Nonlinear renewal theorems for random walks with perturbations of intermediate order

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Abstract: We develop nonlinear renewal theorems for a perturbed random walk without assuming stochastic boundedness of centered perturbation terms. A second order expansion of the expected stopping time is obtained via the uniform integrability of the difference between certain linear and nonlinear stopping rules. An intermediate renewal theorem is obtained which provides expansions between the nonlinear versions of the elementary and regular renewal theorems. The expected sample size of a two-sample rank sequential probability ratio test is considered as the motivating example.

1. Introduction

Let $X, X_n, n \geq 1$, be i.i.d. random variables with a finite, positive mean $EX = \mu$. Let $\{Z_n, n \geq 1\}$ be a perturbed random walk defined as

$$(1.1) \qquad Z_n = S_n + \xi_n, \quad S_n = X_1 + \cdots + X_n, \qquad n \geq 1,$$

where $\{\xi_n, n \geq 1\}$ are random variables such that $\{X_1, \xi_1, \ldots, X_n, \xi_n\}$ are independent of $\{X_{n+j}, j \geq 1\}$ for all $n \geq 1$. Define

$$(1.2) \qquad T_b = \inf\{n \geq 1 : Z_n > b\}, \quad \tau_b = \inf\{n \geq 1 : S_n > b\}.$$

Nonlinear renewal theory studies probabilistic quantities related to stopping rules T_b, especially Blackwell-type theorems for the convergence of the renewal measure $U_b((b, b+h]) = \sum_n P\{b < Z_n \leq b+h\}$, the distribution of the excess over the boundary $R_b = Z_{T_b} - b$, and expansions of ET_b and $\mathrm{Var}(T_b)$, as extensions of the (linear) renewal theorems concerning probabilistic quantities related to τ_b. Many important applications of nonlinear renewal theory come from sequential analysis in which nonlinear renewal theory provides crucial analytical tools and methodologies [8, 19, 22, 25, 28].

Many authors have studied nonlinear renewal theory. See for example Chow and Robbins [6], Chow [3], Siegmund [23, 24], Gut [9], Pollak and Siegmund [18], Woodroofe [26, 27], Lai and Siegmund [15, 16], Chow, Hsiung and Lai [4], Lalley [17], Hagwood and Woodroofe [10], Woodroofe and Keener [31], Zhang [32], Hu [11],

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Aras and Woodroofe [1], Kim and Woodroofe [12, 13], and the books by Woodroofe [28] and Siegmund [25] on the subject. A main condition of the existing Blackwell-type nonlinear renewal theorems is

$$(1.3) \qquad \lim_{\theta \to 0} \limsup_{n \to \infty} P \left\{ \max_{1 \le j \le \theta n^\alpha} \left| \xi_{n+j} - \xi_n \right| > \epsilon \right\} = 0, \quad \forall \epsilon > 0,$$

(Lai and Siegmund [15]). This condition of slowly changing perturbation allows ξ_n with unbounded variability [i.e. stochastically unbounded $\xi_n - c(n)$ for any centering constants $c(n)$]. However, in addition to (1.3) and other regularity conditions, existing results on the second order [i.e. up to $o(1)$] expansion of ET_b requires

$$(1.4) \qquad \text{the uniform integrability of} \max_{N_0 \vee n \le j \le n+n^\alpha} \left| \xi_j - f(j) \right|$$

for certain $1/2 < \alpha \le 1$, where $f(t)$ is a slowly changing deterministic function and N_0 is a random variable with $EN_0 < \infty$ [15]. This condition precludes perturbation processes ξ_n with unbounded variability. In this paper, we remove the restriction on the bounded variability of ξ_n by imposing the condition of

$$(1.5) \qquad \text{the uniform integrability of} \max_{1 \le j \le Mn^\alpha} \left(|\xi_{n+j} - \zeta_n| \wedge n^\alpha \right)$$

for all $M \in \mathbb{R}$, instead of (1.4), where ζ_n are certain truncated ξ_n. This will be done in Section 2.

Blackwell-type nonlinear renewal theorems and second order expansions of ET_b were first obtained by Woodroofe [26] for stopping rules of the form

$$(1.6) \qquad T_b = \inf \left\{ n \ge 1 : S_n > A(n; b) \right\}$$

with certain nonlinear boundaries $A(\cdot; b)$. Lai and Siegmund [15, 16] pointed out that in many applications (1.6) can be written as (1.2) for certain (possibly different) random walk with perturbation and developed nonlinear renewal theorems for (1.2) under much weaker conditions on the distribution of X. Zhang [32] studied nonlinear renewal theorems for both (1.2) and (1.6) through the uniform integrability of $|T_b - \tau_a|^p$, where $a = b - f(b/\mu)$ with the function $f(t)$ in (1.4). He also obtained a second order expansion of $\text{Var}(T_b)$ for stopping rules of the form (1.6), and thus demonstrated certain advantages of investigating (1.6). The methods in Zhang [32] and here can be combined to study a general form of stopping rules with (1.2) and (1.6) as special cases, but for simplicity we confine the rest of our discussion to stopping rules of form (1.2).

The motivating example of this paper is Savage and Sethuraman's [21] two-sample rank sequential probability ratio test (SPRT) for $G = F$ against a Lehmann alternative. In Section 3 we provide formulas for the expansion of the expected sample size of the rank SPRT, with an outline of a proof. Our calculation shows that the rank log-likelihood ratio is of the form (1.1) such that $(\xi_n - E\xi_n)/(\log n)^{1/2}$ converges to a nondegenerate normal distribution under the null hypothesis, so that we are truly dealing with perturbations of intermediate order.

