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Interpolation spaces and the CLT in Banach spaces

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Abstract: Necessary and sufficient conditions for the classical central limit theorem (CLT) for i.i.d. random vectors in an arbitrary separable Banach space require not only assumptions on the original distribution, but also on the sample. What we do here is to continue our study of the CLT in terms of the original distribution. Of course, some new ingredient must be introduced, so we allow slight modifications of the random vectors. In particular, we restrict our modifications to be continuous, and to be no larger than a fixed small number, or in some cases a fixed small proportion of the magnitude of the individual elements of the sample. We find that if we use certain interpolation space norms to measure the magnitude of such modifications, then the CLT can be improved. Examples of our result are also included.

1. Introduction

Let B denote a separable Banach space with norm $\|\cdot\|$ and let μ be a probability measure on B for which continuous linear functionals have mean zero and finite variance. Then there is a Hilbert space H_μ with norm $\|\cdot\|_\mu$ determined by the covariance of μ such that $H_\mu \subseteq B$, and the identity map from H_μ into B is continuous. Furthermore, for all $\epsilon > 0$ and x in the B-norm closure of H_μ, there is a unique point with minimum H_μ-norm in the B-norm ball of radius $\epsilon > 0$ and center x. We denote this point by $T_\epsilon(x)$, and its precise definition appears in (2.1) below. The existence, and the continuity properties of the mapping $T_\epsilon(\cdot)$, can be found in [5]. In addition, if X is a random variable in B with law μ, then under a variety of conditions we obtain the central limit theorem (CLT) for $T_\epsilon(X)$ and certain modifications of such a quantity, even when X itself fails the CLT. The motivation for the use of the mapping $T_\epsilon(\cdot)$ comes from the large deviation rates for the Gaussian measure γ determined by the covariance of X whenever γ exists. However, this is only motivation, and our results apply even when this Gaussian law fails to exist.

One of the drawbacks to the mappings $T_\epsilon(\cdot)$ is that they do not provide universal improvement for the CLT in all Banach spaces. This is particularly true in type-2 Banach spaces, and can be seen from Theorem 4 of [5]. There we show that in type-2 Banach spaces $T_\epsilon(X)$ satisfies the CLT if and only if X does. Another difficulty with these mappings is that they are not easy to compute, so we also provided some alternatives. In particular, the methods used to estimate $\|T_\epsilon(x)\|_\mu$ in [5] are indirect,

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and undoubtedly somewhat imprecise. However, if one tries to compute $\|T_\epsilon(x)\|_\mu$ exactly, one immediately encounters substantial difficulties. That these difficulties are present should perhaps not be suprising, as the computation involves an infimum over an infinite dimensional set. Furthermore, since $T_\epsilon(x)$ has the uniqueness property indicated, we see that if K is the unit ball of H_μ and $\|x\| > \epsilon$, then in determining $\|T_\epsilon(x)\|_\mu = r$, we are actually finding the "best approximation of x" within B-norm distance ϵ in the set rK.

More precisely, letting

$$E(x,r) = \inf\{\|x - rk\| : k \in K\},$$

we see $\|T_\epsilon(x)\|_\mu = r$ if and only if $E(x,r) = \epsilon$. The quantity $E(x,r)$ arises in approximation theory, and is called the E-functional in this context, i.e. see [2]. The use of the E-functional, and its connection with interpolation theory, also appeared recently in the paper [8], where topics in learning theory are addressed. Furthermore, Theorem 3.1 of [8] has implications for the CLT provided we define some additional mappings. However, before we turn to this task we mention that frequently we will say a random variable satisfies the CLT without specifying the required centering. That the centering can always be taken to be the mean vector is not surprising, and some details and suitable references for this in the Banach space setting can be found in [5]. The paper [5] also contains additional motivation that the reader might find of interest.

2. A connection to interpolation spaces and best approximations

Let B denote a separable Banach space over the reals with topological dual space B^* and norm $\|\cdot\|$. Throughout X is assumed to be a Borel measuable, B valued random vector. We say X is weakly square integrable with weak mean zero if

$$E(f(X)) = 0 \text{ and } \mathrm{E(f^2(X))} < \infty \text{ for all } \mathrm{f} \in \mathrm{B}^*.$$

We denote this by writing X is WM_0^2. If $\mu = \mathcal{L}(X)$, we also will say μ is WM_0^2 when that is more appropriate.

