case of intersections. \square

It follows from concavity of $x \mapsto \log x$ that with $p_j \equiv V_j / \sum_{i=1}^m V_i$,

$$\frac{\sum_{j=1}^m V_j \log V_j}{\sum_{j=1}^m V_j} = \sum_1^m p_j \log V_j \leq \log \left(\sum_1^m p_j V_j \right) \leq \log \left(\sum_1^m V_j \right)$$

and hence

$$(1.3) \quad 1 \leq \frac{m}{e^{Ent(\underline{V})/\bar{V}}} \leq m,$$

or $0 \leq Ent(\underline{V})/\bar{V} \leq \log m$, or

$$0 \leq Ent(\underline{V}) \leq \bar{V} \log m.$$

Here are two examples showing that the quantity $m/e^{Ent(\underline{V})/\bar{V}}$ can be very close to 1 (rather than m) if the V_i 's are quite heterogeneous, even if m is large.

Example 1.1. Suppose that $r \in \mathbb{N}$ (large), and that $V_i = r^i$ for $i = 1, \dots, m$. Then it is not hard to show that

$$\frac{m}{e^{Ent(\underline{V})/\bar{V}}} \rightarrow \frac{r}{r-1} r^{1/(r-1)} = \frac{r}{r-1} \exp((r-1)^{-1} \log r)$$

as $m \rightarrow \infty$ where the right side can be made arbitrarily close to 1 by choosing r large.

Example 1.2. Suppose that $m = 2$ and that $V_1 = k$, $V_2 = rk$ for some $r \in \mathbb{N}$. Then

$$Ent(\underline{V})/\bar{V} = \log 2 - \frac{1}{r+1} \log((r+1)(1+1/r)^r) \rightarrow \log 2$$

as $r \rightarrow \infty$ for any fixed k . Therefore

$$\frac{2}{e^{Ent(\underline{V})/\bar{V}}} \rightarrow 1$$

as $r \rightarrow \infty$ for any fixed k .

Our last example shows that the bound of Theorem 1.1 may improve considerably on the bounds resulting from iteration of Dudley's bound $S(1, k) \leq 2k + 1$.

Example 1.3. Suppose $V_1 = V(\mathcal{C}_1) = k$ and $V_j = V(\mathcal{C}_j) = 1$ for $j = 2, \dots, m$. Iterative application of Dudley's bound $S(1, k) \leq 2k + 1$ yields $V(\cap_{j=1}^m \mathcal{C}_j) \leq 2^{m-1}(k+1) - 1$, which grows exponentially as $m \rightarrow \infty$. On the other hand, Theorem 1.1 yields $V(\cap_{j=1}^m \mathcal{C}_j) \leq c_1(m+k-1) \log(c_2 m)$ which is of order $c_1 m \log m$ as $m \rightarrow \infty$.

Although we have succeeded here in providing quantitative bounds for $V(\sqcup_{j=1}^m \mathcal{C}_j)$, $V(\prod_{j=1}^m \mathcal{C}_j)$, and $V(\boxtimes_1^m \mathcal{D}_j)$, it seems that we are far from being able to provide quantitative bounds for the VC - dimensions of the (much larger) classes involved in [6], [7], and [10].

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