Part C

α-Recursion

α-recursion theory lifts classical recursion theory from ω to an arbitrary Σ_1 admissible ordinal α. Many of the classical results lift to every α by means of recursive approximations and fine structure techniques.

Chapter VII
Admissibility and Regularity

The fundamental notions of α-recursion theory are introduced. Metarecursion is shown equivalent to ω_1^{CK}-recursion. The essential properties of the Σ_1 projectum are established and applied in a priority argument to obtain a non-α-recursive, hyperregular, α-recursively enumerable set. A regular sets theorem is proved.

1. Σ_1 Admissibility

The notion of admissible set was invented by R. Platek (1966) in the course of his investigations of the foundations of recursion theory. Here, for reasons soon to be evident, "admissible" is called "Σ_1 admissible".

1.1 Σ_1 Admissible Sets. Let A be a nonempty set. A is said to be transitive if $x \in A$ whenever $x \in y$ and $y \in A$. A is closed under pairing if $\{x, y\} \in A$ whenever $x \in A$ and $y \in A$. A is closed under union if $x \in A$ implies $(\cup x) \in A$. ($\cup x = \{y | (\mathrm{Ez})(y \in z \in x)\}$.)

The formulas of Zermelo–Fraenkel set theory (ZF) are built up from set variables $x, y, z, \ldots$, atomic formulas ($s \in t$, where s and t are set variables), propositional connectives, and set quantifiers ((Ex) (there exists x), (y) (for all y)). A formula is *bounded* if all its quantifiers are bounded, that is of the form $(\mathrm{Ex})_{x \in y}$ or $(y)_{y \in z}$. The Levy hierarchy of formulas is defined by induction on n. A formula F is $\Delta_0 = \Pi_0$ $(=\Sigma_0)$ if F is bounded. $(\mathrm{Ex}_1) \ldots (\mathrm{Ex}_m)G$ is Σ_n if G is Π_{n-1} $(m \geq 0)$. $(y_1) \ldots (y_m)G$ is Π_n if G is Σ_{n-1} $(m \geq 0)$. $\Delta_n = \Pi_n \cap \Sigma_n$.

Typical Δ_0 formulas are: $x \subseteq y$; x is an ordinal. An example of a formula that is Π_1 but not Σ_1 is: x is a cardinal.

A formula G is said to have *parameters* in a set A if G is of the form $H(x_1, \ldots, x_m, a_1, \ldots, a_n)$, where $H(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is a formula of ZF and $a_i \in A$ $(1 \leq i \leq n)$.

A satisfies Δ_0 *separation* if every sentence of the form

$$(\mathrm{Ex})(y)[y \in x \leftrightarrow y \in \underline{a} \quad \& \quad F(y)]$$

is true in $\langle A, \varepsilon \rangle$, where $a \in A$ and $F(y)$ is Δ_0 with parameters in A.

A satisfies Δ_0 *bounding* (or *collection*) if every sentence of the form

$$(x)_{x\in \underline{a}}(\mathrm{Ey})F(x,y) \to (\mathrm{Ez})(x)_{x\in \underline{a}}(\mathrm{Ey})_{y\in z}F(x,y)$$

is true in $\langle A, \varepsilon \rangle$. Again $a \in A$ and $F(x,y)$ is Δ_0 with parameters in A.

A set A is said to be Σ_1 *admissible* if A is transitive, closed under pairing and union, and satisfies Δ_0 separation and Δ_0 bounding. A pairing trick shows a Σ_1 admissible set satisfies Δ_1 separation and Σ_1 bounding (Exercise 1.15). Suppose $B \subseteq C$. The structure $\langle C, B \rangle$ is said to be Σ_1 admissible if C is Σ_1 admissible when $x \in B$ is added to the list of Δ_0 formulas.

The least Σ_1 admissible set is HF, the set of all hereditarily finite sets. (x is hereditarily finite iff x is finite and every member of x is hereditarily finite.) HF can serve as the setting for classical recursion theory. It turns out that a partial function $f\colon \omega \to \omega$ is partial recursive in the classical sense iff the graph of f is Σ_1 definable over HF.

A set $B \subseteq A$ is Σ_n definable over A if there is a Σ_n formula $G(x)$ with parameters in A such that

$$a \in B \leftrightarrow \langle A, \in \rangle \models G(\underline{a})$$

for all $a \in A$. Similarly, Π_n and Δ_n definability.

Call a set A-*recursive* if it is Δ_1 definable over A, and A-*finite* if it belongs to A. Then Δ_1 separation becomes: if B is A-recursive and c is A-finite, then $B \cap c$ is A-finite. Call a function f partial A-recursive if its graph is Σ_1 definable over A. Then Σ_1 bounding and Δ_1 separation imply: if f is partial A-recursive, c is A-finite, and the domain of f includes c, then $f[c]$ is A-finite. In this fashion the beginnings of classical recursion theory can be generalized to every Σ_1 admissible A. A great deal can be lifted if A admits an A-recursive, one-to-one correspondence between A and ord(A), the least ordinal not in A. Such a correspondence is available when A is an initial segment of L, Gödel's universe of constructible sets, the setting for α-recursion theory.

Metarecursion theory was developed in Part B in terms of Π_1^1 sets and unique notations for recursive ordinals. Another approach, perhaps smoother, is via Σ_1 admissibility. There exists a Σ_1 admissible set called HYP such that the metarecursively enumerable subsets of ω_1^{CK} are just those Σ_1 definable over HYP.

The definition of HYP needs the idea of coding hereditarily countable sets (HC) by reals, that is, subsets of ω. $x \in \mathrm{HC}$ iff x is countable and every member of x belongs to HC. The set K of all codes for members of HC is defined by an arithmetic closure condition C, hence is Π_1^1 by Theorem 1.6.I. C has two clauses:

$\{0\}$ is a code;
if $(i)_{i<\omega}((r)_i$ is a code), then r is a code.

Recall that $(r)_i = \{m \mid 2^i \cdot 3^m \in r\}$. Define $u >_K v$ to be the transitive closure of

$$u, v \in K \quad \& \quad (\mathrm{Ei})_{i<\omega}(v = (u)_i).$$

$>_K$ is wellfounded by the natural enumeration approach that showed $<_O$ is wellfounded (Theorem 2.2.I). The set $x(r)$ encoded by $r \in K$ is defined by transfinite recursion on $>_K$:

$$x(r) = \{x((r)_i) \mid i < \omega\}.$$

Thus each member of K encodes a member of HC, and each member of HC has at least one code in K. (The latter is a consequence of AC_ω, the countable axiom of choice.)

Let HC(HYP) be the set of all x such that x has a hyperarithmetic code in K.

1.2 Proposition. HC(HYP) *is* Σ_1 *admissible.*

Proof. To verify Δ_0 bounding, suppose

(1) $$\mathrm{HC(HYP)} \models (x)_{x \in \underline{a}}(\mathrm{E}y) F(x, y)$$

for some $a \in \mathrm{HC(HYP)}$ and $F(x, y) \in \Delta_0$ with parameters in HC(HYP). Since each member of HC(HYP) is countable in HC(HYP), a can be taken to be ω. (1) becomes

(2) $$(n)(\mathrm{E}r)[r \in \mathrm{HYP} \cap K \quad \& \quad \mathrm{HC(HYP)} \models F(\underline{n}, \underline{x}(\underline{r}))].$$

The notion of truth in HC(HYP) for Δ_0 sentences is Π_1^1, because it can be defined inductively by a Σ_1^1 closure condition (cf. proof of Lemma 4.5.III). K is Π_1^1, so the matrix of (2) is Π_1^1. In addition the real variable r of (2) can be regarded as a number variable ranging over indices of hyperarithmetic reals. Then Lemma 2.6.II applies, and r can be construed as a hyperarithmetic function of n. That function is in itself a hyperarithmetic code for a set suitable for bounding y in (1). □

1.3 Proposition. *Let* $A \subseteq \omega_1^{\mathrm{CK}}$. *Then* A *is metarecursively enumerable iff* A *is* Σ_1 *definable over* HC(HYP).

Proof. First suppose A is metarecursively enumerable. Thus

$$(\delta)[\delta \in A \leftrightarrow |\delta| \in A_1]$$

for some Π_1^1 A_1. By Theorem 3.5.III, there is an arithmetic $A(X, b)$ such that

(1) $$\delta \in A \leftrightarrow (\mathrm{E}b)(\mathrm{E}X)[X \in \mathrm{HYP} \quad \& \quad A(X, b) \quad \& \quad |b| = \delta].$$

To say $|b| = \delta$ is the same as saying $W_{q(b)}$ (from Theorem 3.5.I) is isomorphic to $\{\langle u, v\rangle \mid u \in v \in \delta\}$. It follows from 1.2 that the right side of (1) is Σ_1.

