

COUPLING METHODS FOR MULTISTAGE SAMPLING

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Multistage sampling is commonly used for household surveys when there exists no sampling frame, or when the population is scattered over a wide area. Multistage sampling usually introduces a complex dependence in the selection of the final units, which makes asymptotic results quite difficult to prove. In this work, we consider multistage sampling with simple random without replacement sampling at the first stage, and with an arbitrary sampling design for further stages. We consider coupling methods to link this sampling design to sampling designs where the primary sampling units are selected independently. We first generalize a method introduced by [*Magyar Tud. Akad. Mat. Kutató Int. Közl.* **5** (1960) 361–374] to get a coupling with multistage sampling and Bernoulli sampling at the first stage, which leads to a central limit theorem for the Horvitz–Thompson estimator. We then introduce a new coupling method with multistage sampling and simple random with replacement sampling at the first stage. When the first-stage sampling fraction tends to zero, this method is used to prove consistency of a with-replacement bootstrap for simple random without replacement sampling at the first stage, and consistency of bootstrap variance estimators for smooth functions of totals.

1. Introduction. Multistage sampling is widely used for household and health surveys when there exists no sampling frame, or when the population is scattered over a wide area. Three or more stages of sampling may be commonly used. For example, the third National Health and Nutrition Survey (NHANES III) conducted in the United States involved four stages of sampling, with the selection of counties as Primary Sampling Units (PSUs), of segments as Secondary Sampling Units (SSUs) inside the selected counties, of households as Tertiary Sampling Units (TSUs) inside the selected segments, and of individuals inside the selected households, for example, [12]. A detailed treatment of multistage sampling may be found in textbooks like [9, 36] or [13].

Multistage sampling introduces a complex dependence in the selection of the final units, which makes asymptotic properties difficult to prove. In this work, we make use of coupling methods (see [39]) to link multistage sampling designs to sampling designs where the primary sampling units are selected independently.

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The method basically consists in generating a random vector $(X_t, Z_t)^\top$ with appropriate marginal laws, and so that $E(X_t - Z_t)^2$ is smaller than the rate of convergence of X_t . In this case, X_t and Z_t share the same limiting variance and the same limiting distribution. For example, the distribution of Z_t may be that of the Horvitz–Thompson estimator (see [20]) under multistage sampling with simple random without replacement sampling (SI) of PSUs, and the distribution of X_t may be that of the Hansen–Hurwitz estimator (see [19]) under multistage sampling and simple random with replacement sampling (SIR) of PSUs.

In this paper, we derive asymptotic normality results for without-replacement multistage designs, and we prove the consistency of a with-replacement bootstrap of PSUs for SI sampling at the first stage when the sampling fraction tends to zero. Our framework and our assumptions are defined in Section 2. In Section 3, we first give an overview of asymptotic normality results in survey sampling. We then state a central limit theorem for the Horvitz–Thompson estimator in case of multistage sampling with Bernoulli sampling (BE) of PSUs. The theorem follows from standard assumptions and from the independent selections of PSUs. We generalize to the multistage context a coupling algorithm by [16] for the joint selection of a BE sample and an SI sample. This is the main tool to extend the central limit theorem to multistage sampling with SI sampling of PSUs. We also prove the weak consistency of variance estimators (see [38], page 20), which enables to compute normality-based confidence intervals with appropriate coverage. In Section 4, we consider the bootstrap for multistage sampling. We introduce a new coupling algorithm between SI sampling of PSUs and SIR sampling of PSUs. This is the main tool to prove a long-standing issue; namely, that the so-called with-replacement bootstrap of PSUs (see [33]) is consistent in case of SI sampling of PSUs when the first-stage sampling fraction tends to zero. This entails that Studentized bootstrap confidence intervals are valid in such case, and that the bootstrap variance estimators are consistent for smooth functions of totals. The properties of a simplified variance estimator and of the bootstrap procedure are evaluated in Section 5 through a simulation study. An application of the studied bootstrap method on the panel for urban policy survey is presented in Section 6. The proofs of theorems are given in Section 7. Additional proofs are given in the supplement [7].

2. Framework. We consider a finite population U consisting of N sampling units that may be represented by their labels, so that we may simply write $U = \{1, \dots, N\}$. The units are grouped inside N_I nonoverlapping sub-populations u_1, \dots, u_{N_I} called primary sampling units (PSUs). We are interested in estimating the population total

$$Y = \sum_{k \in U} y_k = \sum_{u_i \in U_I} Y_i$$

for some variable of interest y , where $Y_i = \sum_{k \in u_i} y_k$ is the sub-total of the variable y on the PSU u_i . We note $E(\cdot)$ and $V(\cdot)$ for the expectation and the variance of

some estimator. Also, we note $E_{\{X\}}(\cdot)$ and $V_{\{X\}}(\cdot)$ for the expectation and variance conditionally on some random variable X . Throughout the paper, we denote by \hat{Y}_i an unbiased estimator of Y_i , and by $V_i = V(\hat{Y}_i)$ its variance. Also, we denote by \hat{V}_i an unbiased estimator of V_i . In order to study the asymptotic properties of the sampling designs and estimators that we treat below, we consider the asymptotic framework of [21]. We assume that the population U belongs to a nested sequence $\{U_t\}$ of finite populations with increasing sizes N_t , and that the population vector of values $y_{U_t} = (y_{1t}, \dots, y_{N_t})^\top$ belongs to a sequence $\{y_{U_t}\}$ of N_t -vectors. For simplicity, the index t will be suppressed in what follows but all limiting processes will be taken as $t \rightarrow \infty$.