Our expansion of ET_b is obtained via the uniform integrability of $|T_b - \tau_b^*|$ under (1.5), where τ_b^*, defined in (4.1) below, is a linear stopping rule with perturbation at an initial time $n = n_*$. In fact, we will develop sufficient conditions for the uniform integrability of $\{|T_b - \tau_b^*|/\rho(b)\}^p$ for certain normalization constants $1 \le \rho(b) = o(b)$ and consequently an intermediate renewal theory for the expansion of ET_b up to $O(\rho(b))$. These will be done in Section 4.

2. Expectation of stopping rules

In this section, we provide a second order expansion of ET_b under (1.5). Let $\rho(x)$ be a function satisfying

$$(2.1) \qquad 1 \leq \rho(x) = o(x) \text{ as } x \to \infty \ , \quad \sup_{x > 1} \sup_{x \leq t \leq 3x} \rho(t)/\rho(2x) < \infty.$$

We shall first state a general set of regularity conditions on the perturbation ξ_n in (1.1) which we call ρ-regularity. We denote throughout the sequel $\lfloor x \rfloor$ the integer part of x and $\lceil x \rceil$ the smallest integer upper bound of x.

The process $\{\xi_n\}$ in (1.1) is called ρ-*regular*, or regular of order $\rho(\cdot)$, with parameters $p \geq 1$ and $1/2 < \alpha \leq 1$ if the following three conditions hold: there exist constants $\delta_0 > 0$, $\theta > 0$, $\theta < \mu$ for $\alpha = 1$, $K > 0$, $w_0 > 0$ and $0 < \theta^* < K\mu$ such that

$$(2.2) \qquad P\left\{ \max_{\delta_0 n < j \leq n} \xi_j > \theta n^\alpha \right\} = o(\rho(n)/n)^p,$$

$$(2.3) \qquad \sum_{k \geq n + Kn^\alpha} k^{p-1} P\{\xi_k \leq -(k-n)\mu + w_0 k^\alpha\} = o(\rho^p(n)),$$

and that for $\zeta_n = (\xi_n \wedge \theta n^\alpha) \vee (-\theta^* n^\alpha)$ and all $M < \infty$

$$(2.4) \quad \left\{ \max_{0 \leq j \leq Mn^\alpha} \left(\frac{|\zeta_n - \xi_{n+j}| \wedge n^\alpha}{\rho(n)} \right)^p, n \geq 1 \right\} \text{ is uniformly integrable.}$$

We impose the 1-regularity with $\rho(x) = 1$ in Theorem 2.1 below for the expansion of ET_b and the general ρ-regularity in Section 4 for uniform integrability and an intermediate renewal theorem of order $\rho(\cdot)$.

Theorem 2.1. *Suppose $\{\xi_n, n \geq 1\}$ is 1-regular, i.e. $\rho(x) = 1$, with parameters $p = 1$ and $1/2 < \alpha \leq 1$. Suppose X is non-lattice with $EX = \mu > 0$ and $E|X|^{2/\alpha} < \infty$. Suppose $bP\{T_b \leq \delta_0 b/\mu\} = o(1)$ for the δ_0 in (2.2) and the slowly changing condition*

$$(2.5) \qquad \max_{1 \leq j \leq n^\alpha} |\xi_{n+j} - \zeta_n| = o_P(1)$$

holds with the $\zeta_n = (\xi_n \wedge \theta n^\alpha) \vee (-\theta^* n^\alpha)$ in (2.4). Then, as $b \to \infty$,

$$(2.6) \qquad \mu ET_b = b - E\zeta_{n_b} + \frac{ES_{\tau_0}^2}{2ES_{\tau_0}} + o(1),$$

where $n_b = \lfloor b/\mu \rfloor$, S_n is as in (1.1) and τ_0 is as in (1.2).

Remark 2.1. As mentioned earlier, the main difference between the ρ-regularity and the usual regularity conditions in the nonlinear renewal theory literature is (2.4), which allows ξ_n to have unbounded variability as $n \to \infty$.

Remark 2.2. Since $(k-n)/k^\alpha$ is increasing in k, $(k-n)\mu - w_0 k^\alpha \geq (\mu K/(1 + K)^\alpha - w_0)k^\alpha$ for $k \geq n + K(n\mu)^\alpha$. Thus, (2.3) is a consequence of

$$(2.7) \qquad \sum_{k=1}^\infty k^{p-1} P\{-\xi_k \geq w_1 k^\alpha\} < \infty$$

for certain $w_1 > 0$. In typical nonlinear renewal theorems, $\rho(x) = 1$ and (2.7) is imposed (with $w_1 < \mu$ for $\alpha = 1$) instead of (2.3).

Remark 2.3. The condition $bP\{T_b \leq \delta_0 b/\mu\} \to 0$ holds if $EX^2 < \infty$ and $nP\{\max_{j\leq n} \xi_j > w_2 n\} = o(1)$ for some $w_2 < \mu(1/\delta_0 - 1)$.