Now suppose A is Σ_1 is definable over HC(HYP). Then

$$\delta \in A \leftrightarrow (\mathrm{E}r)[r \in \mathrm{HYP} \cap K \quad \& \quad F(x(r), \delta)]$$

for some $\Delta_0\, F$. Let A_1 be the set of all b such that

$$b \in O \quad \& \quad (\mathrm{Er})[r \in \mathrm{HYP} \cap K \quad \& \quad F(x(r), |b|)].$$

Then A_1 is Π_1^1. $\square$

1.4 Let X and Y be sets. X is said to be a *first order definable* subset of Y if there is a formula $F(x)$ of ZF with parameters in Y such that

$$X = \{a \mid a \in Y \quad \& \quad \langle Y, \in \rangle \models F(\underline{a})\}.$$

The set of all first order definable subsets of Y is denoted by Fod(Y). Gödel's universe of constructible sets is defined by iterating Fod through the ordinals.

$$L(0) = \phi.$$
$$L(\delta + 1) = \mathrm{Fod}(L(\delta)).$$
$$L(\lambda) = \cup \{L(\delta) \mid \delta < \lambda\} \ (\lambda \text{ a limit}).$$
$$L = \cup \{L(\delta) \mid \delta \text{ is an ordinal}\}.$$

Some excellent sources of information about L are Jech (1978) and Devlin (1984). Gödel showed that L is a model of ZF, the axiom of choice and the generalized continuum hypothesis. He obtained these extraordinary results by means of ideas which can be refined to yield detailed information about initial segments of L. Some of these refinements will be discussed in future sections. They combine smoothly with ideas from classical recursion theory to produce methods for classifying the Σ_1 definable subsets of $L(\alpha)$ when $L(\alpha)$ is Σ_1 admissible. The ordinal α is said to be Σ_1 admissible if $L(\alpha)$ is a Σ_1 admissible set. α-recursion theory is the study of α-recursively enumerable (Σ_1 over $L(\alpha)$) subsets of α.

Recursion theory on initial segments of ordinals was invented by Takeuti (1960). Σ_1 admissible initial segments of ordinals were first studied by Kripke (1964) and Platek (1966).

1.5 Proposition. *α is Σ_1 admissible $\leftrightarrow L(\alpha)$ satisfies Δ_0 bounding.*

Proof. A straightforward induction on δ shows that $L(\delta)$ is transitive and that the ordinals in $L(\delta)$ are just those less than δ. If α is a limit, then $L(\alpha)$ is closed under pairing and union, and satisfies Δ_0 separation. Suppose $\alpha = \beta + 1$. Consider the Δ_0 function f defined by $f(x) = \beta$ for all $x \in L(\alpha)$. Δ_0 bounding implies $\beta \in z$ for some $z \in L(\alpha)$. But then $z \subseteq L(\beta)$ and so $\beta \in L(\beta)$, an impossibility. $\square$

From now on α is a Σ_1 admissible ordinal. The fundamental notions of α-recursion theory are defined as in subsection 1.1 with $L(\alpha)$ in place of A. A set $B \subseteq \alpha$ is said to be *α-recursively enumerable* if B is Σ_1 definable over $L(\alpha)$. Remember that parameters from $L(\alpha)$ are permitted (in the Σ_1 definition of B). A function $f \subseteq \alpha^2$ is said to be partial α-recursive if its graph is α-recursively enumerable; if the domain of f is α, then f is said to be α-recursive. The *α-finite* subsets of α are those that belong to $L(\alpha)$. There is a natural enumeration of $L(\alpha)$ that maps α onto $L(\alpha)$ (subsection 1.7). It is only slightly more complicated than the natural enumeration of all hereditarily finite sets.

The first two Σ_1 admissible ordinals are ω and ω_1^{CK} (Exercise 1.10). The classical partial recursive functions are the same as the partial ω-recursive functions. Metarecursion theory is equivalent to ω_1^{CK}-recursion theory by Proposition 1.3, once it is seen that $L(\omega_1^{\text{CK}}) = \text{HC(HYP)}$ (Exercise 1.11). ω_1^T is Σ_1 admissible for every $T \subseteq \omega$. Conversely, every countable Σ_1 admissible ordinal greater than ω is of the form ω_1^T for some $T \subseteq \omega$ (Sacks (1976)).

1.6 Recursion on α. The Σ_1 admissibility of $L(\alpha)$ makes it possible to define α-recursive functions by Σ_1-on-$L(\alpha)$ transfinite recursions on α. Let I: $L(\alpha) \to \alpha$ be Σ_1 definable on $L(\alpha)$. There is at most one f such that

$$(1) \qquad (\delta)_{\delta<\alpha}[f(\delta) = I(f \restriction \delta)].$$

Let Z be the set of all α-finite g such that $g(\gamma) = I(g \restriction \gamma)$ for all $\gamma \in \operatorname{dom} g$. Z is α-recursive, and any two functions in Z agree on all arguments in common. Define

$$f(\delta) = y \quad \text{by} \quad (\text{E}g)[g \in Z \quad \& \quad \delta \in \operatorname{dom} g \quad \& \quad f(\delta) = y].$$

f is Σ_1 on $L(\alpha)$. Furthermore, $\operatorname{dom} f = \alpha$, otherwise f belongs to Z and extends to some other member of Z.

The definition of f can be thought of as a process. At stage δ it is assumed that all activity at previous stages is encapsulated in an α-finite object, $s \restriction \delta$. Instruction I is given in advance. It tells how to compute $f(\delta)$ from $s \restriction \delta$. In general it will be necessary to search through $L(\alpha)$ for some existential witness w that bears a specified Δ_0 relationship to $s \restriction \delta$. $s(\delta)$ encodes not only $f(\delta)$ but also w and the location of w. The Σ_1 admissibility of $L(\alpha)$, in particular Σ_1 bounding, is needed to bound the search for w below α so long as δ is bounded below α.

1.7 The Natural Enumeration of $L(\alpha)$. The elements of $L(\alpha)$ lend themselves to an enumeration in α stages. At stage δ the elements of $L(\delta)$ are enumerated one at a time. If δ is a limit ordinal, then the enumerations of $L(\gamma)\,(\gamma<\delta)$ are stacked one on top of another and repeated. Suppose $\delta = \gamma + 1$. A typical element x of $L(\delta)$ is a first order definable subset of $L(\gamma)$. A code for x is a sequence $\langle e, b_1, \ldots, b_m\rangle$. e is the Gödel number of a formula $F(y, z_1, \ldots, z_m)$, $b_i \in L(\gamma)\,(1 \leq i \leq m)$, and

$$x = \{a \mid a \in L(\gamma) \quad \& \quad L(\gamma) \models F(\underline{a}, \underline{b}_1, \ldots, \underline{b}_m)\}.$$

The enumeration of $L(\delta)$ parallels the enumeration of all codes of elements of $L(\delta)$. The latter is derived from the given enumeration of $L(\delta)$.

The main question avoided above is how to pass from a code for x to x. The answer has to do with the definability of truth in $L(\gamma)$. If γ is a limit, then the relation $L(\gamma) \models F$ is definable over $L(\gamma+1)$. In fact, the usual Tarski-style definition of truth works.

1.8 Proposition. *There exists a one-one, α-recursive f that maps α onto $L(\alpha)$.*

Proof. By appeal to the picture of Σ_1 recursion sketched in subsection 1.6 and to the natural enumeration of $L(\alpha)$ given in Subsection 1.7. Let $h(\delta)$ be the enumeration of $L(\delta)$ alluded to in 1.7. The α-finite object $h(\delta)$ is derived from the α-finite object $h \restriction \delta$ by an instruction I easily (but tediously) seen to be Σ_1 over $L(\alpha)$. Let g be the α-recursive function that enumerates $L(\alpha)$, repetitions permitted, by stacking the $h(\delta)$'s one on top of another. The desired f is g with the repetitions omitted. $\square$

1.9 Proposition. *There exists a partial α-recursive function $\phi(\varepsilon, \delta)$ such that for each partial α-recursive $\psi(\delta)$, there exists an ε such that*

$$\phi(\varepsilon, \delta) \simeq \psi(\delta)$$

for all δ. (Enumeration Theorem)

Proof. By appeal to subsections 1.6 and 1.7, as in 1.8. Let ψ be partial α-recursive. There must be a Δ_0 formula $F(u, v, z)$ with parameters in $L(\alpha)$ such that $\psi(\delta) = \gamma$ iff for some $w \in L(\alpha)$,

$$(1) \qquad F(\delta, \gamma, w) \quad \& \quad (y)(z)_{\langle y, z\rangle < \langle \gamma, w\rangle} \sim F(\delta, y, z).$$

In (1) w is a witness to the fact that $\psi(\delta) = \gamma$, and $<$ is the α-recursive wellordering of $L(\alpha)$ provided by Proposition 1.8, and derived from the natural enumeration of $L(\alpha)$. The truth or falsity of (1) can be determined by examining any limit structure $L(\beta)$ whose members include δ, γ, w, the parameters of $F(u, v, z)$, and $<$ below $\langle \gamma, w\rangle$. Thus the natural enumeration of $L(\alpha)$ gives rise to an α-recursive enumeration of T, the set of all true sentences of type (1). A suitable code for (1) is $\langle \varepsilon, \delta, \gamma, w\rangle$, where ε is a code for $F(u, v, z)$. Let T_1 be the set of codes of elements of T. Define

$$\psi(\varepsilon, \delta) \simeq \gamma \quad \text{by} \quad (Ew)[\langle \varepsilon, \delta, \gamma, w\rangle \in T_1].$$

ψ is partial α-recursive since T, hence T_1, is α-recursively enumerable. In fact, T is α-recursive, because all false sentences of type (1) can be enumerated simultaneously with T. $\square$

The immense power of Σ_1 admissibility is not needed to prove Propositions 1.8 and 1.9. They hold for any $L(\beta)$ such that β is a limit. In that case the proof of 1.8 relies on a condensation argument of the sort given in the next section rather than the natural enumeration of subsection 1.7. If $L(\beta)$ is closed with respect to some primitive recursions, then the natural enumeration of 1.7 is available.