In the population $U_I = \{u_1, \dots, u_{N_I}\}$ of PSUs, a first-stage sample S_I is selected according to some sampling design $p_I(\cdot)$. For clarity of exposition, we consider nonstratified sampling designs for $p_I(\cdot)$, but the results may be easily extended to the case of stratified first-stage sampling designs with a finite number of strata, as is illustrated in Section 6. If the PSU u_i is selected in S_I , a second-stage sample S_i is selected in u_i by means of some sampling design $p_i(\cdot|S_I)$. We assume invariance of the second-stage designs: that is, the second stage of sampling is independent of S_I and we may simply write $p_i(\cdot|S_I) = p_i(\cdot)$. Also, we assume that the second-stage designs are independent from one PSU to another, conditionally on S_I . This implies that

$$(2.1) \quad \Pr\left(\bigcup_{u_i \in S_I} \{S_i = s_i\} \middle| S_I\right) = \prod_{u_i \in S_I} p_i(s_i|S_I) = \prod_{u_i \in S_I} p_i(s_i)$$

for any set of samples $s_i \subset u_i, i = 1, \dots, N_I$ (see [36], Chapter 4). The second-stage sampling designs $p_i(\cdot)$ are left arbitrary. For example, they may involve censuses inside some PSUs (which means cluster sampling), or additional stages of sampling.

We will make use of the following assumptions:

H1: $N_I \xrightarrow[t \rightarrow \infty]{} \infty$ and $n_I \xrightarrow[t \rightarrow \infty]{} \infty$. Also, $f_I = n_I/N_I \xrightarrow[t \rightarrow \infty]{} f \in [0, 1[$.

H2: There exists $\delta > 0$ and some constant C_1 such that

$$N_I^{-1} \sum_{u_i \in U_I} E|\hat{Y}_i|^{2+\delta} < C_1.$$

H3: There exists some constant C_2 such that $N_I^{-1} \sum_{u_i \in U_I} E(\hat{V}_i^2) < C_2$.

H4: There exists some constant $C_3 > 0$ such that

$$N_I^{-1} \sum_{u_i \in U_I} (Y_i - \mu_Y)^2 > C_3 \quad \text{where } \mu_Y = N_I^{-1} Y.$$

It is assumed in (H1) that a large number n_I of PSUs is selected. The assumption (H2) implies that the sequence of $\{Y_i\}_{u_i \in U_I}$ has bounded moments of order $2 + \delta$ and that the sequence of $\{V(\hat{Y}_i)\}_{u_i \in U_I}$ has a bounded first moment. This

assumption requires in particular that the numbers of SSUs within PSUs remain bounded. When we establish the mean square consistency of variance estimators, assumption (H2) is strengthened by considering $\delta = 2$, which implies that the sequence of $\{Y_i\}_{u_i \in U_I}$ has bounded moments of order 4. Assumptions (H2) and (H3) are sufficient to have a weakly consistent variance estimator for further stages of sampling. In this regard, assumption (H3) can be relaxed when $f_I \xrightarrow[t \rightarrow \infty]{} 0$ (see Section 3.3). Assumption (H4) requires that the dispersion between PSUs does not vanish. This is a sufficient condition for the first-stage sampling variance of the Horvitz–Thompson estimator to have the usual order $O(N_I^2 n_I^{-1})$, for the sampling designs that we consider in this article.

3. Asymptotic normality for multistage sampling. Unbiased estimators for population totals such as the Horvitz–Thompson estimator are well known; see [20] and [26]. Several results of asymptotic normality have been proved for specific one-stage sampling designs; see, for example, [16, 17] for simple random sampling without replacement, [18] for rejective sampling, [34, 37] and [15] for successive sampling, and [28] for the Rao–Hartley–Cochran procedure proposed by [32]. Brändén and Jonasson [5] state a central limit theorem for the class of sampling algorithms satisfying the strongly Rayleigh property, which includes Sampford sampling, Pareto sampling and ordered pivotal sampling (see [6]). Chen and Rao [8] prove asymptotic normality for a class of estimators under two-phase sampling designs; see also [35]. However, asymptotic normality of estimators resulting from multistage samples has not been much considered in the literature; two notable exceptions are [22] for stratified multistage designs and with-replacement sampling at the first-stage, and [29] who states a martingale central limit theorem for a general two-stage sampling design.

In this section, we confine our attention to Horvitz–Thompson estimators for multistage sampling with BE sampling or SI sampling of PSUs. The central limit Theorems 3.1 and 3.2 are easily extended to cover smooth functions of totals by using the delta method (see [38], Appendix A2).