Proof of Theorem 2.1. Let $T = T_b$ be as in (1.2), $n_* = \lfloor b/\mu - \eta_* b^\alpha \rfloor$, and $\tau^* = \tau_b^* = \inf\{n \geq n_* : S_n + \zeta_{n_*} > b\}$ be as in (4.1) below with $\eta_* > 0$, $\eta_* < 1/\mu$ for $\alpha = 1$. Let $R_b^* = S_{\tau^*} + \zeta_{n_*} - b$ be the overshoot for τ^*. Since ζ_{n_*} is bounded and $b - \zeta_{n_*} \to \infty$,

$$(2.8) \qquad \mu E\tau^* = ES_{\tau^*} = ER_b^* + b - E\zeta_{n_*} = b - E\zeta_{n_*} + \frac{ES_{\tau_0}^2}{2ES_{\tau_0}} + o(1)$$

by the Wald identity and the standard linear renewal theorem [7]. The proof is based on (2.8), (2.4) with $\rho(\cdot) = 1$, (2.5), and the uniform integrability of $|T - \tau^*|$ in Theorem 4.1.

Since $E|T - \tau^*|$ is bounded, $ET < \infty$ and $\mu E(T - \tau^*) = E(S_T - S_{\tau^*})$ by the Wald identity. Let $R_b = Z_T - b$ be the overshoot for T in (1.2). By (1.2) and (4.1)

$$(2.9) \qquad S_T - S_{\tau^*} = (R_b + b - \xi_T) - (R_\tau^* + b - \zeta_{n_*}) = R_b - R_b^* + (\zeta_{n_*} - \xi_T).$$

Since $EX^2 < \infty$, the uniform integrability of $|T - \tau^*|$ implies that of $|S_T - S_{\tau^*}|$. Since $\zeta_{n_*} = O(b^\alpha)$, $|\tau^* - b/\mu| = O_P(b^\alpha)$. This and $T - \tau^* = O_P(1)$ imply

$$(2.10) \qquad |T - b/\mu| + |\tau^* - b/\mu| = O_P(b^\alpha),$$

so that $\zeta_{n_*} - \xi_T = o_P(1)$ by (2.5). Moreover, (2.10) and (2.5) imply the convergence of both R_b and R_b^* in univariate distribution to the same limit [15]. These facts and the uniform integrability of $|S_T - S_{\tau^*}|$ imply $\mu E(T - \tau^*) = E(S_T - S_{\tau^*}) \to 0$ in view of (2.9). It then follows from (2.8) that $ET = b - E\zeta_{n_*} + ES_{\tau_0}^2/(2ES_{\tau_0}) + o(1)$. Since $E|\zeta_{n_*} - \zeta_{n_b}| \to 0$ by (2.4) and (2.5), (2.6) follows. □

3. Rank SPRT

Given a constant $\Delta > 0$ and two independent samples of equal size from continuous distribution functions F and G respectively, the rank likelihood ratio for testing $H_0 : G = F$ against the Lehmann alternative $G = F^\Delta$ is $\Lambda_n = L_n(F, F^\Delta)/L_n(F, F)$, where $L_n(F, G)$ is the probability mass function of the ranks of the F-sample within the combination of the F- and G-samples. Suppose pairs of observations, one from F and one from G, are taken sequentially, the rank SPRT [21] rejects H_0 iff $\Lambda_T > e^b$ with the stopping rule

$$(3.1) \qquad T = T_{a,b} = \inf\left\{n \geq 1 : Z_n < -a \text{ or } Z_n > b\right\}, \quad Z_n = \log \Lambda_n.$$

In this section, we provide formulas for the expansion of the expectation of the sample size (3.1) of the rank SPRT under the following conditions:

$$(3.2) \qquad G = F^A, \quad \mu = \log \Delta + \int \log\left(\frac{F + G}{F + \Delta G}\right) d(F + G) \neq 0.$$

We outline a proof of the expansion via a representation of the rank log-likelihood ratio Z_n in (3.1) as a perturbed random walk (1.1) satisfying the conditions of Theorem 2.1. We prove in (3.9) the convergence of $(\xi_n - E\xi_n)/(\log n)^{1/2}$ in distribution under the null hypothesis, so that the rank log-likelihood is truly a random walk with perturbations of intermediate order.

The rank likelihood ratio was considered earlier by Savage [20] who showed that

$$(3.3) \qquad Z_n = \log\left(\Delta^n(2n)!\right) - n\int \log(nF_n + \Delta nG_n)d(F_n + G_n),$$

where F_n and G_n are the ECDF's of the F- and G-samples. Chernoff and Savage provided [2] representations of certain rank statistics as (1.1) and proved their asymptotic normality, i.e. $\xi_n = o_P(n^{1/2})$. Lai identified [14] the rank log-likelihood ratio Z_n with a Chernoff–Savage statistic and proved the quick convergence $n^{-\epsilon}\xi_n$ for all $\epsilon > 0$ in (1.1). The problem was further investigated by Woodroofe [28, 29] who proved (1.3) for the perturbation term of (3.3) and derived asymptotic formulas for the error probabilities of the rank SPRT. Our results are based on a finer expansion of (3.3) from a slightly different expression (3.4) below.