If μ is WM_0^2, then the Hilbert space H_μ used to determine the way we move points in B and define the mappings $T_\epsilon(\cdot)$ is defined in Lemma 2.1 of [5]. This Hilbert space arises in a number of different contexts, for example, see [4] and its references, but in the generality we employ here this lemma is a useful summary, and provides a convenient source for some of its properties. Furthermore, based on this lemma, Proposition 1 in [5] then allows us to define mappings from B into H_μ which move points continuously a small distance to a uniquely determined point in H_μ which has minimal H_μ-norm. Since we wish to move points by an arbitrarily small distance to points in H_μ, and a simple Hahn-Banach separation argument implies $\mu(\bar{H}_\mu) = 1$, we will henceforth assume that $B = \bar{H}_\mu$. This is no loss of generality for our application to limit theorems, and guarantees that our mappings are defined everywhere on B for arbitrarily small $\epsilon > 0$.

Next we turn to the precise definition of our basic mappings, and recall that throughout we are assuming that $\bar{H}_\mu = B$.

Definition. Let $\epsilon > 0$ and take $x \in B$. Then we define

$$T_\epsilon(x) = 0 \text{ if } \|\mathrm{x}\| \leq \epsilon \quad \text{and} \quad \mathrm{T_\epsilon(x) = b} \text{ if } \|\mathrm{x}\| \geq \epsilon, \tag{2.1}$$

where b is the unique point such that $\|b - x\| \leq \epsilon$,

$$\|b\|_\mu = \inf_{\|y-x\|\leq\epsilon} \|x\|_\mu,$$

and we take $\|x\|_\mu = \infty$ for $x \in B \cap H_\mu^c$.

Then we have $T_\epsilon(\cdot)$ well defined on all of B, with values in H_μ provided μ is WM_0^2, and we are also able to define for $\epsilon > 0,\ r > 0$

$$T_{\epsilon,r}(x) = T_{\epsilon(r\vee 1)}(x), \tag{2.2}$$

where $a \vee b = \max(a, b)$. Hence $T_{\epsilon,r}(\cdot)$ is also well defined on all of B for each $\epsilon > 0$.

Let $0 < \alpha < \infty$. Then using the ideas of the proof of Theorem 4 in [5] it follows that $T_{\epsilon,\|X\|^\alpha}(X)$ fails to improve the CLT in type-2 Banach spaces when $0 < \alpha < 1$. A proof of this is included in Proposition 3.1 at the end of Section 3. Furthermore, if $1 < \alpha < \infty$ it always satisfies the CLT in type-2 spaces, as it is then a bounded random variable for each $\epsilon > 0$. What happens when $\alpha = 1$ is far less understood, and here we consider similar mappings which relate to the approximation error via interpolation spaces. That is, in the theorem below we will see that Theorem 3.1 of [8] implies a sufficient condition on X such that $T_\epsilon(X)$ and various modifications of $T_{\epsilon,r}(X)$ derived from interpolation space norms actually satisfy the CLT on B. However, far less obvious is what this sufficient condition means in terms of explicit examples. Hence we include some examples where X fails the CLT, but our theorem implies a modification of $T_{\epsilon,r}(X)$ derived from interpolation space norms satisfies the CLT. There are also examples dealing with $T_\epsilon(X)$ itself in this setting, but they require we initially assume more about X than one might expect is optimal. Now we turn to some ideas and notation for interpolation spaces.

Suppose B and H_μ are given as in Lemma 2.1 of [5], and that as before we assume $\bar{H}_\mu = B$. Also recall from this lemma that the identity map from H_μ into B is continuous. The Banach spaces that interpolate between B and H_μ can be defined in terms of the K-functional for the pair (B, H_μ), which is defined by

$$K(a,t) = \inf_{b\in H_\mu} \{\|a - b\| + t\|b\|_\mu\}, \quad t > 0. \tag{2.3}$$

For $a \in B$ fixed, the function $K(a,t)$ is continuous, non-decreasing, bounded by $\|a\|$, and tends to zero as t tends to zero. In particular, for $0 < \theta < 1$ and $1 \leq p \leq \infty$, the interpolation space $(B, H_\mu)_{\theta,p}$ consists of the vectors $a \in B$ such that the norm

$$\|a\|_{\theta,p} = \sup_{t>0}\{K(a,t)/t^\theta\}, \quad \text{if } p = \infty, \tag{2.4}$$

or

$$\|a\|_{\theta,p} = \left\{\int_0^\infty (K(a,t)/t^\theta)^p dt/t\right\}^{1/p}, \quad \text{if } 1 \leq p < \infty, \tag{2.5}$$

is finite.