3.1. Bernoulli sampling of PSUs. We first consider the case when a first-stage sample S_I^B is selected in U_I by means of Bernoulli sampling (BE) with expected size n_I , which we note as $S_I^B \sim \text{BE}(U_I; n_I)$. The PSUs are independently selected in S_I^B with inclusion probabilities $f_I = n_I/N_I$, and the size n_I^B of S_I^B is random. The Horvitz–Thompson estimator

$$(3.1) \quad \hat{Y}_B = \frac{N_I}{n_I} \sum_{u_i \in U_I} I_i^B \hat{Y}_i = \frac{N_I}{n_I} \sum_{u_i \in S_I^B} \hat{Y}_i$$

is unbiased for Y , with I_i^B the sample membership indicator for the PSU u_i in the sample S_I^B . The variance of \hat{Y}_B is

$$(3.2) \quad V(\hat{Y}_B) = \frac{N_I^2}{n_I} \left\{ (1 - f_I) \frac{1}{N_I} \sum_{u_i \in U_I} Y_i^2 + \frac{1}{N_I} \sum_{u_i \in U_I} V_i \right\},$$

where $V_i = V(\hat{Y}_i)$. We consider the variance estimator

$$(3.3) \quad v_B(\hat{Y}_B) = \frac{N_I^2}{n_I} \left(\frac{1 - f_I}{n_I^B} \sum_{u_i \in S_I^B} \hat{Y}_i^2 + \frac{f_I}{n_I^B} \sum_{u_i \in S_I^B} \hat{V}_i \right)$$

if $n_I^B > 0$ and $v_B(\hat{Y}_B) = 0$ if $n_I^B = 0$, with \hat{V}_i an unbiased estimator of V_i . Conditionally on n_I^B , S_I^B may be seen as an SI sample of size n_I^B selected in U_I . It follows that

$$(3.4) \quad \begin{aligned} E_{\{n_I^B\}} \left(\frac{1}{n_I^B} \sum_{u_i \in S_I^B} \hat{Y}_i^2 \right) &= \frac{1}{N_I} \sum_{u_i \in U_I} (Y_i^2 + V_i), \\ E_{\{n_I^B\}} \left(\frac{1}{n_I^B} \sum_{u_i \in S_I^B} \hat{V}_i \right) &= \frac{1}{N_I} \sum_{u_i \in U_I} V_i, \end{aligned}$$

and $v_B(\hat{Y}_B)$ is unbiased for $V(\hat{Y}_B)$ conditionally on $n_I^B = k > 0$.

THEOREM 3.1. *Assume that (H1) and (H2) hold. Then the Horvitz–Thompson estimator $\hat{Y}_B = N_I n_I^{-1} \sum_{u_i \in S_I^B} \hat{Y}_i$ is asymptotically normally distributed, that is,*

$$(3.5) \quad \{V(\hat{Y}_B)\}^{-0.5} (\hat{Y}_B - Y) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1),$$

where $\xrightarrow{\mathcal{L}}$ stands for the convergence in distribution. Assume further that (H2) holds with $\delta = 2$ and that (H3) holds. Then $v_B(\hat{Y}_B)$ is mean-square consistent for $V(\hat{Y}_B)$ conditionally on $n_I^B > 0$, that is,

$$(3.6) \quad E_{\{n_I^B > 0\}} |N_I^{-2} n_I \{v_B(\hat{Y}_B) - V(\hat{Y}_B)\}|^2 \xrightarrow{t \rightarrow \infty} 0.$$

Also, $v_B(\hat{Y}_B)$ is mean-square consistent unconditionally:

$$(3.7) \quad E |N_I^{-2} n_I \{v_B(\hat{Y}_B) - V(\hat{Y}_B)\}|^2 \xrightarrow{t \rightarrow \infty} 0.$$

If $n_I^B > 0$, we define $T_B \equiv \{v_B(\hat{Y}_B)\}^{-0.5} (\hat{Y}_B - Y)$. It follows by the mean-square consistency of $v_B(\hat{Y}_B)$ in (3.6) that under assumption (H4), $v_B(\hat{Y}_B)$ is weakly consistent for $V(\hat{Y}_B)$, namely

$$(3.8) \quad \{V(\hat{Y}_B)\}^{-1} v_B(\hat{Y}_B) \xrightarrow{\Pr_{\{n_I^B > 0\}}} 1,$$

where $\xrightarrow{\Pr_{\{n_I^B > 0\}}}$ stands for the convergence in probability, conditionally on $n_I^B > 0$.

It follows by (3.8) and by the central limit theorem in (3.5) that the pivotal quantity T_B has a limiting standard normal distribution. An approximate two-sided $100(1 - 2\alpha)\%$ confidence interval for Y is thus given by $[\hat{Y}_B \pm u_{1-\alpha}\{v_B(\hat{Y}_B)\}^{0.5}]$, with $u_{1-\alpha}$ the quantile of order $1 - \alpha$ of the standard normal distribution.