Let $H = F + G$, $W = F + \Delta G$, and H_n and W_n be their empirical versions. Since $\log((2n)!) = n\int \log(nH_n)dH_n$, (3.3) can be written as

$$(3.4) \qquad Z_n = n\log(\Delta) + n\int \log\left(\frac{H_n}{W_n}\right)dH_n.$$

Let $\psi_n = H_n/W_n$ and $\psi = H/W$. We write (3.4) as a perturbed random walk (1.1) with

$$(3.5) \qquad S_n = n\log\Delta + n\int \log(\psi)dH_n + n\int (H_n/H - W_n/W)dH$$

and the perturbation term

$$(3.6) \qquad \xi_n = \int n\left\{ \log(\psi_n/\psi)dH_n - (H_n/H - W_n/W)dH\right\}.$$

Let y_k be hypergeometric(n, n, k) variables and $\eta = (1 - \Delta)/(1 + \Delta)$. Define

$$C(\eta) = \lim_n \sum_{k=1}^{2n} E\log\left(1 + \eta\left(\frac{2y_k}{k} - 1\right)\right) + \frac{\eta^2}{2}\log(2n).$$

The above limit exists since $Ey_k = k/2$ and $\mathrm{Var}(y_k) = (k/4)(2n - k)/(2n - 1)$. We state our expansion of the expectation of the sample size (3.1) as follows.

Theorem 3.1. *Set $\eta = (\Delta - 1)/(\Delta + 1)$. Let $T_{a,b}$, S_n, and τ_0 be as in (3.1), (3.5), and (1.2) respectively. Suppose $\mu > 0$ in (3.2) and $a/\log b \to \infty$ as $b \to \infty$. Then,*

$$(3.7) \qquad \mu ET_{a,b} = b - E\xi_{n_b} + ES_{\tau_0}^2/(2ES_{\tau_0}) + o(1),$$

where $n_b = \lfloor b/\mu \rfloor$ as in Theorem 2.1. Moreover, if $G = F^A$, then

$$\mu ET_{a,b} = \begin{cases} b - \int h(x)d(x + x^A) + ES_{\tau_0}^2/(2ES_{\tau_0}) + o(1), & A \neq 1 \\ b - (\eta^2/2)\log(2b/\mu) + C(\eta) + ES_{\tau_0}^2/(2ES_{\tau_0}) + o(1), & A = 1, \end{cases}$$

where $h(x) = (1 - \Delta)^2 x^{1+A}/\{2(x + \Delta x^A)^2(x + x^A)\}$.

Remark 3.1. Since $\Lambda_n = L_n(G^{1/\Delta}, G)/L_n(G, G)$, Theorem 3.1 also provides expansions of $ET_{a,b}$ for $\mu < 0$ as $(a, b/\log a) \to (\infty, \infty)$ by symmetry.

Here is an outline of a proof of Theorem 3.1. For $\epsilon_0 \le \mu$, $P(Z_n \le 0) \le P(|Z_n - n\mu| > \epsilon_0 n) \to 0$ exponentially fast [21]. Since $P(|Z_n| \le n|\log \Delta|) = 1$, this implies $P(Z_{T_{a,b}} < 0) = o(b^2)$, so that $ET_{a,b} = ET_b + o(1)$. Thus, it suffices to consider $a = \infty$ ($T_{a,b} = T_b$). Moreover, the same argument proves $bP\{T_b \le \delta_0 b/\mu\} = o(1)$ for all $\delta_0 < 1$.

Let M^* denote universal constants. For $c_n \ge 1/n$, we split the integral for ξ_j in (3.6) according to whether $H(x) \le c_n$ so that

$$\xi_j = \left(\int_{H \le c_n} + \int_{H > c_n} \right) \left\{ \log \left(\frac{\psi_j}{\psi} \right) d(jH_j) - \left(\frac{jH_j}{H} - \frac{jW_j}{W} \right) dH \right\} = \xi_{n,j}^{(1)} + \xi_{n,j}^{(2)}.$$

Since $\psi_j = (jH_j)/(jW_j)$ with jH_j and jW_j being sums of iid measures, $\xi_{n,j+1}^{(1)} = \xi_{n,j}^{(1)}$ if $\int_{H \le c_n} d\{(j+1)H_{j+1} - jH_j\} = 0$. Thus, by algebra and the Taylor expansion of $\log(\psi_{j+1}/\psi_j)$,

$$|\xi_{n,j+1}^{(1)} - \xi_{n,j}^{(1)}| \le M^* \int_{H \le c_n} \{1 + \log n - \log(H/2)\} d\{(j+1)H_{j+1} - jH_j\}$$

due to $d\{(j+1)W_{j+1} - jW_j\} \le (1 \vee \Delta) d\{(j+1)H_{j+1} - jH_j\}$ and $\int (1/H_n) dH_n = \sum_{k=1}^{2n} 1/k$. For $\xi_{n,j}^{(2)}$, the four term Taylor expansion of $\log(\psi_j/\psi)$ yields

$$E \max_{n \le j \le 2n} \left| \xi_{n,j}^{(2)} - U_{n,j}^{[1]} - (j-1)U_{n,j}^{[2]} \right|^2 \le M^*/(nc_n),$$

where $\{U_{n,j}^{[k]}, j \ge 1\}$ are U-statistics with kernels $U_{n,k}^{[k]}$ such that

$$E\left| U_{n,1}^{[1]} \right| \le M^* \log \left(\frac{2}{c_n} \right), \quad \mathrm{Var}\left(U_{n,1}^{[1]} \right) \le \frac{M^*}{nc_n}, \quad E\left(U_{n,2}^{[2]} \right)^2 \le M^* \log \left(\frac{2}{c_n} \right),$$

with completely degenerate $U_{n,2}^{[2]}$. Choose c_n satisfying $(\log n)c_n n^\alpha + 1/(nc_n) = o(1)$, we find that the above five inequalities imply the 1-regularity of ξ_n for $p = 1$ and all $1/2 < \alpha < 1$.