In our next theorem we use modifications of $T_{\epsilon,r}(\cdot)$ based on the $\|\cdot\|_{\theta,p}$-interpolation norms which provide a filtration between the μ-norm and the B-norm. In fact, standard properties of interpolation spaces, see [2], pages 40-47, and/or [3], pages 293-300, imply that for $0 < \theta < 1$ and $1 \leq p < \infty$ we have

$$H_\mu \hookrightarrow (B, H_\mu)_{\theta,p} \hookrightarrow (B, H_\mu)_{\theta,\infty} \hookrightarrow B, \tag{2.6}$$

with the spaces increasing in p, and $E \hookrightarrow$ F meaning that $E \subseteq F$ and E is continuously embedded in F under the identity map. Hence there are constants $C_1, C_2, C_3 \in (0, \infty)$ such that

$$\|x\|_B \le C_1 \|x\|_{\theta,\infty} \le C_2 \|x\|_{\theta,p} \le C_3 \|x\|_\mu \tag{2.7}$$

for all $x \in B$, with some of the quantities sometimes possibly being infinite in (2.7).

The modification of $T_\epsilon(\cdot)$ for the θ, p interpolation norms is given in terms of $T_{\epsilon,r}(\cdot)$, and is as follows.

Definition. If $0 < \epsilon < \infty, 0 < \theta < 1, 1 \le p \le \infty$, and $0 \le \alpha < \infty$, we define

$$T_{\epsilon,\|x\|^\alpha_{\theta,p}}(x) = T_{\epsilon(\|x\|^\alpha_{\theta,p} \vee 1)}(x), \ x \in B. \tag{2.8}$$

In particular, if $\alpha = 0$ in (2.8), then obviously $\|x\|^\alpha_{\theta,p} = 1$ when $0 < \|x\|_{\theta,p} < \infty$, and we use continuity in x for its definition at $x = 0$ and when $\|x\|_{\theta,p} = \infty$, i.e. it is one for all values of $\|x\|_{\theta,p}$. Thus we have

$$T_{\epsilon,\|x\|^\alpha_{\theta,p}}(x) = T_\epsilon(x), \ x \in B$$

when $\alpha = 0$. If $0 < \alpha < \infty$, and $\|x\|_{\theta,p} = \infty$, then we define

$$T_{\epsilon,\|x\|^\alpha_{\theta,p}}(x) = T_\infty(x) = 0.$$

Thus (2.8), as interpreted in the previous definition, defines $T_{\epsilon,\|x\|^\alpha_{\theta,p}}(x)$ for all $x \in B$. In order to apply it in our next theorem, we now turn to measurability properties of these mappings.

Lemma 2.1. *Fix $0 < \epsilon < \infty$, $p \in [0, \infty]$, $0 \le \alpha < \infty$ and $0 < \theta < 1$. Then the mapping*

$$T_{\epsilon,\|x\|^\alpha_{\theta,p}}(x), \ x \in B$$

takes B into H_μ, and it is Borel measurable from B into H_μ.

Proof. If $\alpha = 0$, then by definition $T_{\epsilon,\|x\|^\alpha_{\theta,p}}(x) = T_\epsilon(x)$, and the lemma follows from part-a of Proposition 2 in [5]. Hence take $0 < \alpha < \infty$. The next thing to observe is that $T_{\epsilon,r}(x)$ can be extended to be continuous in (x, r), $x \in B$, $r \in [0, \infty]$, into H_μ, with the μ-norm topology on H_μ. That is, if we define $T_{\epsilon,0}(x) = T_\epsilon(x)$ and $T_{\epsilon,\infty}(x) = 0$, then by part-c of Proposition 2 in [5], the continuity we claimed is immediate. Hence the lemma will follow if we check that the map $f(x) = \|x\|^\alpha_{\theta,p}$ is Borel measurable from B into $[0, \infty]$ for $0 < \alpha < \infty$. That is, if we define $\phi(r, x) = T_{\epsilon,r}(x)$, then

$$T_{\epsilon,\|x\|^\alpha_{\theta,p}}(x) = \phi(f(x), x) = \phi(h(x)),$$

where $h(x) = (f(x), x)$. Now ϕ continuous in (r, x) for $r \in [0, \infty], x \in B$, and $h(\cdot)$ Borel measurable from B into the product space determined by $[0, \infty]$ and B, with all spaces separable, implies $\phi(h(\cdot))$ is Borel measurable as indicated. What remains is to show $f(x) = \|x\|^\alpha_{\theta,p}$ is Borel measurable from B into $[0, \infty]$.

To check this, let A be a Borel measurable subset of $[0, \infty]$. Then we need that $E = f^{-1}(A)$ is a Borel subset of B. Now $E = f^{-1}(A \cap \{\infty\}) \cup f^{-1}(A \cap [0, \infty))$, and $f^{-1}(A \cap \{\infty\}) = \{x \in B : f(x) = \infty\} = B - (B, H_\mu)_{\theta,p}$ since $(B, H_\mu)_{\theta,p} = \{x \in B : \|x\|_{\theta,p} < \infty\}$, or it is trivially empty. Moreover,

$$f^{-1}(A \cap [0, \infty)) = \{x \in (B, H_\mu)_{\theta,p} : \|x\|^\alpha_{\theta,p} \in A, \|x\|^\alpha_{\theta,p} < \infty\},$$

and this last set is a Borel subset of $(B, H_\mu)_{\theta,p}$, as $x \to \|x\|^\alpha_{\theta,p}$ is continuous on $(B, H_\mu)_{\theta,p}$. Since the identity map embeds $(B, H_\mu)_{\theta,p}$ into B, Kuratowski's Theorem [7], page 21, implies that the image of Borel subsets of $(B, H_\mu)_{\theta,p}$ are Borel subsets of B. Hence E is a Borel subset of B, and the lemma is proven. ☐

Now we are ready to state and prove our theorem involving interpolation spaces.