3.2. *Without replacement simple random sampling of PSUs.* We consider the case when a first-stage sample S_I is selected in U_I by means of simple random sampling without replacement (SI) of size n_I , which we note as $S_I \sim \text{SI}(U_I; n_I)$. The Horvitz–Thompson estimator is

$$(3.9) \quad \hat{Y} = \frac{N_I}{n_I} \sum_{u_i \in U_I} I_i \hat{Y}_i = \frac{N_I}{n_I} \sum_{u_i \in S_I} \hat{Y}_i,$$

with I_i the sample membership indicator for the PSU u_i in the sample S_I . We may alternatively rewrite the Horvitz–Thompson estimator as

$$(3.10) \quad \hat{Y} = N_I \bar{Z} \quad \text{with} \quad \bar{Z} = \frac{1}{n_I} \sum_{j=1}^{n_I} Z_j,$$

where the sample S_I of PSUs is obtained by drawing n_I times without replacement one PSU in U_I , and where Z_j stands for the estimator of the total for the PSU selected at the j th draw. The variance of \hat{Y} is

$$(3.11) \quad V(\hat{Y}) = \frac{N_I^2}{n_I} \left\{ (1 - f_I) S_{Y,U_I}^2 + \frac{1}{N_I} \sum_{u_i \in U_I} V_i \right\},$$

with $S_{Y,U_I}^2 = (N_I - 1)^{-1} \sum_{u_i \in U_I} (Y_i - \mu_Y)^2$ the population dispersion of the sub-totals Y_i . Under (H1) and (H2), \hat{Y} is mean-square consistent for Y in the sense that

$$(3.12) \quad E\{N_I^{-1}(\hat{Y} - Y)\}^2 \xrightarrow[t \rightarrow \infty]{} 0.$$

This implies that $N_I^{-1}(\hat{Y} - Y) \xrightarrow[\text{Pr}]{\text{Pr}} 0$ where $\xrightarrow[\text{Pr}]{\text{Pr}}$ stands for the convergence in probability.

Hajek (1960) proposed a coupling procedure to draw simultaneously a BE sample and an SI sample. This procedure is adapted in Algorithm 3.1 to the context of multistage sampling, and Proposition 3.1 below generalizes the Lemma 2.1 in [16].

PROPOSITION 3.1. *Assume that the samples S_I^B and S_I are selected according to Algorithm 3.1. We note $\Delta_2 \equiv \sum_{u_i \in S_I} (\hat{Y}_i - \mu_Y) - \sum_{u_i \in S_I^B} (\hat{Y}_i - \mu_Y)$. Then*

$$(3.13) \quad \frac{E\{\Delta_2\}^2}{V\{\sum_{u_i \in S_I^B} (\hat{Y}_i - \mu_Y)\}} \leq \left\{ \frac{1}{n_I} + \frac{1}{N_I - n_I} \right\}^{0.5}.$$

Algorithm 3.1 A coupling procedure for Bernoulli sampling of PSUs and simple random sampling without replacement of PSUs

1. Draw the sample $S_I^B \sim \text{BE}(U_I; n_I)$. Denote by n_I^B the (random) size of S_I^B .
 2. Draw the sample S_I as follows:
 - if $n_I^B = n_I$, take $S_I = S_I^B$;
 - if $n_I^B < n_I$, draw $S_I^+ \sim \text{SI}(U_I \setminus S_I^B; n_I - n_I^B)$ and take $S_I = S_I^B \cup S_I^+$;
 - if $n_I^B > n_I$, draw $S_I^+ \sim \text{SI}(S_I^B; n_I^B - n_I)$ and take $S_I = S_I^B \setminus S_I^+$.
 3. For any PSU u_i :
 - if $u_i \in S_I^B \cap S_I$, select the same second-stage sample S_i for both \hat{Y}_B and \hat{Y} ;
 - if $u_i \in S_I^B \setminus S_I$, select a second-stage sample S_i for \hat{Y}_B ;
 - if $u_i \in S_I \setminus S_I^B$, select a second-stage sample S_i for \hat{Y} .
-

The result in Proposition 3.1 can be easily generalized to the multivariate case: if $y_k = (y_{1k}, \dots, y_{qk})^\top$ denotes the value taken for unit k by some q -vector of interest y , we have

$$V\{\Delta_2\} \leq \left\{ \frac{1}{n_I} + \frac{1}{N_I - n_I} \right\}^{0.5} V \left\{ \sum_{u_i \in S_I^B} (\hat{Y}_i - \mu_Y) \right\},$$

where for symmetric matrices A and B of size q , $A \leq B$ means that $B - A$ is nonnegative definite.