Proposition 3.1. *The rank log-likelihood ratio Z_n in (3.1) and (3.3) can be written as a perturbed random walk $Z_n = S_n + \xi_n$ with the random walk S_n in (3.5), drift μ in (3.2), and the perturbation ξ_n in (3.6). Moreover, for all $1/2 < \alpha < 1$ and $\epsilon > 0$, $E \max_{j \le n^\alpha} |\xi_{n+j} - \xi_n| = o(1)$, $E(|\xi_n| - n^\alpha)^+ = o(1)$, $nP\{ \max_{0 \le j \le n} |\xi_{n+j}| > \epsilon n^\alpha \} = o(1)$, and $\sum_n P\{|\xi_n| > \epsilon n^\alpha\} < \infty$.*

It follows from (3.5) that S_1 is non-lattice with $E|S_1|^p < \infty$ for all p. Since we have already proved $bP\{T_b \le \delta_0 b/\mu\} = o(1)$, all conditions of Theorem 2.1 hold by Proposition 3.1. Thus, (3.7) holds and it remains to compute $E\xi_n$ up to $o(1)$. We sketch below the calculation in the case of $F = G$.

Assume $F = G$ are uniform in $(0, 1)$. Let $u_1 \le \cdots \le u_{2n}$ be the ordered observations in the combined sample. Let $\varepsilon_k = 1$ if u_k is from G and $\varepsilon_k = 0$ otherwise. Set $y_k = \sum_{i=1}^k \varepsilon_i$. Since $y_k = nG_n(u_k)$ and $k = nH_n(u_k)$, $(\psi/\psi_n)(u_k) = 1 + \eta(2y_k/k - 1)$, so that by (3.6)

$$(3.8) \qquad \xi_n = -\sum_{k=1}^{2n} \log \left(1 + \eta(2y_k/k - 1) \right) + \sum_{k=1}^{2n} \eta(2y_k - k) \log(u_{k+1}/u_k)$$

with $u_{2n+1} = 1$. Since $y_k \sim \text{hypergeometric}(n, n, k)$, $Ey_k = k/2$. This and the independence of $\{y_k\}$ and $\{u_k\}$ imply

$$E\xi_n = (\eta^2/2) \log(2n) - C(\eta) + o(1).$$

Thus, the expansion in Theorem 3.1 holds for $A = 1$, in view of (3.7).

Since $(u_k - u_{k-1})/u_k$ are independent beta$(1, k-1)$ variables, (3.8) implies

$$
\begin{aligned}
\xi_n &\approx \sum_{k=1}^{2n} \frac{\eta^2}{2} \Big(\frac{2y_k}{k} - 1 \Big)^2 + \sum_{k=1}^{2n} \eta \Big(\frac{2y_k}{k} - 1 \Big) \Big\{ (k+1) \frac{u_{k+1} - u_k}{u_{k+1}} - 1 \Big\} \\
&\approx \frac{\eta^2}{2} \int_{1/(2n)}^{1} \frac{B_0^2(t)}{t^2} dt + \eta \int_{1/(2n)}^{1} \frac{B_0(t)}{t} dW_1(t) \\
&\approx E\xi_n + N\Big(0, (\eta^4 + \eta^2) \log n \Big),
\end{aligned}
$$

(3.9)

where $B_0(\cdot)$ is a Brownian bridge, $W_1(\cdot)$ is a Brownian motion independent of $B_0(\cdot)$, and $\xi_n \approx \xi_n'$ means ξ_n has the same limiting distribution as ξ_n' after centering by $E\xi_n$ and normalization by $\sqrt{\log n}$. Thus, condition (1.4) of the existing nonlinear renewal theorems does not hold.

4. Uniform integrability and an intermediate nonlinear renewal theorem

The results in this section are obtained by comparing T_b in (1.2) with

(4.1)
$$\tau_b^* = \inf \big\{ n \ge n_* : S_n + \zeta_{n_*} > b \big\},$$

where ζ_n is as in (2.4) and $n_* = \lfloor b/\mu - \eta_* b^\alpha \rfloor$ for certain $\eta_* > \theta/\mu^{1+\alpha}$, $\eta_* < 1/\mu$ for $\alpha = 1$. We shall state all the results before proceeding to the proofs.

Theorem 4.1. *Suppose that $\{\xi_n, n \ge 1\}$ is ρ-regular with parameters $p \ge 1$ and $1/2 < \alpha \le 1$. Suppose that the stopping time T_b in (1.2) satisfies*

(4.2)
$$\lim_{b \to \infty} \big(\rho(b)/b \big)^{-p} P\{T_b \le \delta_0 b/\mu\} = 0$$

for the δ_0 in (2.2). If $E|X|^{(p+1)/\alpha} < \infty$, then

(4.3)
$$\Big\{ \Big| \frac{T_b - \tau_b^*}{\rho(b)} \Big|^p : b \ge 1 \Big\} \quad \text{is uniformly integrable.}$$

For $\rho(x) = 1$ and $p = 1$, this uniform integrability result is a crucial component in our proof of the second order expansion of ET_b in Theorem 2.1. It can be also used to derive expansions of the expectations of the renewal measure U_b and the last exit time N_b^*, where

(4.4)
$$U_b = \sum_{n=1}^{\infty} I\{Z_n \le b\}, \qquad N_b^* = 1 + \sup \big\{ n \ge 1 : Z_n \le b \big\}.$$

For general $\rho(\cdot)$, it yields an expansion of ET_b up to $O(\rho(b))$. Since $\rho(b) = o(b)$ (e.g. $\rho(b)/b^\epsilon \to 0$ for all $\epsilon > 0$ with the rank SPRT by Lai [14], as shown in Section 3), such an expansion is typically sharper than direct extensions of the elementary renewal theorem but cruder than Theorem 2.1 as an extension of the standard renewal theorem. Thus, we call Theorem 4.2 (i) below an intermediate nonlinear renewal theorem.