1.10–1.15 Exercises

1.10. Show ω_1^{CK} is the first Σ_1 admissible ordinal after ω.

1.11. Show $L(\omega_1^{\mathrm{CK}}) = \mathrm{HC(HYP)}$.

1.12. Suppose $A \subseteq \omega_1^{\mathrm{CK}}$. Show A is metarecursively enumerable iff A is ω_1^{CK}-recursively enumerable.

1.13. Show ω_1^T is Σ_1 admissible for all $T \subseteq \omega$.

1.14. Suppose $A \subseteq \alpha$ is α-recursively enumerable. Find a partial, one-one, α-recursive g whose range is A and whose domain is an initial segment of α.

1.15. Show every Σ_1 admissible set satisfies Δ_1 separation and Σ_1 bounding.

1.16. Let f be partial and α-recursive. Find a one-one, partial α-recursive g with the same range as f.

2. *The Σ_1 Projectum*

In part B considerable use was made of a metarecursive, one-one map of ω_1^{CK} into ω. The priority arguments of Section 2.VI relied strongly on the fact that each requirement was injured only for the sake of finitely many requirements of higher priority. Thus some of the combinatoric tricks of classical recursion theory were lifted to ω_1^{CK} by mapping ω_1^{CK} into ω. In general there is no hope of mapping α down to ω, or even down to something less than α, but the idea of mapping α downward makes sense for every α. The Σ_1 *projectum of* α, denoted by $\sigma 1 p\alpha$ or α^*, is defined to be

$$\mu\beta(\mathrm{Ef})\left[f \text{ is } \alpha\text{-recursive: } \alpha \xrightarrow[\text{into}]{\text{one-one}} \beta\right].$$

Thus $(\omega_1^{\mathrm{CK}})^* = \omega$. Mapping ω_1^{CK} into ω proved useful because every metarecursively enumerable subset of a finite set is finite, hence metafinite. In this manner some troublesome, bounded, metarecursively enumerable, but not metafinite sets were avoided. The Σ_1 projectum of α has a similar virtue for every α according to the next proposition.

2.1 Proposition. *Assume $A \subseteq \delta < \sigma 1 p\alpha$. If A is α-recursively enumerable, then A is α-finite.*

Proof. Suppose not. Let g be a partial, one-one, α-recursive function whose range is A, and whose domain is an initial segment of α (Exercise 1.14). Since $\delta < \sigma 1 p\alpha$, the domain of g must be less than α. But then the range of g is α-finite. □

2.2 Proposition. *$\sigma 1 p\alpha$ is the least β such that some α-recursively enumerable subset of β is not α-finite.*

Proof. Let β_0 be the least β. By Proposition 2.1, $\sigma 1 p\alpha \leq \beta_0$. To see that $\sigma 1 p\alpha \geq \beta_0$, let f be a one-one, α-recursive map of α into $\sigma 1 p\alpha$. If $f[\alpha]$ were α-finite, then $f^{-1} f[\alpha]$ would be α-finite, hence bounded below α. □

If $\sigma 1 p\alpha = \omega$, then there is little difference between metarecursion theory, that is ω_1^{CK}-recursion theory, and α-recursion theory. If $\sigma 1 p\alpha > \omega$, then $L(\alpha)$ merits a finer analysis in terms of the notion of α-cardinal. Suppose $\beta < \alpha$; β is said to be an *α-cardinal* if there is no α-finite, one-to-one correspondence between β and some $\delta < \beta$. In other words,

$$L(\alpha) \models [\beta \text{ is a cardinal}].$$

Note that $\sigma 1 p\alpha < \alpha$ implies $\sigma 1 p\alpha$ is the greatest α-cardinal. Each α-finite set has an α-cardinality, namely the least ordinal with which it can be put into one-to-one, α-finite correspondence. If an α-finite set H has α-cardinality less than $\sigma 1 p\alpha$, then every α-recursively enumerable subset of H is α-finite by Proposition 2.1.

Each α-cardinal is either regular or singular. β is a *regular* α-cardinal if

$$L(\alpha) \models [\beta \text{ is a regular cardinal}].$$

Equivalently, there is no α-finite function with domain $p < \beta$ and range an unbounded subset of β.

A collection $\{A_\delta | \delta < \gamma\}$ of subsets of α is said to be *simultaneously α-recursively enumerable* if there exists a partial α-recursive function $f(\delta, \sigma)$ such that

$$A_\delta = \{f(\delta, \sigma) | \sigma < \alpha\}$$

for all $\delta < \alpha$.

2.3 Lemma (Sacks & Simpson 1972). *Suppose $\gamma < \beta$ and β is a regular α-cardinal. Let $\{A_\delta | \delta < \gamma\}$ be a simultaneously α-recursively enumerable collection of subsets of α. If each A_δ is α-finite and has α-cardinality less than β, then $\cup \{A_\delta | \delta < \gamma\}$ is α-finite and has α-cardinality less than β.*

Proof. It is safe to assume the A_δ's are pairwise disjoint, since otherwise each can be replaced by $A_\delta \times \{\delta\}$. The given simultaneous enumeration of the A_δ's gives rise to an overall enumeration of $\cup \{A_\delta | \delta < \gamma\}$ without repetitions. At each stage of the overall enumeration, just one element of just one A_δ is enumerated. If the number of stages needed is less than β, then the lemma is proved. Otherwise, let A_δ^β be that part of A_δ enumerated during the first β stages of the overall enumeration. Then $\{A_\delta^\beta | \delta < \gamma\}$ is an α-finite collection of α-finite sets whose union has α-cardinality equal to β. But $\gamma < \beta$ and each A_δ^β has α-cardinality less than β, so β is not regular. □

Lemma 2.3 says that a regular α-cardinal is more regular than it appears; it is in fact $\mu - \Pi_1^\alpha$ regular (Exercise 2.12). Note the strong use of Σ_1 admissibility in the proof of 2.3. In the next Chapter, Lemma 2.3 will be needed to show that the

α-recursively enumerable sets of "injuries" that arise in the solution to Post's problem are α-finite.

2.4 α-Stability. Let M and N be sets. M is said to be a Σ_1 *substructure* of N (in symbols $M \prec_1 N$) if $M \subseteq N$ and

$$(1) \qquad [\langle N,\in\rangle \models F] \rightarrow [\langle M,\in\rangle \models F]$$

for every Σ_1 sentence of ZF with parameters in M. If $M \prec_1 N$, then (1) holds for every Π_2 sentence of ZF with parameters in M. β is said to be α-*stable* if $\beta < \alpha$ and $L(\beta) \prec_1 L(\alpha)$. α-stable ordinals provide bounds for partial α-recursive functions. Suppose ϕ is partial α-recursive and the parameters occurring in the Σ_1 definition of ϕ are ordinals less than β. If β is α-stable, then

$$(2) \qquad \gamma < \beta \ \& \ \phi(\gamma) \text{ defined} \rightarrow \phi(\gamma) < \beta.$$

It will be seen very shortly that every α-cardinal beyond ω is α-stable by arguments similar to those invented by Gödel to show the generalized continuum hypothesis. The first result of this sort was obtained by Takeuti (1960). He showed (2) holds if β is a cardinal of L and ϕ is a partial recursive function of ordinals.

2.5 Lemma (H. Putnam). *There exists a Π_2 sentence, $[V = L]$, such that for every transitive set M,*

$$M \models [V = L] \quad \textit{iff} \quad M = L(\lambda)$$

for some limit ordinal λ.

Proof. The sentence $[V = L]$ is

$$(1) \qquad (x)(E\delta)(\mathrm{E}y)[y = L(\delta) \ \& \ x \in y].$$

The function $\lambda\delta | L(\delta)$ is defined by a Σ_1 transfinite recursion. (Cf. Subsection 1.6.) Hence (1) is Π_2.

If $x \in L(\lambda)$ and λ is a limit, then $x \in L(\delta)$ for some $\delta < \lambda$. Also $L(\delta) \in L(\lambda)$. Thus $[V = L]$ is true in $L(\lambda)$.

Suppose M is transitive and $M \models [V = L]$. Let λ be the least ordinal not in M. Then $\lambda = M \cap \lambda$. For each $\delta < \lambda$, let $L(\delta)^M$ be $L(\delta)$ in the sense of M, that is, the result of executing within M the Σ_1 recursion that defines $\lambda\gamma | L(\gamma)$. By induction on δ, $L(\delta)^M = L(\delta)$ for all $\delta \in M$. So $L(\lambda) \subseteq M$. On the other hand, if $x \in M$, then $x \in L(\delta)^M$ for some $\delta < \lambda$. Hence $x \in L(\delta) \subseteq L(\lambda)$. $\square$

2.6 Lemma. *Suppose $\omega \leq \delta < \alpha^*$. Then there exists an α-stable ordinal $\delta_0 \geq \delta$ of the same α-cardinality as δ.*

Proof. The natural enumeration of $L(\alpha)$ outlined in subsection 1.7 yields a partial α-recursive function $h(e, \langle x_1, \ldots, x_n \rangle)$ $(n \geq 0)$ with properties similar to those of the enumeration function ϕ of Proposition 1.9. Suppose $P(x_1, \ldots, x_n, y)$ is the Σ_1 formula of ZF whose Gödel number is e. If

$$L(\alpha) \models (\mathrm{E}y)\, P(\underline{a}_1, \ldots, \underline{a}_n, y)$$

for some $a_1, \ldots, a_n \in L(\alpha)$, then $h(e, \langle a_1, \ldots, a_n \rangle)$ is defined and

$$L(\alpha) \models P(\underline{a}_1, \ldots, \underline{a}_n, \underline{h(e, \langle a_1, \ldots, a_n \rangle)});$$

otherwise h is undefined. $h(e, \langle a_1, \ldots, a_n \rangle)$ is the "first" y that is seen to satisfy $P(a_1, \ldots, a_n, y)$ as the natural enumeration of $L(\alpha)$ unfolds.

h is *lightface* Σ_1; its Σ_1-on-$L(\alpha)$ definition does not require any parameters from $L(\alpha)$. For any $x \subseteq L(\alpha)$, let

$$h[x] \text{ be } \{h(e, \langle a_1, \ldots, a_n \rangle) \mid e < \omega \;\;\&\;\; a_i \in x\}.$$

$h[x]$ is called the Σ_1 hull of x. Note that $h^2[x] = h[x] \subseteq x$. By design h is a universal, partial Σ_1 Skolem function for $L(\alpha)$.