THEOREM 3.2. Assume that (H1) and (H2) hold. Then the Horvitz–Thompson estimator $\hat{Y} = N_I n_I^{-1} \sum_{u_i \in S_I} \hat{Y}_i$ is asymptotically normally distributed, that is,

$$(3.14) \quad \{V(\hat{Y})\}^{-0.5} (\hat{Y} - Y) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1).$$

3.3. Variance estimation for SI sampling of PSUs. We first consider the usual, unbiased variance estimator for \hat{Y} :

$$(3.15) \quad v(\hat{Y}) = \frac{N_I^2}{n_I} \left\{ (1 - f_I) s_Z^2 + \frac{1}{N_I} \sum_{u_i \in S_I} \hat{V}_i \right\}$$

$$\text{with } s_Z^2 = \frac{1}{n_I - 1} \sum_{j=1}^{n_I} (Z_j - \bar{Z})^2.$$

PROPOSITION 3.2. Assume that (H1) and (H3) hold, and that (H2) holds with $\delta = 2$. Then $v(\hat{Y})$ is mean-square consistent for $V(\hat{Y})$:

$$(3.16) \quad E |N_I^{-2} n_I \{v(\hat{Y}) - V(\hat{Y})\}|^2 \xrightarrow{t \rightarrow \infty} 0.$$

It follows by Proposition 3.2 that under the assumption (H4), $v(\hat{Y})$ is weakly consistent for $V(\hat{Y})$, namely

$$(3.17) \quad \{V(\hat{Y})\}^{-1} v(\hat{Y}) \xrightarrow{\text{Pr}} 1.$$

From the central limit theorem in (3.14), $T \equiv \{v(\hat{Y})\}^{-0.5}(\hat{Y} - Y)$ has a limiting standard normal distribution. Therefore, an approximate two-sided $100(1 - 2\alpha)\%$ confidence interval for Y is given by

$$(3.18) \quad [\hat{Y} \pm u_{1-\alpha} \{v(\hat{Y})\}^{0.5}].$$

In proving Proposition 3.2, assumption (H3) is needed, requiring that an unbiased variance estimator \hat{V}_i can be computed inside PSUs. This assumption may be cumbersome, particularly if the sampling design implies additional stages of sampling inside PSUs. It is thus desirable to provide simplified variance estimators which do not require assumption (H3) while remaining consistent. We are able to do so in the particular important case when the first-stage sampling rate tends to zero. A simplified variance estimator (see [36], equation (4.6.1)) is obtained by simply dropping the term involving the variance estimators inside PSUs \hat{V}_i . This leads to

$$(3.19) \quad v_{\text{SIMP}}(\hat{Y}) = \frac{N_I^2}{n_I} (1 - f_I) s_Z^2.$$

PROPOSITION 3.3. *Assume that (H1) holds, and that (H2) holds with $\delta = 2$. Assume that $f_I \xrightarrow[t \rightarrow \infty]{} 0$. Then $v_{\text{SIMP}}(\hat{Y})$ is mean-square consistent for $V(\hat{Y})$:*

$$(3.20) \quad E|N_I^{-2} n_I \{v_{\text{SIMP}}(\hat{Y}) - V(\hat{Y})\}|^2 \xrightarrow[t \rightarrow \infty]{} 0.$$

The proof (omitted) follows from the fact that when $f_I \xrightarrow[t \rightarrow \infty]{} 0$, $V(\hat{Y})$ is asymptotically equivalent to

$$(3.21) \quad V_{\text{app}}(\hat{Y}) = \frac{N_I^2}{n_I} (1 - f_I) \left\{ S_{Y,U_I}^2 + \frac{1}{N_I} \sum_{u_i \in U_I} V_i \right\}$$

under assumption (H2). It is easily seen from equation (3.15) that $v_{\text{SIMP}}(\hat{Y})$ tends to underestimate the true variance, with a bias equal to $-\sum_{u_i \in U_I} V_i$. An alternative simplified estimator is obtained by estimating the variance as if the PSUs were selected with replacement [see equation (4.5)]. This leads to the second simplified variance estimator

$$(3.22) \quad v_{\text{WR}}(\hat{Y}) = \frac{N_I^2}{n_I} s_Z^2.$$

It is easily shown that $v_{WR}(\hat{Y})$ tends to overestimate the true variance, with a bias equal to $N_I S_{Y,U_I}^2$. Under the conditions of Proposition 3.3, $v_{WR}(\hat{Y})$ is also mean-square consistent for the true variance since it only differs from $v_{SIMP}(\hat{Y})$ with the factor $(1 - f_I)$. Under the additional assumption (H4), the variance estimators $v_{SIMP}(\hat{Y})$ and $v_{WR}(\hat{Y})$ are therefore weakly consistent for $V(\hat{Y})$. When $f_I \xrightarrow{t \rightarrow \infty} 0$, an approximate two-sided $100(1 - 2\alpha)\%$ confidence interval for Y is therefore obtained from (3.18) by replacing $v(\hat{Y})$ with $v_{SIMP}(\hat{Y})$ or $v_{WR}(\hat{Y})$.

4. With-replacement bootstrap for multistage sampling. The use of bootstrap techniques in survey sampling has been widely studied in the literature. Most of them may be thought as particular cases of the weighted bootstrap [1–3]; see also [10, 23, 38] and [11] for detailed reviews.