Theorem 4.2. (i) *Suppose the conditions of Theorem 4.1 hold with $p = 1$. Then, for certain finite M,*

(4.5)
$$\mu ET_b = b - E\zeta_{n_b} + O(1)\left\{1 + E \max_{1 \le j \le Mb^\alpha}\left(|\xi_{n_*+j} - \zeta_{n_*}| \wedge n^\alpha\right)\right\}$$
$$= b - E\zeta_{n_b} + O(\rho(b)).$$

(ii) *Suppose the conditions of Theorem 4.1 hold with $p = 2$. Then,*

(4.6)
$$\mathrm{Var}(T_b) = \frac{\sigma^2 b}{\mu^3} + O(\sqrt{b}\rho(b) + \rho^2(b)), \quad \text{as } b \to \infty.$$

Remark 4.1. The conclusions of Theorem 4.2 remain valid if we replace the $o(1)$ in (2.2) and (2.3) with $O(1)$ and (2.4) with the weaker

$$E \max_{1 \le j \le Mn^\alpha}\left(|\zeta_n - \xi_{n+j}|^p \wedge n^{\alpha p}\right) = O(\rho^p(b)).$$

Theorem 4.3. *Let U_b and N_b^* be as in (4.4). Suppose*

(4.7)
$$\sum_{k \ge n+Kn^\alpha} k^{p-1} P\left\{\sup_{j \ge k} j^{-\alpha}(\xi_j + (j-n)\mu) \le w_0\right\} = o(\rho^p(n)),$$

for some $K > 0$ and $w_0 > 0$ as in (2.3). Then, (4.3), (4.5) and (4.6) hold with T_b replaced by either U_b or N_b^* under their respective conditions in Theorems 4.1 and 4.2.

Proofs of of Theorems 4.2 and 4.3 are omitted since they follow from standard methods in nonlinear renewal theory, cf. [28] or [32]. We need three lemmas for the proof of Theorem 4.1.

Lemma 4.1. *Suppose $E|X|^{(p+1)/\alpha} < \infty$. If (2.2) and (4.2) hold for certain $\theta > 0$, then $P\{T_b \le b/\mu - \eta_* b^\alpha\} = o(\rho(b)/b)^p$ for all $\eta_* > \theta/\mu^{1+\alpha}$.*

Proof. Let $w > 0$ satisfy $(1-w)\mu^{1+\alpha}\eta_* = \theta$ with the θ in (2.2). Let $b > 1$ and $n_* = \lfloor b/\mu - \eta_* b^\alpha \rfloor$. Let δ_0 be as in (2.2) and (4.2). By (1.2)

(4.8)
$$P\{\delta_0 b/\mu < T_b \le b/\mu - \eta_* b^\alpha\}$$
$$\le P\{S_n + \xi_n > b \quad \text{for some} \quad \delta_0 b/\mu < n \le n_*\}$$
$$\le P\left\{\max_{1 \le n \le n_*} S_n - n\mu > w\mu\eta_* b^\alpha\right\}$$
$$+ P\left\{\max_{\delta_0 n_* < n \le n_*}(\xi_n + n\mu) > b - w\mu\eta_* b^\alpha\right\}.$$

Since $E|X|^{(p+1)/\alpha} < \infty$, by Theorem 1 of Chow and Lai [5]

(4.9)
$$P\left\{\max_{1 \le n \le b}|S_n - n\mu| > \lambda b^\alpha\right\} = o(b^{-p}), \quad \forall \lambda > 0.$$

Thus, the first term on the right-hand side of (4.8) is of the order $o(b^{-p})$ since $n_* = O(b)$. Since $b - w\mu\eta_* b^\alpha - n_*\mu \ge (1-w)\mu\eta_* b^\alpha \ge (1-w)\mu^{1+\alpha}\eta_* n_*^\alpha = \theta n_*^\alpha$ and $\rho(n_*)$ is of the same order as $\rho(b)$ by (2.1), it follows from (2.2) that the second term on the right-hand side of (4.8) is bounded by

$$P\left\{\max_{\delta_0 n_* < n \le n_*}\xi_n > \theta n_*^\alpha\right\} = o(\rho(n_*)/n_*)^p = o\left(\rho(b)/b\right)^p.$$

Hence, (4.8) is $o\left(\rho(b)/b\right)^p$ and the conclusion follows from (4.2). $\square$

Lemma 4.2. *Suppose $E|X|^{(p+1)/\alpha} < \infty$. If (2.3) holds for certain $K > 0$, then*

$$\lim_{b\to\infty} \frac{1}{\rho^p(b)} \sum_{k\geq b/\mu + \eta^* b^\alpha} k^{p-1} P\{T_b > k\} = 0, \quad \eta^* = K/\mu^\alpha.$$