Theorem 1. *Let X be WM_0^2 on B, and assume $\bar{H}_\mu = B$. Fix $0 < \theta < 1$ and $1 \le p < \infty$. If $0 \le \alpha \le 1/(1-\theta)$, $\beta = 2(\alpha + (1-\alpha)/\theta)$, and*

$$E(\|X\|^\beta_{\theta,p}) < \infty, \tag{2.9}$$

then for all $\epsilon > 0$ both

$$T_{\epsilon,\|X\|^\alpha_{\theta,p}}(X) \text{ and } \mathrm{T}_{\epsilon,\|\mathrm{X}\|^\alpha_{\theta,\infty}}(\mathrm{X}), \tag{2.10}$$

satisfy the CLT on B. If the integrability condition (2.9) is replaced by

$$E(\|X\|^\beta_{\theta,\infty}) < \infty, \tag{2.11}$$

then

$$T_{\epsilon,\|X\|^\alpha_{\theta,\infty}}(X) \tag{2.12}$$

satisfies the CLT on B for all $\epsilon > 0$. Furthermore, under (2.9), for all $\epsilon > 0$ sufficiently small the random vectors in (2.10) are non-degenerate. Similarly, (2.11) and $\epsilon > 0$ small implies the random vector in (2.12) is non-degenerate.

After one sees the proof, it is clear the result is strongest when we use the space $(B, H_\mu)_{\theta,\infty}$. This follows by (2.7), which shows an integrability assumption on $\|X\|_{\theta,\infty}$ is weaker than the equivalent integrability assumption for $\|X\|_{\theta,p}$, $1 \le p < \infty$. In addition, from (2.7) we also see that up to a constant multiple, points of B are moved a smaller distance by $T_{\epsilon,\|x\|^\alpha_{\theta,\infty}}(\cdot)$ than by $T_{\epsilon,\|x\|^\alpha_{\theta,p}}(\cdot)$. Hence the maps and spaces indexed by the pair θ, ∞ turn out to be optimal for our purposes. Therefore, one might ask, why do we include the θ, p maps and spaces? The answer is that in the examples that follow, and possibly in other settings as well, the θ, p spaces and their norms are more easily recognizable, and hence useful in doing the analysis. Thus we thought it important to formulate results for them as well. Of course, if one is able to compute the space $(B, H_\mu)_{\theta,\infty}$ and its norm well for certain B and H_μ, then for the reasons mentioned above one would work only with them.

Remark. It should be mentioned that the relationship between H_μ and B in our setting is more special than that which is typical in interpolation theory, or in [8]. That is, our space H_μ is always a Hilbert space, and it arises from the covariance structure of a WM_0^2 measure μ on the Borel sets of B, whereas in [8] the space $\mathcal{H}$ is an arbitrary continuously embedded dense subspace of B. Of course, since we assume $\bar{H}_\mu = B$, the denseness is a common element, and from Lemma 2.1 of [5] we also have the continuous embedding of H_μ into B.

Remark. Some cases of Theorem 1 that bear special mention include the following. (a) The first is that if $\alpha = 0$, then $\beta = 2/\theta$. Hence $E(\|X\|^{2/\theta}_{\theta,\infty}) < \infty$ implies $T_\epsilon(X)$ satisfies the CLT on B for all $\epsilon > 0$. (b) If $\alpha = 1$, then $\beta = 2$ and hence $E(\|X\|^2_{\theta,\infty}) < \infty$ implies $T_{\epsilon,\|X\|_{\theta,\infty}}(X)$ satisfies the CLT for all $\epsilon > 0$. Of course, a similar result holds for the θ, p norms. (c) If $\alpha = 1/(1-\theta)$, then $\beta = 0$, and

$$P(\|X\|_{\theta,\infty} < \infty) = 1,$$

implies that

$$T_{\epsilon,\|X\|_{\theta,\infty}^{1/(1-\theta)}}(X)$$

satisfies the CLT for all $\epsilon > 0$ and is non-degenerate for small ϵ. Of course, a similar result holds for the θ, p norms.