Let $\omega \leq \delta < \alpha^*$. Then

$$h[\delta] \prec_1 L(\alpha), \quad \text{and so} \quad h[\delta] \models [V = L].$$

Suppose for the moment that $h[\delta]$ is transitive. By Lemma 2.5, $h[\delta] = L(\delta_0)$ for some limit δ_0. Clearly $\delta \leq \delta_0$ and δ_0 is α-stable. The domain of $h \upharpoonright \delta$ is α-finite by Proposition 2.1, so $h[\delta]$ is α-finite and δ_0 has the same α-cardinality as δ.

To check that $h[\delta]$ is transitive, define an α-recursive map

$$t\colon h[\delta] \to L(\alpha)$$

by transfinite recursion:

$$t(x) = \{t(y) \mid y \in x \quad \& \quad y \in h[\delta]\}.$$

An induction on rank shows: for all $x, y \in h[\delta]$,

$$y \in x \leftrightarrow t(y) \in t(x).$$

The induction succeeds because $h[\delta]$ is *extensional*:

$$u, v \in h[\delta] \;\;\&\;\; u \neq v \to (\mathrm{E}w)[w \in h[\delta] \;\;\&\;\; w \in (u - v) \cup (v - u)].$$

Thus t maps $\langle h[\delta], \in \rangle$ isomorphically to $\langle M, \in \rangle$, where M is $t[h[\delta]]$. The definition of t implies M is transitive. t is often called (Mostowski's) collapsing map, since it isomorphically collapses any set satisfying extensionality to a transitive set.

$t \upharpoonright \delta$ is the identity, because $\delta \subseteq h[\delta]$. Fix $b \in h[\delta]$. Then $b = h(\gamma)$ for some $\gamma < \delta$. The sentence "$\underline{b} = \underline{h(\gamma)}$" is Σ_1 and true in $L(\alpha)$, hence true in $h[\delta]$, and consequently true in M when b and γ are replaced by their t-images. $t(\gamma) = \gamma$ so "$\underline{t(b)} = \underline{h(\gamma)}$" is true in M. The transitivity of M implies $t(b) = h(\gamma)$ is outright true, and so $t(b) = b$. Thus $t \upharpoonright h[\delta]$ is the identity. □

Corollary 2.7. *Every α-cardinal greater than ω is α-stable.*

Proof. Every α-cardinal is less than or equal to α^*, hence by Lemma 2.6 the limit of stable ordinals. □

The method of proof of Lemma 2.6 was originated by Gödel. It consists of forming a Skolem hull in L and identifying its transitive collapse with an initial segment of L. The method possesses great power, but must be applied with caution. The proof of 2.6 shows that the Σ_1 hull of a transitive set is transitive. This principle can fail for Σ_2 hulls of transitive sets, and for Σ_1 hulls of intransitive sets. It is worth noting that the collapse of a Σ_1 substructure of $L(\alpha)$ need not be a Σ_1 substructure of $L(\alpha)$.

2.8–2.12 Exercises

2.8. Assume σ is α-stable and show σ is Σ_1 admissible.

2.9. Give an example of a Σ_1 admissible α such that $\omega < \alpha^* < \alpha$ and α^* is not the greatest α-stable ordinal.

2.10. (R. Jensen). Prove Proposition 2.2 without assuming α is Σ_1 admissible. Assume only that α is a limit.

2.11. (S. Friedman). Show Lemma 2.3 is false for some limit α that is not Σ_1 admissible.

2.12. A function $f: \alpha \to \alpha$ is said to be $\mu - \Pi_1^\alpha$ if

$$f(x) = \mu y P(x, y)$$

for some $P(x, y) \in \Pi_1^\alpha$. Show: $g \in \Sigma_1^\alpha \to g \in \mu - \Pi_1^\alpha \to g \in \Sigma_2^\alpha$. Suppose β is a regular α-cardinal, $\gamma < \beta$, and $f \in \mu - \Pi_1^\alpha$. Show $f[\gamma]$ is bounded below β if $f[\gamma] \subseteq \beta$. In short: a regular α-cardinal is $\mu - \Pi_1^\alpha$ regular.

3. Relative α-Recursiveness

In this section the notion of α-degree is defined, and the first steps are taken towards the proof of the regular sets theorem of Section 4.

3.1 Indices for α-Finite Sets. The proof of Proposition 1.8 is easily modified to yield a one-one, α-recursive function f_0 that maps α onto the collection of all α-finite sets. $f_0(\delta)$ is simply the δ-th α-finite set to occur in the natural enumeration of $L(\alpha)$.

Define

$$K_\delta \text{ to be } f_0(\delta).$$

If K is α-finite, then the unique δ such that $K_\delta = K$ is said to be the index of K as an α-finite set. The 2-place predicate, $\gamma \in K_\delta$, is α-recursive. A collection of α-finite sets,

(1) $$\{K_\delta \mid \delta \in J\},$$

is said to be α-finite if J is. In that event the union of (1) is α-finite.

3.2 α-Recursive In. R is called a *reduction procedure* if R is α-recursively enumerable and every member of R is of the form $\langle H, J, \gamma, \delta\rangle$, where H and J are disjoint α-finite sets. $f\colon \alpha \to \alpha$ is *weakly α-recursive in* $B \subseteq \alpha$ if there exists a reduction procedure R such that

(1) $$f(\gamma) = \delta \leftrightarrow (\text{EH})(\text{EJ})[\langle H, J, \delta, \gamma\rangle \in R \ \&\ H \subseteq B \ \&\ J \subseteq cB]$$

for all γ, $\delta < \alpha$. (cB is $\alpha - B$). (1) is rendered in symbols as $f \leq_{w\alpha} B$. Let ϕ be the enumerating partial function of Theorem 1.9. For each $\varepsilon < \alpha$, define

$$R_\varepsilon = \{\delta \mid \phi(\varepsilon, \delta) \text{ is defined}\}.$$

R_ε is called the ε-th α-recursively enumerable set. If R is α-recursively enumerable, then $R = R_\varepsilon$ for some ε.

For any $B \subseteq \alpha$ the expression $\{\varepsilon\}^B(\delta)$ is defined and equal to γ iff

(2) $$(\text{EH})(\text{EJ})[\langle H, J, \delta, \gamma\rangle \in R_\varepsilon \ \&\ H \subseteq B \ \&\ J \subseteq cB].$$

Note that $\{\varepsilon\}^B(\delta)$ may have more than one value. If f is a function, then

$$f \leq_{w\alpha} B \leftrightarrow f = \{\varepsilon\}^B \text{ for some } \varepsilon.$$

The many-valued character of $\{\varepsilon\}$ is a fact of life that can be avoided for some, but not all, α. This oddity was discussed in subsection 4.V.

A is said to be *weakly α-recursive in* B (symbolically $A \leq_{w\alpha} B$) if $c_A \leq_{w\alpha} B$, where c_A is the characteristic function of A. (It is customary to write $A(n)$ in place of $c_A(n)$, and $A = \{\varepsilon\}^B$ in place of $c_A = \{e\}^B$.) It is a result of Driscoll (1968) (Corollary 2.3.VI) that $\leq_{w\alpha}$ fails to be transitive on the α-recursively enumerable sets when $\alpha = \omega_1^{\text{CK}}$. Shore (1975) characterizes those α's for which $\leq_{w\alpha}$ does not have the above failing.

The following notion of reducibility, suggested by Kreisel, is transitive by virtue of its symmetry: an α-finite neighborhood condition on A is established by one on B. A is *α-recursive in* B (symbolically $A \leq_\alpha B$) if there exist reduction procedures R_0 and R_1 such that

(3) $$K \subseteq A \leftrightarrow (\text{EH})(\text{EJ})[\langle H, J, K\rangle \in R_0 \ \&\ H \subseteq B \ \&\ J \subseteq cB]$$

(4) $$K \subseteq cA \leftrightarrow (\mathrm{EH})(\mathrm{EJ})[\langle H,J,K\rangle \in R_1 \ \& \ H \subseteq B \ \& \ J \subseteq cB]$$

for all α-finite K. Note that $A \leq_\alpha B$ implies $A \leq_{w\alpha} B$. Define A and B to be of the same α-degree (symbolically $A \equiv_\alpha B$) if $A \leq_\alpha B$ and $B \leq_\alpha A$.