Proof. Let $n^* = \lceil b/\mu + \eta^* b^\alpha \rceil$. By (1.2)

$$(4.10) \qquad \sum_{k=n^*}^{\infty} k^{p-1} P\{T_b > k\} \leq \sum_{k=n^*}^{\infty} k^{p-1} P\{S_k + \xi_k \leq b\}$$

$$\leq \sum_{k=n^*}^{\infty} k^{p-1} P\{S_k \leq k\mu - w_0 k^\alpha/2\}$$

$$+ \sum_{k=n^*}^{\infty} k^{p-1} P\{\xi_k \leq b - k\mu + w_0 k^\alpha/2\}.$$

Since $E|X|^{(p+1)/\alpha} < \infty$, by Theorem 1 of Chow and Lai [5]

$$(4.11) \qquad \sum_{n=1}^{\infty} P\Big\{|S_n - n\mu| > \lambda n^\alpha\Big\} < \infty, \quad \forall \lambda > 0.$$

Thus, since $w_0 > 0$, the first term on the right-hand side of (4.10) is $o(1)$ as $b \to \infty$. Let $n_b = \lfloor b/\mu \rfloor$ as in Theorem 2.1. Since $n_b\mu \leq b < (n_b + 1)\mu$ and $n^* \geq n_b + \eta^*(n_b\mu)^\alpha = n_b + K n_b^\alpha$, in view of (2.3) the second term is bounded by

$$\sum_{k\geq n_b + K n_b^\alpha} k^{p-1} P\{\xi_k \leq (n_b + 1)\mu - k\mu + w_0 k^\alpha/2\} = o(\rho^p(n_b)).$$

The conclusion follows since $\rho(n_b)$ and $\rho(b)$ are of the same order. □

Lemma 4.3. *Suppose $E|X|^{(p+1)/\alpha} < \infty$. Let τ_b^* be as in (4.1) with the η_* in Lemma 4.1. Then,*

$$(4.12) \qquad P\{b/\mu - \eta_* b^\alpha < \tau_b^* < b/\mu + \eta_\tau^* b^\alpha\} \geq 1 + o(b^{-p})$$

for all $\eta_\tau^* > \theta^*/\mu^{1+\alpha}$. Moreover, for such η_τ^*

$$(4.13) \qquad \lim_{b\to\infty} \sum_{k\geq b/\mu + \eta_\tau^* b^\alpha} k^{p-1} P\{\tau_b^* > k\} = 0.$$

Proof. Since $b - n_*\mu - \theta n_*^\alpha \geq \mu\eta_* b^\alpha - \theta n_*^\alpha \geq (\eta_*\mu^{1+\alpha} - \theta)n_*^\alpha > 0$, it follows from (4.9) that

$$P\{\tau_b^* = n_*\} \leq P\{S_{n_*} + \theta n_*^\alpha > b\}$$
$$= P\{S_{n_*} - n_*\mu > b - n_*\mu - \theta n_*^\alpha\} = o(b^{-p}).$$

Let $n_\tau^* = \lceil b/\mu + \eta_\tau^* b^\alpha \rceil$. Since $b - n_\tau^*\mu + \theta^* n_*^\alpha \leq -\mu\eta_\tau^* b^\alpha + \theta^* n_*^\alpha \leq -(\eta_\tau^*\mu^{1+\alpha} - \theta^*)n_*^\alpha < 0$, we have

$$P\{\tau_b^* \geq n_\tau^*\} \leq P\{S_{n_\tau^*} - \theta^* n_*^\alpha \leq b\}$$
$$= P\{S_{n_\tau^*} - n_\tau^*\mu \leq b - n_\tau^*\mu + \theta^* n_*^\alpha\} = o(b^{-p}).$$

The above calculations prove (4.12). The proof of (4.13) is simpler than that of Lemma 4.2 and omitted. □

Proof of Theorem 4.1. Let $\eta_*, \eta^* = K/\mu^\alpha$ and $\eta^* > \eta^*_\tau > \theta^*/\mu^{1+\alpha}$ be as in Lemmas 4.1, 4.2 and 4.3 respectively. This is possible since $\theta^* < K\mu$ with the ρ-regularity conditions. Set $n_* = \lfloor b/\mu - \eta_* b^\alpha \rfloor$, $n^* = \lceil b/\mu + \eta^* b^\alpha \rceil$ and $n^*_\tau = \lceil b/\mu + \eta^*_\tau b^\alpha \rceil$.

Step 1: We first prove that for all integers $k_* \geq 1$

$$(4.14) \quad \begin{aligned} &\sum_{k=k_*}^\infty k^{p-1} P\{T_b - \tau^*_b > k\} \\ &\leq 2 \sum_{k \geq n^*} k^{p-1} P\{T_b > k\} + \sum_{n^* - n_* \leq k < n^*} k^{p-1} P\{\tau^*_b \leq n_*\} \\ &\quad + \sum_{k_* \leq k < n^* - n_*} k^{p-1} P\{A_\tau\} + \sum_{k_* \leq k < n^* - n_*} k^{p-1} P\{S_k \leq \mu k/2\} \\ &\quad + \sum_{k_* \leq k < n^* - n_*} k^{p-1} P\Big\{ \max_{1 \leq j \leq 2 + M_\tau b^\alpha} (\zeta_{n_*} - \xi_{j+n_*}) > \mu k/2 \Big\} \end{aligned}$$