3. Proof of Theorem 1 and some examples

Proof of Theorem 1. First we assume (2.9) and take $0 \leq \alpha \leq 1/(1-\theta)$. Then $P(\|X\|_{\theta,p} < \infty) = 1$ and again using Kuratowski's Theorem we have $\mathcal{L}(X)$ a Borel measure on $(B, H_\mu)_{\theta,p}$, i.e. the Borel subsets of $(B, H_\mu)_{\theta,p}$ consist of the Borel subsets of B intersected with $(B, H_\mu)_{\theta,p}$. The separability and completeness of these spaces then says there are compact subsets of $(B, H_\mu)_{\theta,p}$, and hence also B via continuous embedding, with arbitrarly large probability. Now on these compacts, the norm topologies are equivalent, and hence if $\epsilon > 0$ is sufficiently small we have $\epsilon\|X\|_{\theta,p}^\alpha < \|X\|_B$ except if X=0 where they are equal. Hence we have the non-degeneracy for small $\epsilon > 0$ as claimed.

Let $r > 0$. Then for $\|X\|_{\theta,p} > \epsilon(\|X\|_{\theta,p}^\alpha \vee 1)$ we have $\|T_{\epsilon,\|X\|_{\theta,p}^\alpha}(X)\|_\mu = r$ if and only if $E(X,r) = \epsilon(\|X\|_{\theta,p}^\alpha \vee 1)$, and it equals zero otherwise. Hence for all $r > 0$

$$P(\|T_{\epsilon,\|X\|_{\theta,p}^\alpha}(X)\|_\mu > r) = P(E(X,r) > \epsilon(\|X\|_{\theta,p}^\alpha \vee 1)). \tag{3.1}$$

Therefore by (2.7) for $r > 0$

$$P(\|T_{\epsilon,\|X\|_{\theta,p}^\alpha}(X)\|_\mu > r) \leq P(E(X,r) > \epsilon((C_1/C_2)^\alpha \|X\|_{\theta,\infty}^\alpha \vee 1)),$$

and by Theorem 3.1 of [8] we thus have

$$P(\|T_{\epsilon,\|X\|_{\theta,p}^\alpha}(X)\|_\mu > r) \leq P(\|X\|_{\theta,\infty}^{1/(1-\theta)} > r^{\theta/(1-\theta)}\epsilon((C_1/C_2)^\alpha \|X\|_{\theta,\infty}^\alpha \vee 1)).$$

Dividing by $((C_1/C_2)^\alpha \|X\|_{\theta,\infty}^\alpha \vee 1)$ within the probability in the last term of the previous line, and replacing it by the possibly smaller quantity $((C_1/C_2)^\alpha \|X\|_{\theta,\infty}^\alpha)$, we see that

$$P(\|T_{\epsilon,\|X\|_{\theta,p}^\alpha}(X)\|_\mu > r) \leq P(\|X\|_{\theta,\infty}^{1/(1-\theta)-\alpha} > r^{\theta/(1-\theta)}\epsilon((C_1/C_2)^\alpha)). \tag{3.2}$$

Hence for $r > 0$ we have

$$P(\|T_{\epsilon,\|X\|_{\theta,p}^\alpha}(X)\|_\mu > r) \leq P(\|X\|_{\theta,\infty}^{\alpha+(1-\alpha)/\theta} > k_1^{(1-\theta)/\theta} r), \tag{3.3}$$

where $k_1 = \epsilon((C_1/C_2)^\alpha)$. Thus (2.7) and (2.9) implies (2.11), and hence (3.3) with $\beta = 2(\alpha + (1-\alpha)/\theta)$ implies

$$E(\|T_{\epsilon,\|X\|_{\theta,p}^\alpha}(X)\|_\mu^2) < \infty.$$

Thus $T_{\epsilon,\|X\|_{\theta,p}^\alpha}(X)$ satisfies the CLT in H_μ for all $\epsilon > 0$, and since the identity map embeds H_μ continuously into B, it also satisfies the CLT in B for all $\epsilon > 0$.

Repeating the previous argument, starting with $T_{\epsilon,\|X\|_{\theta,\infty}^\alpha}(X)$, we thus have

$$E(\|T_{\epsilon,\|X\|_{\theta,\infty}^\alpha}(X)\|_\mu^2) < \infty.$$

Thus (2.9) implies (2.10) holds. Finally, if we assume (2.11), then the same argument gives (2.12) and the theorem is proved. $\square$

Examples. In this section we provide two types of examples - those which indicate the differences between the classical CLT and the modifications we present in this paper and others which deal with $T_\epsilon(X)$ itself. In order to make the comparisons, we separate the individual conditions of Theorem 1 as well as the conditions used in describing classical CLT behavior.