Bootstrap for multistage sampling under without-replacement sampling of PSUs has been considered, for example, in [14, 24, 27, 30, 31, 33], among others. In this section, we consider the so-called with-replacement bootstrap of PSUs (see [33]). This method is suitable for with-replacement sampling of PSUs, and basic results for such sampling designs are therefore reminded in Section 4.1. A new coupling algorithm between SI sampling of PSUs and SIR sampling of PSUs is given in Section 4.2. This is the main tool to study the bootstrap of PSUs for multistage sampling with SI sampling of PSUs when the first-stage sampling fraction tends to zero. In Section 4.3, we prove that Studentized bootstrap confidence intervals are valid. In Section 4.4, we prove that the bootstrap variance estimator is consistent for smooth functions of means whenever it is consistent in case of SIR sampling of PSUs.

4.1. *With replacement sampling of PSUs.* We consider the case when a first-stage sample S_I^{WR} is selected in U_I according to simple random sample with replacement (SIR) of size n_I inside U_I , which we note as $S_I^{WR} \sim \text{SIR}(U_I; n_I)$. Denote by W_i the number of selections of the PSU u_i in S_I^{WR} , and by S_I^d of size n_I^d the set of distinct PSUs associated to S_I^{WR} . Each time $j = 1, \dots, W_i$ that unit u_i is drawn in S_I^{WR} , a second-stage sample $S_{i[j]}$ is selected in u_i . The total Y is unbiasedly estimated by the Hansen–Hurwitz estimator

$$(4.1) \quad \hat{Y}_{WR} = \sum_{u_i \in S_I^d} \frac{1}{E(W_i)} \sum_{j=1}^{W_i} \hat{Y}_{i[j]} = \frac{N_I}{n_I} \sum_{u_i \in S_I^d} \sum_{j=1}^{W_i} \hat{Y}_{i[j]},$$

where $\hat{Y}_{i[j]}$ stands for an unbiased estimator of Y_i computed on $S_{i[j]}$. We may alternatively rewrite the Hansen–Hurwitz estimator as

$$(4.2) \quad \hat{Y}_{WR} = N_I \bar{X} \quad \text{with} \quad \bar{X} = \frac{1}{n_I} \sum_{j=1}^{n_I} X_j,$$

where the sample S_I^{WR} of PSUs is obtained by drawing n_I times with replacement one PSU in U_I and where X_j stands for the estimator of the total for the PSU selected at the j th draw.

The variance of \hat{Y}_{WR} is

$$(4.3) \quad V(\hat{Y}_{WR}) = \frac{N_I^2}{n_I} \left\{ \frac{N_I - 1}{N_I} S_{Y,U_I}^2 + \frac{1}{N_I} \sum_{u_i \in U_I} V_i \right\}.$$

Under (H1) and (H2), \hat{Y}_{WR} is mean-square consistent for Y in the sense that

$$(4.4) \quad E\{N_I^{-1}(\hat{Y}_{WR} - Y)\}^2 \xrightarrow{t \rightarrow \infty} 0.$$

This implies that $N_I^{-1}(\hat{Y}_{WR} - Y) \xrightarrow{\text{Pr}} 0$.

An unbiased variance estimator for $V(\hat{Y}_{WR})$ is

$$(4.5) \quad v_{WR}(\hat{Y}_{WR}) = \frac{N_I^2}{n_I} s_{\bar{X}}^2 \quad \text{with } s_{\bar{X}}^2 = \frac{1}{n_I - 1} \sum_{j=1}^{n_I} (X_j - \bar{X})^2.$$

The simple form of the variance estimator in (4.5) is primarily due to (4.2), where \hat{Y}_{WR} is written as a sum of independent and identically distributed random variables (see also [36], page 151).

THEOREM 4.1. *Assume that (H1) and (H2) hold. Then the Hansen–Hurwitz estimator $\hat{Y}_{WR} = N_I n_I^{-1} \sum_{u_i \in S_I^d} \sum_{j=1}^{W_i} \hat{Y}_{i[j]}$ is asymptotically normally distributed, that is,*

$$(4.6) \quad \{V(\hat{Y}_{WR})\}^{-0.5}(\hat{Y}_{WR} - Y) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1).$$

Assume further that (H2) holds with $\delta = 2$. Then $v_{WR}(\hat{Y}_{WR})$ is mean-square consistent for $V(\hat{Y}_{WR})$:

$$(4.7) \quad E|N_I^{-2} n_I \{v_{WR}(\hat{Y}_{WR}) - V(\hat{Y}_{WR})\}|^2 \xrightarrow{t \rightarrow \infty} 0.$$

In proving the consistency of $v_{WR}(\hat{Y}_{WR})$, assumption (H3) is not needed. In particular, unbiased variance estimators \hat{V}_i inside PSUs are not mandatory. This appealing property leads to consider $v_{WR}(\cdot)$ as a possible simplified variance estimator when the PSUs are selected without replacement with a first-stage sampling fraction tending to zero; see equation (3.22).

4.2. A coupling procedure between SIR sampling of PSUs and SI sampling of PSUs. The procedure is described in Algorithm 4.1. Conditionally on n_I^d , the sample S_I^d obtained in step 1 is by symmetry an SI sample of size n_I^d from U_I , which implies that $S_I^d \cup S_I^c$ is an SI sample of size n_I from U_I . Consequently, this procedure leads to a sample S_I drawn by means of SI sampling of PSUs.