with $A_\tau = \{\tau^*_b \notin (n_*, n^*_\tau)\}$ and $M_\tau = \eta^*_\tau + \eta^* + 2\eta_*$, and

$$(4.15) \quad \begin{aligned} &\sum_{k=k_*}^\infty k^{p-1} P\{\tau^*_b - T_b > k\} \\ &\leq 2 \sum_{k \geq n^*} k^{p-1} P\{\tau^*_b > k\} + \sum_{n^* - n_* \leq k < n^*} k^{p-1} P\{T_b \leq n_*\} \\ &\quad + \sum_{k_* \leq k < n^* - n_*} k^{p-1} P\{\widetilde{A}_T\} + \sum_{k_* \leq k < n^* - n_*} k^{p-1} P\{S_k \leq \mu k/2\} \\ &\quad + \sum_{k_* \leq k < n^* - n_*} k^{p-1} P\Big\{ \max_{1 \leq j \leq 2 + M_T b^\alpha} (\xi_{j+n_*} - \zeta_{n_*}) > \mu k/2 \Big\} \end{aligned}$$

with $\widetilde{A}_T = \{T_b \notin (n_*, n^*), \tau^*_b > T_b\}$ and $M_T = \eta^* + \eta_*$.

For the proof of (4.14), we divide $[k_*, \infty)$ into three intervals at $n^* - n_*$ and n^*. For the first interval $k \in [k_*, n^* - n_*)$, we have $n_* < \tau^*_b + k \leq n^*_\tau + n^* - n_* - 2 < n_* + 2 + M_\tau b^\alpha$ in A^c_τ and

$$\begin{aligned} \{T_b - \tau^*_b > k\} &\subseteq \{S_{\tau^*_b + k} + \xi_{\tau^*_b + k} \leq b < S_{\tau^*_b} + \zeta_{n_*}\} \\ &\subseteq \{S_{\tau^*_b + k} - S_{\tau^*_b} \leq \mu k/2\} \cup \{\xi_{\tau^*_b + k} - \zeta_{n_*} < -\mu k/2\}. \end{aligned}$$

For the second interval $k \in [n^* - n_*, n^*)$, we have

$$\begin{aligned} \sum_{k \geq n^* - n_*} k^{p-1} P\{T_b > k + \tau^*_b, \tau^*_b > n_*\} &\leq \sum_{k \geq n^* - n_*} k^{p-1} P\{T_b > k + n_*\} \\ &\leq \sum_{j \geq n^*} j^{p-1} P\{T_b \geq j\}. \end{aligned}$$

The proof of (4.15) is nearly identical, with $n_* < T_b < n^* \leq n_* + (\eta^* + \eta_*) b^\alpha + 2$ in $\widetilde{A}^c_T$ and

$$\begin{aligned} \{\tau^*_b - T_b > k\} &\subseteq \{S_{T_b+k} + \zeta_{n_*} \leq b < S_{T_b} + \xi_{T_b}\} \\ &\subseteq \{S_{T_b+k} - S_{T_b} \leq \mu k/2\} \cup \{\zeta_{n_*} - \xi_{T_b} < -\mu k/2\}. \end{aligned}$$

Step 2: Prove the uniform integrability of $\{(T_b - \tau^*_b)^+/\rho(b)\}^p$. For $k_* = \lfloor C\rho(b) \rfloor$, we have

$$J(b, C) = E\left(\frac{T_b - \tau^*_b}{\rho(b)} - C\right)^p I\left\{\frac{T_b - \tau^*_b}{\rho(b)} > C\right\}$$

$$\leq \frac{C_p}{\rho^p(b)} \sum_{k=k_*}^{\infty} k^{p-1} P\{T_b - \tau_b^* > k\},$$

where C_p is a universal constant. For the current step, it suffices to show $J(b,C) \to 0$ as $b \to \infty$ and then $C \to \infty$. Since $n^* = O(b)$ and $n^* - n_* = O(b^\alpha)$, it follows from (4.14), Lemmas 4.2 and 4.3, and (4.11) that

$$J(b,C) = o(1) + \frac{C_p}{\rho^p(b)} \sum_{k_* \leq k < n^* - n_*} k^{p-1} P\Big\{ \max_{1 \leq j \leq 2 + M_\tau b^\alpha} (\zeta_{n_*} - \xi_{j+n_*}) > \mu k/2 \Big\}.$$

Since $b = O(n_*)$, we may choose M in (2.4) satisfying $Mn_*^\alpha > 2 + M_\tau b^\alpha$ for all $b > 1$. Thus, by the uniform integrability in (2.4), $J(b,C) = o(1)$ as $b \to \infty$ and then $C \to \infty$.

Step 3: Prove the uniform integrability of $\{(T_b - \tau_b^*)^- / \rho(b)\}^p$. This step is nearly identical to Step 2, with (4.15) instead of (4.14). In fact, since $M_T < M_\tau$, the same M in (2.4) works. Although Lemma 4.1 does not provide $(b^\alpha/\rho(b))^p P(T_b \geq n^*) = o(1)$, we have $P(\widetilde{A}_T) \leq P(T_n \leq n_*) + P(\tau_b^* \geq n^* \geq n_\tau^*)$, so that Lemmas 4.1 and 4.3 can be used to control the third sum on the right-hand side of (4.15). This completes the proof of Step 3 and thus the entire theorem. □

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