For our examples we use the spaces

$$L_r(w_0) = \left\{ \{x_j\} : \sum_{j=1}^{\infty} \beta_j |x_j|^r < \infty \right\}, \tag{3.4}$$

where the weights $w_0(j) = \beta_j > 0$ for $j \geq 1$ and $1 < r < 2$. Then $L_r(w_0)$, together with the norm norm $\|\{x_j\}\|_r = (\sum_{j=1}^{\infty} \beta_j |x_j|^r)^{1/r}$, is a Banach space. If $\{e_j\}$ denotes the canonical basis of these sequence spaces, we describe a point in the space both by $x = \{x_j\}$ and $x = \sum_{j=1}^{\infty} x_j e_j$.

Let $X = \sum_{j=1}^{\infty} \lambda_j^{1/2} \eta_j e_j$, where $\lambda_j > 0$ and $\{\eta_j\}$ are orthogonal, mean zero random variables with $E(\eta_j^2) = 1,\ j \geq 1$. If

$$E(\|X\|_r^r) = \sum_{j=1}^{\infty} \beta_j \lambda_j^{r/2} E(|\eta_j|^r) < \infty, \tag{3.5}$$

then X is $L_r(w_0)$.

Also, since $1 < r < 2$ our normalization, $E(\eta_j^2) = 1$, implies $\sup_{j \geq 1} E(|\eta_j|^r) \leq 1$. Hence, (3.5) holds if

$$\sum_{j=1}^{\infty} \beta_j \lambda_j^{r/2} < \infty. \tag{3.6}$$

Since $1 < r < 2$, the condition (3.6) is necessary and sufficient for X to satisfy the CLT in $B = L_r(w_0)$, i.e. see, for example, [1], page 205.

While (3.6) implies X is WM_0^2, it is not a necessary condition for X to be WM_0^2, and this is what we examine next.

A necessary condition for WM_0^2. If X is WM_0^2 and $\mu = \mathcal{L}(X)$, then our Hilbert space H_μ is the sequence space $L_2(w_1)$ given by

$$L_2(w_1) = \left\{ \{x_j\} : \sum_{j=1}^{\infty} w_1(j) x_j^2 < \infty \right\}, \tag{3.7}$$

where the weights $w_1(j) = \lambda_j^{-1},\ j \geq 1$, and

$$\|\{x_j\}\|_\mu = \left(\sum_{j=1}^{\infty} \lambda_j^{-1} x_j^2 \right)^{1/2}.$$

Claim. X is WM_0^2 if

$$\sum_{j=1}^{\infty} (\lambda_j \beta_j^{2/r})^{r/(2-r)} = \sum_{j=1}^{\infty} \lambda_j^{r/(2-r)} \beta_j^{2/(2-r)} < \infty.$$

To see this first observe that since $B = L_r(w_0)$, then $B^* = L_s(w_0)$ where $s = r/(r-1)$. Hence we have $E(g^2(X)) < \infty$ for all $g = \{g_j\} \in B^*$ if

$$\sum_{j=1}^{\infty} \lambda_j g^2(e_j) < \infty, \tag{3.8}$$

for all such $g \in L_{r/(r-1)}(w_0)$. Since the sequence $e_k = \{\delta(k,j) : j \geq 1\} \in B$, then for $g = \{g_j\} \in B^*$ we have via the usual pairing of g and e_k over $L_r(w_0)$ that

$$g(e_k) = \sum_{j=1}^{\infty} \beta_j g_j \delta(j,k) = \beta_k g_k$$

for all $k \geq 1$. Therefore, (3.8) requires that

$$\sum_{j=1}^{\infty} \lambda_j \beta_j^2 g_j^2 < \infty \tag{3.9}$$

whenever $g \in L_s(w_0) = L_{r/(r-1)}(w_0)$, i.e. whenever

$$\sum_{j=1}^{\infty} \beta_j |g_j|^{r/(r-1)} < \infty. \tag{3.10}$$

Now (3.10) implies $\{\beta_j^{2(r-1)/r} g_j^2\} \in \ell^{r/(2(r-1))}$ for all $g = \{g_j\} \in L_{r/(r-1)}(w_0)$, and hence (3.9) converges for all such $\{g_j\}$ if

$$\{\lambda_j \beta_j^{2-2(r-1)/r}\} = \{\lambda_j \beta_j^{2/r}\} \in \ell^{s^*},$$

where $1/s^* + 2(r-1)/r = 1$. Hence $s^* = r/(2-r)$ and (3.9) holds for all $g \in B^*$ if

$$\sum_{j=1}^{\infty} (\lambda_j \beta_j^{2/r})^{r/(2-r)} = \sum_{j=1}^{\infty} \lambda_j^{r/(2-r)} \beta_j^{2/(2-r)} < \infty. \tag{3.11}$$

Summary. if (3.6) holds then X satisfies the CLT on B, and X is WM_0^2 if the weaker condition in (3.11) holds, i.e. recall $1 < r < 2$.