Algorithm 4.1 A coupling procedure for simple random sampling with-replacement of PSUs and simple random sampling without replacement of PSUs for multistage sampling

1. Draw the sample $S_I^{WR} \sim SIR(U_I; n_I)$. Denote by S_I^d of (random) size n_I^d the set of distinct PSUs in S_I^{WR} .
 2. Draw a complementary sample $S_I^c \sim SI(U_I \setminus S_I^d; n_I - n_I^d)$ and take $S_I = S_I^d \cup S_I^c$.
 3. For any $u_i \in S_I^d$:
 - Each time $j = 1, \dots, W_i$ that unit u_i is drawn in S_I^{WR} , select a second-stage sample $S_{i[j]}$ with associated estimator $\hat{Y}_{i[j]}$ for \hat{Y}_{WR} .
 - Take $S_i = S_{i[1]}$ and $\hat{Y}_i = \hat{Y}_{i[1]}$ for \hat{Y} .
 4. For any $u_i \in S_I^c$, select a second-stage sample S_i with associated estimator \hat{Y}_i for \hat{Y} .
-

PROPOSITION 4.1. Assume that the samples S_I^{WR} and S_I are selected according to Algorithm 4.1. Then

$$(4.8) \quad \frac{E(\hat{Y}_{WR} - \hat{Y})^2}{V(\hat{Y}_{WR})} \leq \frac{n_I - 1}{N_I - 1}.$$

The right bound in (4.8) is mainly of interest when $f_I \xrightarrow{t \rightarrow \infty} 0$. In this case, from the trivial inequality $\frac{n_I - 1}{N_I - 1} \leq \frac{n_I}{N_I}$, Algorithm 4.1 may be used to select the samples S_I^{WR} and S_I so that the difference between \hat{Y}_{WR} and \hat{Y} is asymptotically negligible. A similar result holds for the dispersions between the estimated totals inside PSUs, as stated in Proposition 4.2 below.

PROPOSITION 4.2. Assume that the samples S_I^{WR} and S_I are selected according to Algorithm 4.1. Assume that (H1) and (H2) hold, and that $f_I \xrightarrow{t \rightarrow \infty} 0$. Then

$$(4.9) \quad E(\bar{Z} - \bar{X})^2 = o(n_I^{-1}),$$

$$(4.10) \quad E|s_{\bar{Z}}^2 - s_{\bar{X}}^2| \xrightarrow{t \rightarrow \infty} 0,$$

where \bar{X} and $s_{\bar{X}}^2$ are defined in equations (4.2) and (4.5), and \bar{Z} and $s_{\bar{Z}}^2$ are defined in equations (3.10) and (3.15),

4.3. *With replacement bootstrap of PSUs.* We consider the with-replacement bootstrap of PSUs described in [33]. Using the notation introduced in equation (3.10), let $(Z_1, \dots, Z_{n_I})^\top$ denote the sample of estimators under SI sampling of PSUs. Also, let $(Z_1^*, \dots, Z_m^*)^\top$ be obtained by sampling m times inde-

pendently in $(Z_1, \dots, Z_{n_I})^\top$. Similarly, using the notation introduced in equation (4.2), let $(X_1, \dots, X_{n_I})^\top$ denote the sample of estimators under SIR sampling of PSUs. Also, let $(X_1^*, \dots, X_m^*)^\top$ be obtained by sampling m times independently in $(X_1, \dots, X_{n_I})^\top$.

We first demonstrate the bootstrap consistency. We note

$$\bar{Z}_m^* = \frac{1}{m} \sum_{j=1}^m Z_j^* \quad \text{and} \quad s_Z^{*2} = \frac{1}{m-1} \sum_{j=1}^m (Z_j^* - \bar{Z}_m^*)^2,$$

$$\bar{X}_m^* = \frac{1}{m} \sum_{j=1}^m X_j^* \quad \text{and} \quad s_X^{*2} = \frac{1}{m-1} \sum_{j=1}^m (X_j^* - \bar{X}_m^*)^2.$$

We proceed by showing that, using Algorithm 4.1, the samples S_I and S_I^{WR} can be drawn so that the pivotal statistics

$$(4.11) \quad m^{0.5}(s_Z^*)^{-1}(\bar{Z}_m^* - \bar{Z}) \quad \text{and} \quad m^{0.5}(s_X^*)^{-1}(\bar{X}_m^* - \bar{X})$$

are close. More precisely, we make use of the Mallows metric (see [25] and [4]), also known as the Wasserstein metric. Let $1 \leq q < \infty$, and let α and β denote two distributions on \mathbb{R}^s with finite moments of order q . Then

$$(4.12) \quad d_q(\alpha, \beta) = \inf\{E\|X - Z\|^q\}^{1/q},$$

where the infimum is taken over all couples (X, Z) with marginal distributions α and β . For two random vectors X and Z , we note $d_q(\alpha, \beta)$ for the d_q -distance between the distributions of X and Z . In what follows, we consider $q = 1$ or $q = 2$.