3.3 Proposition. *Assume A is α-recursively enumerable. Then for all B: $A \leq_\alpha B$ iff there exists a reduction procedure R such that for all α-finite K,*

$$K \subseteq cA \leftrightarrow (\mathrm{EH})(\mathrm{EJ})[\langle H,J,K\rangle \in R \ \& \ H \subseteq B \ \& \ J \subseteq cB].$$

Proof. Since A is α-recursively enumerable, so is

$$\{K \mid K \subseteq A \quad \& \quad K \text{ is } \alpha\text{-finite}\}.$$

Thus clause (3) of the definition of $\leq_\alpha$ can be expressed without reference to B. □

3.4 Proposition. *Let $A^* = \{\delta \mid K_\delta \cap A \neq \varnothing\}$. If A is α-recursively enumerable, then A^* is α-recursively enumerable, $A^* \equiv_\alpha A$, and*

$$(B)[A^* \leq_{w\alpha} B \leftrightarrow A \leq_\alpha B].$$

Proof. The α-recursive enumerability of A^* follows from that of A and the α-recursiveness of the predicate $\gamma \in K_\delta$. Proposition 3.3 implies

(1) $$A^* \leq_{w\alpha} B \leftrightarrow A \leq_\alpha B$$

for all B. (1), with A^* in place of B, yields $A \leq_\alpha A^*$. (1), with A in place of B, yields $A^* \leq_{w\alpha} A$. Let t be an α-recursive function such that

$$K_{t(\delta)} = \cup\{K_\gamma \mid \gamma \in K_\delta\}$$

for all δ. Then

$$K_\delta \subseteq c(A^*) \leftrightarrow K_{t(\delta)} \subseteq cA \leftrightarrow t(\delta) \in c(A^*).$$

Hence $A^* \leq_{w\alpha} A$ implies $A^* \leq_\alpha A$ with the aid of Proposition 3.3. □

3.5 Regularity. Let $A \subseteq \alpha$. A is said to be *regular* if $A \cap \delta$ is α-finite for every $\delta < \alpha$. The notion of regular set was inspired by Gödel's notion of constructible class, that is, a class whose intersection with each constructible set is constructible. In Jensen's terminology A is amenable. In the next section it will be shown that each α-recursively enumerable set has the same α-degree as some regular, α-recursively enumerable set.

It is immediate from Proposition 2.2 that there exists a non-regular, α-recursively enumerable set iff $\sigma 1 p(\alpha) < \alpha$.

For a non-α-finite, α-recursively enumerable A, regularity is equivalent to a dynamic property important in priority arguments. Assume $A = f[\alpha]$ for some one-one, α-recursive f. Then A is regular iff

$$(1) \qquad (\delta)(E\sigma)(\tau)_{\tau \geq \sigma}[f(\tau) \geq \delta].$$

(1) says that for each α-finite K, there is a stage in the enumeration (without repetitions) of A after which no member of K is added to A.

For all $A \subseteq \alpha$, regularity can be expressed in static, set theoretic terms. Introduce a predicate $x \in \underset{\sim}{A}$. (Of course $\underset{\sim}{\delta} \in \underset{\sim}{A}$ is true iff $\delta \in A$.) The formulas of ZF^A are defined as were the formulas of ZF in subsection 1.1 save that $x \in \underset{\sim}{A}$ is added to the list of atomic formulas. Fod^A is defined as in subsection 1.4 with ZF^A in place of ZF. The definition of $L[A]$ parallels that of L.

$$L[A,0] = \phi.$$
$$L[A,\delta+1] = \mathrm{Fod}^A(L[A,\delta]).$$
$$L[A,\lambda] = \cup\{L[A,\delta] \mid \delta < \lambda\} \quad (\lambda \text{ a limit}).$$
$$L[A] = \cup\{L[A,\delta] \mid \delta \text{ an ordinal}\}.$$

$L[A]$ is said to be *constructible from $\underset{\sim}{A}$ as an additional predicate*, and is not to be confused with $L(A)$, said to be constructible from A as a set, and to be defined later. The structure $\langle L[A,\alpha], \in, A\rangle$ has $L[A,\alpha]$ as universe, and $x \in y$ and $x \in A \cap \alpha$ as atomic predicates.

3.6 Proposition. *Assume $A \subseteq \alpha$. A is regular iff $L[A,\alpha] = L(\alpha)$.*

Proof. By induction on δ the ordinals in $L[A,\delta]$ are just those less than δ. A further induction shows

$$L[A,\delta] = L[A \cap \delta, \delta].$$

Hence A is regular iff $L[A,\delta]$ is α-finite for all $\delta < \alpha$. $\square$

The formulas of ZFC^A fall into quantifier complexity classes ($\Delta_0^A, \Sigma_1^A, \Pi_1^A$, etc.) in the same manner as the formulas of ZF in subsection 1.1. If F is a Σ_n^A formula with parameters in $L[A,\alpha]$, then F is said to be $\Sigma_n^{\alpha,A}$. Suppose $B \subseteq L[A,\alpha]$. If B is first order definable over $\langle L[A,\alpha], \in, A\rangle$ by means of a formula in $\Sigma_n^{\alpha,A}$, then B is said to be $\Sigma_n^{\alpha,A}$. Similarly for Π_n and Δ_n. If $A = \phi$, then $\Sigma_n^{\alpha,A}$ is written Σ_n^α. Thus B is α-recursively enumerable iff $B \subseteq \alpha$ and $B \in \Sigma_1^\alpha$.

3.7–3.8 Exercises

3.7. Suppose $B \leq_{w\alpha} A$. Show $B \in \Delta_1^{\alpha,A}$.

3.8. Suppose A is regular, $B \subseteq \alpha$, and $B \in \Delta_1^{\alpha,A}$. Show $B \leq_{w\alpha} A$.

4. Existence of Regular Sets

Let f be a one-one, α-recursive map of α into α. The deficiency set of f, in symbols D_f, is defined by:

$$\gamma \in D_f \leftrightarrow (\mathrm{E}\delta)[\gamma < \delta \ \& \ f(\delta) < f(\gamma)].$$

Deficiency sets are the source of regularity in α-recursion theory. They were invented by Dekker (1954) in order to prove theorems about simple sets in the setting of classical recursion theory.

4.1 Lemma

(1) *D_f is a regular, α-recursively enumerable set.*
(2) $D_f \leq_\alpha f[\alpha]$.
(3) *If $f[\alpha]$ is regular, then $f[\alpha] \leq_\alpha D_f$.*

Proof. Let $g(\tau)$ be the unique z such that

$$f(z) = \mu x[x \in f[\alpha - \tau]].$$

Then $g \leq_{w\alpha} f[\alpha]$ and range $g = cD_f$.

D_f is regular, because

$$\gamma \in D_f \cap \tau \leftrightarrow \gamma < \tau \ \& \ (\mathrm{E}y)[\gamma < y \leq g(\tau) \ \& \ f(y) < f(\gamma)].$$

Proposition 3.3 implies $D_f \leq_\alpha f[\alpha]$, since

$$K \subseteq cD_f \leftrightarrow (\gamma)_{\gamma \in K}(y)[\gamma < y \leq g(\sup K) \rightarrow f(y) > f(\gamma)].$$

cD_f is unbounded; otherwise there would be an infinite descending sequence of ordinals. $f \upharpoonright (cD_f)$ is strictly increasing. If $f[\alpha]$ is regular, then $f|(cD_f)$ is unbounded, and consequently

$$K \subseteq cf[\alpha] \leftrightarrow (\mathrm{E}v)[v \notin D_f \ \& \ \sup K < f(v) \quad \& \quad K \subseteq cf[v]].$$

4.2 Theorem (Sacks 1966). *Let A be α-recursively enumerable. Then there exists a regular, α-recursively enumerable B of the same α-degree as A.*

Proof. Following Maass (1978a). Let $A^* = \{\delta | K_\delta \cap A \neq \phi\}$. It is safe to assume $0 \in A$. Then A^* is unbounded, and there exists a one-one, α-recursive map f of α onto A^*. Let $p: \alpha \rightarrow \alpha^*$ be α-recursive, one-one and into. α^* is the Σ_1 projectum of α as in section 2. Define

$$\langle w,x \rangle \in B \text{ by}$$

$$(\mathrm{E}y)[x < y \ \& \ pf(y) < pf(x) \ \& \ f(y) < w].$$

Note that $\{x \mid (\mathrm{Ew})(\langle w,x\rangle \in B\}$ is $D_{p\circ f}$, the deficiency set of $p\circ f$. A typical member of B consists of a deficiency point x with an upper bound w on a witness $f(y)$ to the deficiency of x. The regularity of B will be a consequence of its connection with $D_{p\circ f}$. The use of $D_{p\circ f}$, rather than D_f, makes it possible to show $A \leq_\alpha B$ without the assumption, as in Lemma 4.1, that A is regular. The bound w is needed to show $B \leq_\alpha A$.

To see that B is regular, fix τ and define

$$\langle w,x\rangle \in B_y^\tau \text{ by}$$

$$\langle w,x\rangle \in \tau^2 \;\; \& \;\; x < y \;\; \& \;\; pf(y) < pf(x) \;\; \& \;\; f(y) < w.$$

Observe that

$$\tau \leq y_0 < y_1 \;\; \& \;\; B_{y_0}^\tau \not\supseteq B_{y_1}^\tau \to pf(y_1) < pf(y_0) \vee f(y_1) < f(y_0). \tag{1}$$

Suppose $B \cap \tau^2$ is not α-finite. Then there exists an infinite sequence $y_0 < y_1 < y_2 < \ldots$ such that for all $i < j$,

$$\tau \leq y_i < y_j \quad \& \quad B_{y_i}^\tau \not\supseteq B_{y_j}^\tau.$$

But then (1) yields an infinite descending sequence of ordinals.