The interpolation spaces. If we consider the interpolation spaces for the pair $(B, H_\mu) = (L_r(w_0), L_2(w_1))$, where $1 < r < 2$, then for $0 < \theta < 1$ Theorem 5.5.1 of [2] implies

$$(B, H_\mu)_{\theta,q} = (L_r(w_0), L_2(w_1))_{\theta,q} = L_q(w), \tag{3.12}$$

where $w(j) = (w_0(j))^{q(1-\theta)/r}(w_1(j))^{q\theta/2}$, and $q = q(\theta, r)$ is determined by $1/q = (1-\theta)/r + \theta/2$. Then a simple calculation shows $1 < r < q < 2$, and we have

$$(B, H_\mu)_{\theta,q} = L_q(w) = \left\{ \{x_j\} : \sum_{j=1}^{\infty} \beta_j^{q(1-\theta)/r} \lambda_j^{-q\theta/2} |x_j|^q < \infty \right\}, \tag{3.13}$$

with norm

$$\|x\|_{\theta,q} = \left(\sum_{j=1}^{\infty} \beta_j^{q(1-\theta)/r} \lambda_j^{-q\theta/2} |x_j|^q \right)^{1/q}.$$

Hence if X is given by $\sum_{j=1}^{\infty} \lambda_j^{1/2}\eta_j e_j$, where the $\{\eta_j\}$ are orthonormal as above, then

$$(3.14) \qquad \|X\|_{\theta,q} = \left(\sum_{j=1}^{\infty} \beta_j^{q(1-\theta)/r} \lambda_j^{q(1-\theta)/2} |\eta_j|^q\right)^{1/q},$$

and

$$\|X\|_B = \|X\|_r = \left(\sum_{j=1}^{\infty} \beta_j \lambda_j^{r/2} |\eta_j|^r\right)^{1/r}.$$

Therefore, if (3.6) holds, then an easy application of Minkowski's inequality implies we have $E(\|X\|_B^2) < \infty$. Similarly, (3.14) implies

$$(3.15) \qquad E(\|X\|_{\theta,q}^2) \leq \sum_{j=1}^{\infty} \beta_j^{q(1-\theta)/r} \lambda_j^{q(1-\theta)/2},$$

and

$$(3.16) \qquad E(\|X\|_{\theta,q}^{2/\theta}) \leq \sum_{j=1}^{\infty} \beta_j^{q(1-\theta)/r} \lambda_j^{q(1-\theta)/2} (E(|\eta_j|^{2/\theta}))^{q\theta/2}.$$

Finally, if $1/q = (1-\theta)/r + \theta/2$, with $0 < \theta < 1$ and $1 < r < 2$ fixed, then with β of Theorem 1 such that $\beta = 2(\alpha + (1-\theta)/2) = q$ we have from (3.14) that

$$(3.17) \qquad E(\|X\|_{\theta,q}^{\beta}) = E(\|X\|_{\theta,q}^{q}) = \sum_{j=1}^{\infty} \beta_j^{q(1-\theta)/r} \lambda_j^{q(1-\theta)/2} E(|\eta_j|^q),$$

where $1 < r < q < 2$. To see we can choose $\alpha \in [0, 1/(1-\theta)]$ such that $\beta = \beta(\alpha) = 2(\alpha + (1-\alpha)/\theta) = q$, notice that $\beta(\alpha)$ is linear in α with $\beta(0) = 2/\theta > 2$ and $\beta(1/(1-\theta)) = 0$. Thus $1 < q < 2$ implies there exists $\alpha = \alpha_q$ such that $\beta(\alpha_q) = q$ and (3.17) holds.

Hence if $\lambda_j = 1/Lj, \beta_j = 1/j$ for $j \geq 1$, and $E(|\eta_j|^q) \leq j^{-q/2}$, then we have

(a) $P(X \in B) = 1$ since (3.5) holds.
(b) (3.6) fails since $r < 2$.
(c) X is WM_0^2 since (3.11) holds and $r/(2-r) > 1, 2/(2-r) > 2$.
(d) $E(\|X\|_{\theta,q}^q) < \infty$ since (3.17) holds and $1 < q(1-\theta)/r + q/2$.

To see there are $\{\eta_j : j \geq 1\}$ such that $E(|\eta_j|^q) \leq j^{-q/2}$, take $c_j = j^{q/(2-q)}$ and $\{\eta_j\}$ independent such that $P(\eta_j = \pm c_j^{1/2}) = 1/(2c_j)$ and $P(\eta_j = 0) = 1/c_j$. Then $E(\eta_j) = 0, E(\eta_j^2) = 1$ and $E(|\eta_j|^q) = j^{-q/2}$.

Conclusions concerning $T_{\epsilon,\|X\|_{\theta,\infty}^{\alpha}}(X)$. Hence with these choices of $\{\lambda_j\}$, $\{\beta_j\}$, $\{\eta_j\}$, and $\alpha = \alpha_q$ as above, the random vector X is B-valued, X is WM_0^2, X is not pre-Gaussian, and by Theorem 1 and (d) above we have $T_{\epsilon,\|X\|_{\theta,\infty}^{\alpha}}(X)$ satisfies the CLT in B for all $\epsilon > 0$. That X is not pre-Gaussian on B when (3.6) fails follows from [9], or our previous reference to [1] following (3.6).

Some additional examples. For some examples X where $T_\epsilon(\cdot)$ satisfies the CLT, we take $\{\eta_j\}$ orthogonal with $E(\eta_j) = 0,\ E(\eta_j^2) = 1$ and such that we also have $\sup_{j\geq 1} E(|\eta_j|^{2/\theta})$ is finite. In addition we take

$$\beta_j = j^{-r/(q(1-\theta))},\ \lambda_j = (Lj)^{-2(1+\delta)/(q(1-\theta))},$$

where $\delta > 0$. Then (3.15) and (3.16) are both finite. Furthermore, (3.5) and (3.6) hold since $r/(q(1-\theta)) > 1$, and X is WM_0^2 since (3.11) holds. Of course, X is also WM_0^2 since we have $E(\|X\|_B^2)$ finite by (3.6). In addition, since $1 < r < 2$, we recall that (3.6) implies X satisfies the CLT. Furthermore, by Theorem 1 with $\alpha = 0$ we also have $T_\epsilon(X)$ satisfies the CLT, i.e. recall (2.7), so (3.16) finite implies $E(\|X\|_{\theta,\infty}^{2/\theta}) < \infty$ and hence (2.11) yields (2.12) with $\alpha = 0$.

The following proposition shows that for $0 < \alpha < 1$, the analogues of the mappings used in Theorem 1 when the interpolation norms are replaced by the B-norm do not improve the CLT in a type-2 Banach space.

Proposition 3.1. *Let B be a separable type-2 Banach space and X be a Borel measurable random vector with values in B. If X satisfies the CLT in B, then for all $\epsilon > 0$ and $0 < \alpha < 1$ we have $T_{\epsilon,\|X\|^\alpha}(X)$ satisfies the CLT. Conversely, if $T_{\epsilon,\|X\|^\alpha}(X)$ satisfies the CLT for some $\epsilon > 0$, then X satisfies the CLT in B.*

Proof. If X satisfies the CLT in B, then as in the proof of Theorem 4 in [5] we have $T_{\epsilon,\|X\|^\alpha}(X)$ satisfies the CLT provided $X - T_{\epsilon,\|X\|^\alpha}(X)$ satisfies the CLT there. Now

$$\|X - T_{\epsilon,\|X\|^\alpha}(X)\| \leq \epsilon(\|X\|^\alpha \vee 1), \tag{3.18}$$

and hence $E(\|X - T_{\epsilon,\|X\|^\alpha}(X)\|^2) \leq \epsilon^2 E((\|X\|^{2\alpha} \vee 1)) < \infty$, where the finiteness of the last inequality holds since X satisfies the CLT and $0 < \alpha < 1$.

If $T_{\epsilon,\|X\|^\alpha}(X)$ satisfies the CLT, then again it suffices to show that $X - T_{\epsilon,\|X\|^\alpha}(X)$ satisfies the CLT. Since

$$\|X\| \leq [\|T_{\epsilon,\|X\|^\alpha}(X)\| + \epsilon(\|X\|^\alpha \vee 1)],$$

we have $\|X\|^{2\alpha} \leq [\|T_{\epsilon,\|X\|^\alpha}(X)\| + \epsilon(\|X\|^\alpha \vee 1)]^{2\alpha}$, and hence there exists $C_\alpha < \infty$ such that

$$\|X\|^{2\alpha} \leq C_\alpha[\|T_{\epsilon,\|X\|^\alpha}(X)\|^{2\alpha} + \epsilon^{2\alpha}(\|X\|^{2\alpha^2} \vee 1)].$$

Since $0 < \alpha < 1$ and $T_{\epsilon,\|X\|^\alpha}(X)$ satisfies the CLT, we have from the previous inequality that

$$E(\|X\|^{2\alpha}) < \infty.$$

Therefore (3.18) implies $E(\|X - T_{\epsilon,\|X\|^\alpha}(X)\|^2) < \infty$, and when B is type-2 this implies $X - T_{\epsilon,\|X\|^\alpha}(X)$ satisfies the CLT. Thus the proposition is proven. $\square$

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