To show $A \leq_\alpha B$ it suffices by Proposition 3.4 to show $A^* \leq_{w\alpha} B$. Fix z. The set

$$K = \{v \mid v \leq p(z) \quad \& \quad v \in p[A^*]\}$$

is α-finite by Proposition 2.1, and so $p^{-1}[K]$ is α-finite. It follows there is a y_0 such that

$$(x)[x > y_0 \to pf(x) > p(z)].$$

Let $x_0 > y_0$ be such that $pf(x_0) = \min\{pf(x) \mid x > y_0\}$. Then

$$pf(x_0) > p(z) \;\; \& \;\; \langle z+1, x_0\rangle \notin B.$$

Observe that for all x,

$$pf(x) > p(z) \;\; \& \;\; \langle z+1, x\rangle \notin B \to z \notin A^* - f[x].$$

Thus

$$z \notin A^* \leftrightarrow (\mathrm{Ex})\,[pf(x) > p(z) \;\; \& \;\; \langle z+1,x\rangle \notin B \quad \& \quad z \notin f[x]].$$

To prove $B \leq_\alpha A$ let J be an α-finite subset of α^2 and define

$$J_p = \bigcup_{\langle w,x\rangle \in J} (\{v \mid v < w \;\; \& \;\; pv < pf(x)\} - f[x]).$$

J_p is α-finite thanks to the bound on v. Then

$$J \subseteq cB \leftrightarrow J_p \subseteq cA^*,$$

and so $B \leq_\alpha A$ by Propositions 3.3 and 3.4. □

The above proof of the regular sets theorem provides a uniformity absent from earlier versions: there exists an α-recursive function t such that for all ε, $R_{t(\varepsilon)}$ is regular and of the same α-degree as R_ε. Maass has obtained a further uniformity by finding a definition of t independent of α. The above proof has a troublesome parameter in the Σ_1^α definition of p: $\alpha \to \alpha^*$.

5. *Hyperregularity*

The notion of hyperregularity is useful in the study of relative α-recursiveness. The main result of this section is the existence of a non-α-recursive, hyperregular, α-recursively enumerable set via a mild priority argument. Here "mild" means that the assignment of priorities is based on properties of α^*, mild in contrast to Section 2.VIII, where a tame Σ_2 projection of α is invoked to solve Post's problem.

Suppose $A \subseteq \alpha$. A is said to be *hyperregular* if

(1) $$(\delta)_{\delta<\alpha}(f)[f \leq_{w\alpha} A \to (E\gamma)_{\gamma<\alpha}(f[\delta] \subseteq \gamma].$$

If $B \leq_\alpha A$ and A is hyperregular, then B is hyperregular. The hyperregularity of A can be viewed as a weak form of Σ_1 admissibility relative to A. A strong form would be:

(2) $$(\delta)_{\delta<\alpha}(f)[f \leq_{w\alpha} A \to f \upharpoonright \delta \text{ is } \alpha\text{-finite}].$$

In general (1) and (2) are distinct, but they do coincide, as will be seen shortly, when A is α-recursively enumerable.

5.1 Proposition. *If A is α-recursively enumerable and hyperregular, then A is regular.*

Proof. Fix δ to see $A \cap \delta$ is α-finite. Assume A is non-empty, hence the range of some α-recursive function g. Let

$$f(x) = \begin{cases} \mu\sigma(g(\sigma) = x) & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Since $f \leq_{w\alpha} A$, there is a $\gamma \supseteq f[\delta]$. But then $A \cap \delta = g[\gamma] \cap \delta$. □

Sets that are both regular and hyperregular figure prominently in the study of α-degrees. They make it possible to lift certain constructions of classical recursion theory to α. Friedberg's jump theorem (1957a) is a good example: each Turing degree above $0'$ is the Turing jump of some lesser degree. Simpson's jump theorem (1974a) for α says: if an α-degree above $0'$, the α jump of the empty set, has a regular representative, then it is the α-jump of some lesser α-degree with a regular, hyperregular representative. Simpson's result is proved in Chapter VIII.

Lemma 5.2 captures the property of regular, hyperregular sets that makes them so welcome in recursion-theoretic constructions.

5.2 Lemma. *A is regular and hyperregular iff*

$$(\delta)_{\delta<\alpha}(f)[f \leq_{w\alpha} A \rightarrow f \restriction \delta \text{ is } \alpha\text{-finite}].$$

Proof. Suppose A has the latter property. Then A is hyperregular since each α-finite function is bounded below α. Let c be the characteristic function of A. Then $c \restriction \delta$ is α-finite, hence $A \cap \delta$ is α-finite.

Now suppose A has the former property. Fix $\delta < \alpha$ and $f \leq_{w\alpha} A$. Let R be α-recursively enumerable and such that

$$(1) \quad f(x) = y \leftrightarrow (\mathrm{E}H)(\mathrm{E}K)[\langle H, K, x, y\rangle \in R \quad \& \quad H \subseteq A \quad \& \quad K \subseteq cA].$$

R is the range of some α-recursive t. For simplicity write

$$t(\gamma) = \langle H^{\gamma}, K^{\gamma}, x^{\gamma}, y^{\gamma}\rangle.$$

Define: $g(x) = \mu\gamma[x^{\gamma} = x \;\&\; H^{\gamma} \subseteq A \;\&\; K^{\gamma} \subseteq cA]$;

$$h(x) = \sup\{H^{\rho} \cup K^{\rho} \mid \rho \leq g(x)\}.$$

$h(x) < \alpha$ since t is α-recursive. The regularity of A implies $h(x) \cap A$ and $h(x) - A$ are α-finite. Hence $g \leq_{w\alpha} A$, since the value of $g(x)$ is determined by α-finitely much of A.

Since A is hyperregular there is a β such that

$$\cup\{H^{g(x)} \cup K^{g(x)} \mid x < \delta\} \subseteq \beta.$$

$A \cap \beta$ is α-finite. Consequently an α-finite definition of $f \restriction \delta$ is obtained from the right side of (1) by replacing R by $t\left[\sup_{x<\delta} g(x)+1\right]$, A by $\beta \cap A$, and cA by $\beta - A$. $\square$

It is time for a mild priority argument in the setting of α-recursion theory. The construction of a non-α-recursive, hyperregular set is an excellent choice, since the associated injury sets are simpler than those developed in the solution to Post's problem in the next Chapter. It first appears that the proof of Theorem 5.5 follows the same lines as that of Theorem 2.1.VI with ω_1^{CK} replaced by α, and ω by α^*. But some unexpected difficulties arise when $\alpha^* > \omega$. They are overcome by Lemma 2.1 of the present chapter.

5.3 Theorem (Sacks 1966). *There exists a non-α-recursive, hyperregular, α-recursively enumerable set.*

Proof. Let p be a one-one, α-recursive map of α into α^*. Recall R_ε and $\{\varepsilon\}^B$ from subsection 3.2. Let

$$x \in R_{p^{-1}\varepsilon} \text{ mean } p^{-1}\varepsilon \text{ is defined and equal to } \gamma \quad \& \quad x \in R_\gamma.$$

Then "$x \in R_{p^{-1}\varepsilon}$" is α-recursively enumerable, and so $\{R_{p^{-1}\varepsilon} | \varepsilon < \alpha^*\}$ is a simultaneous α-recursive enumeration of all α-recursively enumerable sets. Define $\{p^{-1}\varepsilon\}^B$ as in subsection 3.2, formula (2), with R_ε replaced by $R_{p^{-1}\varepsilon}$. Then $f \leq_{w\alpha} B$ iff $f = \{p^{-1}\varepsilon\}^B$ for some $\varepsilon < \alpha^*$.

To prove the theorem a set B will be α-recursively enumerated with three objectives in mind.

(1) cB is unbounded.
(2) If $R_{p^{-1}\varepsilon}$ is unbounded, then $R_{p^{-1}\varepsilon} \cap B \neq \varnothing$.
(3) Let t_ε be α^* if α^* is a regular α-cardinal, and ε otherwise. If $\{p^{-1}\varepsilon\}^B(\gamma)$ is defined and single-valued for all $\gamma < t_\varepsilon$, then $\{p^{-1}\varepsilon\}^B \restriction t_\varepsilon$ is α-finite.

(1) and (2) combine to make B non-α-recursive. (3) will imply B is hyperregular. B will be the union of an increasing, α-recursive sequence $\{B^\sigma | \sigma < \alpha\}$ of α-finite sets. B^σ is that part of B enumerated by the end of stage σ. Let

$$B^{<\sigma} \text{ be } \cup \{B^\delta | \delta < \sigma\}.$$

E enumerates the even ordinals: $E(0) = 0$; $E(\varepsilon + 1) = E(\varepsilon) + 2$; and $E(\lambda) = \lambda$ for limit λ. Od enumerates the odd ordinals: $\mathrm{Od}(\varepsilon) = E(\varepsilon) + 1$.

For the sake of (2) and (3) attempts are made during the enumeration of B to satisfy the following requirements.

Req $E(\varepsilon)$: If $R_{p^{-1}\varepsilon}$ is unbounded, then $R_{p^{-1}\varepsilon} \cap B^\sigma \neq \varnothing$ for some σ.

Req $\mathrm{Od}(\varepsilon)$: If $\{p^{-1}\varepsilon\}^B(\gamma)$ is defined and single-valued for all $\gamma < t_\varepsilon$, then $(E\sigma)(\tau)_{\tau \geq \sigma}(\gamma)_{\gamma < t_\varepsilon}$ such that

$$\{p^{-1}\varepsilon\}_\tau^{B^{<\tau}}(\gamma) = \{p^{-1}\varepsilon\}_\sigma^{B^{<\sigma}}(\gamma). \tag{4}$$

The right side of (4) is an α-recursive approximation of $\{p^{-1}\varepsilon\}^B(\gamma)$ at the beginning of stage σ of the enumeration of B, and is defined as follows.

Suppose R is α-recursively enumerable. Then there exists a Δ_0^α formula $F(x, y)$ such that for all $\delta < \alpha$,

$$\delta \in R \leftrightarrow L(\alpha) \models (\mathrm{E}y) F(\underline{\delta}, y).$$

Define R^σ by

$$\delta \in R^\sigma \leftrightarrow L(\sigma) \models (\mathrm{E}y) F(\underline{\delta}, y).$$

Recall that $F(x, y)$ may contain some parameter $q \in L(\alpha)$. Take R^σ to be empty if $q \notin L(\sigma)$. Each R^σ is α-finite, the function $\lambda\sigma|R^\sigma$ is α-recursive, and $R = \cup\{R^\sigma|\sigma < \alpha\}$. The *right side of* (4) is defined and equal to δ if

$$\text{(5)} \qquad (\mathrm{EH})(\mathrm{EJ})[\langle H, J, \gamma, \delta\rangle \in R^\sigma_{p^{-1}\varepsilon} \quad \& \quad H \subseteq B^{<\sigma} \quad \& \quad J \subseteq cB^{<\sigma}].$$

If $\{p^{-1}\varepsilon\}^B(\gamma)$ is defined and equal to δ, then for all sufficiently large σ,

$$\langle H, J, \gamma, \delta\rangle \in R^\sigma_{p^{-1}\varepsilon} \quad \& \quad H \subseteq B^{<\sigma} \quad \& \quad J \subseteq cB^{<\sigma}.$$

Requirement $E(\varepsilon)$ is said to be *positive* because it is met by adding an element to B. Requirement $\mathrm{Od}(\varepsilon)$ is said to be *negative* because it is met by excluding α-finitely many elements from B. If a stage σ can be found such that the right side of (4) is defined for all $\gamma < t_\varepsilon$, then requirement $\mathrm{Od}(\varepsilon)$ can be met by keeping out of B any element that would falsify any of the facts about $cB^{<\sigma}$ used in the computation of the right side of (4) for any $\gamma < t_\varepsilon$. The preservation of $\{p^{-1}\varepsilon\}^{B^{<\sigma}}_\sigma \upharpoonright t_\varepsilon$ during all stages $\tau \geq \sigma$ implies that (3) holds.

Positive and negative requirements tend to conflict. If an ordinal is added to B at stage τ for the sake of requirement $E(\varepsilon_0)$, then that addition may falsify some computation developed at stage $\sigma < \tau$ and being preserved for the sake of requirement $\mathrm{Od}(\varepsilon)$. Such an event is termed an *injury* to req $\mathrm{Od}(\varepsilon)$ for the sake of req $E(\varepsilon_0)$. The adverse effect of injuries is limited by a system of Friedberg-Muchnik priorities. Req $E(\varepsilon_0)$ is said to be of *higher priority than* req $\mathrm{Od}(\varepsilon)$ if $\varepsilon_0 \leq \varepsilon$. An injury occurs only for the sake of a requirement of higher priority. It will follow from the fact that $\varepsilon < \alpha^*$ that req $\mathrm{Od}(\varepsilon)$ is injured only α-finitely often.

The construction of B consists of simultaneous α-recursive definitions of $\lambda\sigma|B^\sigma$, $\lambda\sigma|n(\sigma, \varepsilon)$, $\lambda\sigma\varepsilon|r(\sigma, \varepsilon)$ and $\lambda\sigma\varepsilon|m(\sigma, \varepsilon)$. r insures the unboundedness of B. m defines the preservation requirements on B.

Stage σ. $r(\sigma, \varepsilon)$ is the least β such that

$$\beta \geq \sup_{\delta<\sigma} r(\delta, \varepsilon) \quad \& \quad \beta \notin B^{<\sigma}$$
$$\& \quad [\varepsilon \in p[\sigma] \rightarrow \beta \geq p^{-1}\varepsilon].$$

$n(\sigma, \varepsilon)$ is the greatest $m \leq t_\varepsilon$ such that $\{p^{-1}\varepsilon\}^{B^{<\sigma}}_\sigma(\gamma)$ is defined for all $\gamma < m$. $J^\sigma_{\varepsilon,\gamma}$ is the least J (least according to the ordering of $L(\alpha)$ supplied by Proposition 1.8) such that: $\langle H, J, \gamma, \delta\rangle$ satisfies the matrix of (5) and $\sup J$ is as small as possible. $m(\sigma, \varepsilon)$ is the least β such that

$$\beta \geq \sup_{\delta<\sigma} m(\delta, \varepsilon) \quad \& \quad \beta \geq \sup_{\gamma<n(\sigma,\varepsilon)} J^\sigma_{\varepsilon,\gamma}.$$

When possible, ordinals less than or equal to $m(\sigma, \varepsilon)$ will be kept out of $B^\tau (\tau \geq \sigma)$ in order to preserve

$$\{\{p^{-1}\varepsilon\}^{B^{<\sigma}}_\sigma(\gamma) \mid \gamma < n(\sigma, \varepsilon)\}.$$

Stage σ is completed by attacking req $E(\varepsilon_0)$, where $\varepsilon_0 = h(r)$ and h is an α-recursive function that enumerates every ordinal less than α^* unboundedly often. If $R^\sigma_{p^{-1}\varepsilon_0} \cap B^{<\sigma} = \varnothing$ and there exists a β such that

(6) $$\beta \in R^\sigma_{p^{-1}\varepsilon_0} \quad \& \quad (\varepsilon)_{\varepsilon<\varepsilon_0}[m(\sigma, \varepsilon) < \beta \quad \& \quad r(\sigma, \varepsilon) < \beta],$$

then put the least such β in B. Otherwise $B^\sigma = B^{<\sigma}$. End of stage σ.

The clause, $R^\sigma_{p^{-1}\varepsilon_0} \cap B = \varnothing$, is important. It implies:

(7) For each ε_0 there is at most one stage σ at which an ordinal is added to B^σ for the sake of req $E(\varepsilon_0)$.

If β is added to B at stage σ and $\beta \leq m(\sigma, \varepsilon)$, then $\mathrm{Od}(\varepsilon)$ is said to be injured. Define

$$I_\varepsilon = \{\sigma \mid \text{req}\ \mathrm{Od}(\varepsilon) \text{ is injured at stage } \sigma\}.$$

I_ε is α-recursively enumerable. The priorities were designed to insure

(8) $$I_\varepsilon \text{ is } \alpha\text{-finite.}$$

To prove (8) fix $\varepsilon < \alpha^*$. (6) implies $h(\sigma) \leq \varepsilon$ for all $\sigma \in I_\varepsilon$. $\varepsilon < \alpha^*$, so $h[I_\varepsilon]$ is α-finite by Proposition 2.1. $h \upharpoonright I_\varepsilon$ is one-one by (7). Hence

$$I_\varepsilon = h^{-1}[h[I_\varepsilon]]$$

is α-finite (and has α-cardinality less than α^*).

$\lim_\sigma r(\sigma, \varepsilon)$ is said to exist if there are τ and ℓ such that $r(\sigma, \varepsilon) = \ell$ for all $\sigma \geq \tau$.

$\lambda\sigma \mid r(\sigma, \varepsilon)$ is non-decreasing, so $\lim_\sigma r(\sigma, \varepsilon)$ exists if $\lambda\sigma \mid r(\sigma, \varepsilon)$ increases only α-finitely often. Suppose $r(\sigma-1, \varepsilon) \neq r(\sigma, \varepsilon)$. This could be because $\varepsilon = p(\sigma) > r(\sigma-1, \varepsilon)$. Otherwise $r(\sigma-1, \varepsilon)$ was put in $B^{\sigma-1}$ for the sake of req $E(\varepsilon_0)$ for some $\varepsilon_0 \leq \varepsilon$. The set of all such σ's is α-finite by the same reasoning used to prove (8). Thus

$$r(\varepsilon) = \lim_\sigma r(\sigma, \varepsilon)$$

exists for all $\varepsilon < \alpha^*$.

To prove (1) observe that $r(\varepsilon) \notin B$, and that $r(p\gamma) \geq \gamma$ for all γ.

To prove (3) assume $\{p^{-1}\varepsilon\}^B(\gamma)$ is defined and single-valued for all $\gamma < t_\varepsilon$. (8) implies there is a stage σ_ε after which req $\mathrm{Od}(\varepsilon)$ is not injured. It follows that

(9) $$\sigma > \sigma_\varepsilon \quad \& \quad \gamma < n(\sigma, \varepsilon) \rightarrow \{p^{-1}\varepsilon\}^{B<\sigma}_\sigma(\gamma) = \{p^{-1}\varepsilon\}^B(\gamma).$$

The function $\lambda\sigma \mid n(\sigma, \varepsilon)$, restricted to $\sigma > \sigma_\varepsilon$, is non-decreasing. The set S_ε of stages after stage σ_ε at which $n(\sigma, \varepsilon)$ increases is α-recursive. Since $n(\sigma, \varepsilon)$ is bounded above by $t_\varepsilon < \alpha$, it follows that the ordertype of S_ε is less than α. Hence $n(\sigma, \varepsilon)$ increases only α-finitely often and

$$n(\varepsilon)=\lim_{\sigma} n(\sigma, \varepsilon)$$

exists. Suppose $n(\varepsilon)<t_{\varepsilon}$. Then $\{p^{-1}\varepsilon\}^{B}(n(\varepsilon))$ is defined and single-valued. By the remark following (5),

$$\{p^{-1}\varepsilon\}^{B}(n(\varepsilon))=\{p^{-1}\varepsilon\}_{\sigma}^{B<\sigma}(n(\varepsilon))$$

for all sufficiently large σ. But then $n(\varepsilon)<n(\sigma, \varepsilon)$ for all sufficiently large σ. So $n(\varepsilon)$ must equal t_{ε}, and (3) follows from (9).

The remainder of the proof of Theorem 5.5 splits into two lemmas.

5.4 Lemma. *B is hyperregular.*

Proof. Assume $\{p^{-1}\varepsilon\}^{B}\restriction \delta$ is defined and single-valued for some $\delta<\alpha$. The proof that $\{p^{-1}\varepsilon\}^{B}\restriction \delta$ is α-finite breaks into three cases.

Case 1: $\alpha^*=\alpha$. Then p can be the identity map. Choose $\eta>\delta$ so that $\{\varepsilon\}^{X}\simeq\{n\}^{X}$ for all $X\subseteq\alpha$. η exists because of the endless repetitions that inevitably occur in any standard enumeration of the α-recursively enumerable sets. η codes up the same instructions as ε. Thus $\{\varepsilon\}^{B}\restriction\delta=\{\eta\}^{B}\restriction\delta$. $t_{\eta}=\eta>\delta$, so by (3), proved above, $\{\eta\}^{B}\restriction\delta$ is α-finite.

Case 2: $\alpha^*<\alpha$ and α^* is a regular α-cardinal. Then α^* is the greatest α-cardinal and there is an α-finite map v of α^* onto δ. Choose $\varepsilon_0<\alpha^*$ so that

$$\{p^{-1}\varepsilon_0\}^{B}(x)\simeq\{p^{-1}\varepsilon\}^{B}(vx)$$

for all $x<\alpha^*$. $t_{\varepsilon_0}=\alpha^*$, so by (3), $\{p^{-1}\varepsilon_0\}^{B}\restriction\alpha^*(\equiv\{p^{-1}\varepsilon\}^{B}\restriction\delta)$ is α-finite.

Case 3: $\alpha^*<\alpha$ and α^* is a singular α-cardinal. As in case 2, it suffices to show $\{p^{-1}\varepsilon_0\}^{B}\restriction\alpha^*$ is α-finite. Let λ_0 be the cofinality of α^* in $L(\alpha)$. Then for some $\lambda_0<\alpha^*$, there is a sequence $\{\gamma_i\mid i<\lambda_0\}$ of α-cardinals whose limit is α^*. For each $i<\lambda_0$, there is an $n_i>\gamma_i$ such that $\{p^{-1}n_i\}^{X}\simeq\{p^{-1}\gamma_i\}^{X}$ for all $X\subseteq\alpha$ (as in case 1). By (3) $\{p^{-1}n_i\}^{B}\restriction\gamma_i$ is α-finite.

Choose ε_1 so that

$$\{p^{-1}\varepsilon_1\}^{B}(i)=\{p^{-1}\varepsilon_0\}^{B}\restriction\gamma_i$$

for all $i<\lambda_0$. Then $\{p^{-1}\varepsilon_1\}^{B}\restriction\lambda_0$ is α-finite since $\lambda_0<\alpha^*$. □

So far no reason has been given for setting t_{ε} equal to ε when α^* is a singular α-cardinal. Thus case 3 of the proof of Lemma 5.6 may appear to be an unnecessary complication that could have been eliminated by setting t_{ε} equal to α^* whenever $\alpha^*<\alpha$. The reason for distinguishing between regular and singular α^*'s is made evident in the next lemma.

5.5 Lemma. *If R_{ε_0} is unbounded, then $R_{\varepsilon_0}\cap B\neq\varnothing$.*

Proof. It suffices to show

$$\{m(\sigma, \varepsilon) \mid \varepsilon < \varepsilon_0 \quad \& \quad \sigma < \alpha\} \quad \text{and} \quad \{r(\sigma, \varepsilon) \mid \varepsilon < \varepsilon_0 \quad \& \quad \sigma < \alpha\}$$

are bounded sets. Recall from Theorem 5.5 the proof that $\lim_\sigma r(\sigma, \varepsilon)$ exists. $\lambda\varepsilon \mid r(\varepsilon)$ is Σ_2^α and unbounded; nonetheless the restriction of $\lambda\varepsilon \mid r(\varepsilon)$ to ε_0 is bounded. It follows from Proposition 2.1 and (8) of Theorem 5.5 that there is a σ_0 such that:

(10) $$\{\varepsilon \mid \varepsilon < \varepsilon_0 \quad \& \quad \varepsilon \in p[\alpha]\} \subseteq p[\sigma_0];$$

and no element of R_ε is added to B at or after stage σ_0 for the sake of req $E(\varepsilon)$ for any $\varepsilon < \varepsilon_0$.

The bounds on m are less apparent than those on r. Two cases have to be considered. The second contains the principal twist in the proof of Theorem 5.5.
Case 1: $\alpha^* = \alpha$ or α is a singular α-cardinal. Fix $\varepsilon < \varepsilon_0$. Let σ_0 be as in (10) above. A change in the value of $m(\sigma, \varepsilon)$ occurs at stage $\sigma > \sigma_0$ only if $\{p^{-1}\varepsilon\}_\sigma^{B<\sigma}$ is defined on a longer initial segment than it was at any previous stage after σ_0. All such initial segments are bounded by $\varepsilon + 1$. Thus the set of stages after σ_0 at which $m(\sigma, \varepsilon)$ increases for some $\varepsilon < \varepsilon_0$ is correlated with an α-recursively enumerable subset of $(\varepsilon_0 + 1)^2$. That set is α-finite, since $\varepsilon_0 < \alpha^*$.
Case 2: α^* is a regular α-cardinal. If $\varepsilon < \varepsilon_0$ and $\sigma > \sigma_0$, then changes in $m(\sigma, \varepsilon)$ at stage σ occur as in case 1 save that $n(\sigma, \varepsilon)$ is now bounded by $\alpha^* + 1$ instead of $\varepsilon + 1$. Let

$$Z = \{\varepsilon \mid \varepsilon < \varepsilon_0 \quad \& \quad \lim_\sigma n(\sigma, \varepsilon) = \alpha^*\}.$$

Z is α-recursively enumerable because $\alpha^* < \alpha$. Hence Z is α-finite, since $\varepsilon_0 < \alpha^*$. Choose $\sigma_1 > \sigma_0$ so that

$$(\varepsilon)\left[\varepsilon \in Z \rightarrow n(\sigma_1, \varepsilon) = \lim_\sigma n(\sigma, \varepsilon)\right].$$

Then $m(\sigma_1, \varepsilon) = \lim_\sigma m(\sigma, \varepsilon)$ for all $\varepsilon \in Z$.

For each $\varepsilon \in \varepsilon_0 - Z$, let K_ε be the set of stages after σ_0 at which $m(\sigma, \varepsilon)$ increases. K_ε is α-finite and has α-cardinality less than α^*. It follows from Lemma 2.3 that

$$\cup \{K_\varepsilon \mid \varepsilon \in (\varepsilon_0 - Z)\}$$

is α-finite, hence bounded by some σ_2. Then $m(\sigma_2, \varepsilon) = \lim_\sigma m(\sigma, \varepsilon)$ for all $\varepsilon \in (\varepsilon_0 - Z)$. □

Exercises 5.6 and 5.7 are intended to clarify the proof of Theorem 5.3, particularly the nature of the bounds on m and r obtained in the proof of Lemma 5.5.

These matters are discussed further in Chapter VIII. For now note that some of the functions developed in the course of a priority argument are Σ_2^α, for example, $\lambda\varepsilon \mid r(\varepsilon)$ in the proof of Lemma 5.5. Bounds on these Σ_2^α functions have to be derived somehow from the Σ_1 admissibility of $L(\alpha)$. For Theorem 5.5 the job was done by α^* and by Lemma 2.3 (cf. Exercise 2.12).

5.6–5.9 Exercises

5.6. α is said to be Σ_2 *admissible* if for all $f \in \Sigma_2^\alpha$,

$$(\gamma)_{\gamma<\alpha}[f \upharpoonright \gamma \text{ is } \alpha\text{-finite}].$$

Show α is Σ_2 admissible iff every α-recursively enumerable set is hyperregular.

5.7. Let $m(\sigma, \varepsilon)$ be the α-recursive function defined in the construction of Theorem 5.5. Let $m(\varepsilon) = \lim_\sigma m(\sigma, \varepsilon)$. Show $\lambda\varepsilon \mid m(\varepsilon)$ is Σ_2^α. Show that $(\lambda\varepsilon \mid m(\varepsilon)) \upharpoonright \varepsilon_0$ is α-finite when $\varepsilon_0 < \alpha^*$.

5.8. Suppose A is regular and hyperregular. Show

$$(B)[B \leq_{w\alpha} A \leftrightarrow B \leq_\alpha A \leftrightarrow B \in \Delta_1^{\alpha, A}].$$

Show $\langle L[A, \alpha], \varepsilon, A \rangle$ is Σ_1 admissible.

5.9. Assume $V = L$. Let α be ω_ω and A be $\alpha - \{\omega_n \mid n < \omega\}$. Verify that A is α-recursively enumerable and regular, but not hyperregular. Show A is complete, that is, $B \leq_\alpha A$ for every α-recursively enumerable B. Show C is hyperregular iff $A \not\leq_\alpha C$, for every $C \subseteq \alpha$.