Let $D = (D_1, \dots, D_{n_I})^\top$ be generated according to a multinomial distribution with parameters $(m; n_I^{-1}, \dots, n_I^{-1})$. The same multinomial weights D are used in the selection of $(Z_1^*, \dots, Z_m^*)^\top$ and $(X_1^*, \dots, X_m^*)^\top$, so that we may write

$$(4.13) \quad \bar{Z}_m^* = \frac{1}{m} \sum_{j=1}^{n_I} D_j Z_j \quad \text{and} \quad \bar{X}_m^* = \frac{1}{m} \sum_{j=1}^{n_I} D_j X_j.$$

PROPOSITION 4.3. *Assume that (H1) and (H2) hold. Assume that $f_I \xrightarrow[t \rightarrow \infty]{} 0$ and that $m \xrightarrow[t \rightarrow \infty]{} \infty$. Then*

$$(4.14) \quad E(\bar{Z}_m^* - \bar{X}_m^*)^2 = o(m^{-1}) + o(n_I^{-1}).$$

PROPOSITION 4.4. *Assume that (H1) and (H2) hold. Assume that $f_I \xrightarrow[t \rightarrow \infty]{} 0$ and that $m \xrightarrow[t \rightarrow \infty]{} \infty$. Then*

$$(4.15) \quad d_2[m^{0.5}(\bar{Z}_m^* - \bar{Z}), m^{0.5}(\bar{X}_m^* - \bar{X})] \xrightarrow[t \rightarrow \infty]{} 0,$$

$$(4.16) \quad d_1[s_Z^{*2}, s_X^{*2}] \xrightarrow[t \rightarrow \infty]{} 0,$$

where the distance $d_q(\cdot, \cdot)$ is defined in (4.12).

From Proposition 4.4, the pivotal statistics in (4.11) share the same limiting distribution. Theorem 4.2 below follows from Theorem 2.1 of [4].

THEOREM 4.2. *Assume that (H1) and (H2) hold. Assume that $f_I \xrightarrow[t \rightarrow \infty]{} 0$ and that $m \xrightarrow[t \rightarrow \infty]{} \infty$. Then*

$$(4.17) \quad m^{0.5}(s_Z^*)^{-1}(\bar{Z}_m^* - \bar{Z}) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1).$$

Theorem 4.2 implies that the normality-based confidence interval for Y given in (3.18) may be replaced by the Studentized bootstrap confidence interval (see [10], page 194)

$$(4.18) \quad [\hat{Y} - u_{1-\alpha}^* \{v(\hat{Y})\}^{0.5}, \hat{Y} - u_{\alpha}^* \{v(\hat{Y})\}^{0.5}],$$

where the quantiles $u_{1-\alpha}$ and u_{α} of the normal distribution are replaced by the corresponding quantiles $u_{1-\alpha}^*$ and u_{α}^* of the bootstrap pivotal quantity in (4.17). The simplified variance estimators $v_{\text{SIMP}}(\hat{Y})$ and $v_{\text{WR}}(\hat{Y})$ can also be used in (4.18).

4.4. Bootstrap variance estimation for functions of totals. We now consider the case when $y_k = (y_{1k}, \dots, y_{qk})^\top$ is multivariate, and denotes the value taken for unit k by some q -vector of interest y . We are interested in a parameter $\theta = f(Y)$ for some function $f : \mathbb{R}^q \rightarrow \mathbb{R}$. Under SI sampling of PSUs, the plug-in estimator of θ is $\hat{\theta} \equiv f(N_I \bar{Z})$. Under SIR sampling of PSUs, the plug-in estimator of θ is $\hat{\theta}_{\text{WR}} \equiv f(N_I \bar{X})$. Also, we note $\hat{\theta}^* \equiv f(\bar{Z}_m^*)$ and $\hat{\theta}_{\text{WR}}^* \equiv f(\bar{X}_m^*)$ for the bootstrap estimators, where \bar{Z}_m^* and \bar{X}_m^* are defined in (4.13). We consider the additional regularity assumptions:

H5: $f(\cdot)$ is homogeneous of degree $\beta \geq 0$, in that $f(ry) = r^\beta f(y)$ for any real $r > 0$ and q -vector y . Also, f is a differentiable function on \mathbb{R}^q with bounded partial derivatives.

H6: There exists some constant $C_4 > 0$ such that $V(\hat{\theta}_{\text{WR}}) > C_4 N_I^{2\beta} n_I^{-1}$.

Assumption (H6) is similar to (H4), and requires the variance of the plug-in estimator $\hat{\theta}_{\text{WR}}$ to have the usual order $O(N_I^{2\beta} n_I^{-1})$.

PROPOSITION 4.5. *Assume that the samples S_I^{WR} and S_I are selected according to Algorithm 4.1. Assume that assumptions (H1), (H2) and (H5) hold. Assume that $f_I \xrightarrow[t \rightarrow \infty]{} 0$. Then*

$$(4.19) \quad E(\|\bar{Z} - \bar{X}\|^2) = o(n_I^{-1}),$$

$$(4.20) \quad E(\hat{\theta} - \hat{\theta}_{\text{WR}})^2 = o(N_I^{2\beta} n_I^{-1}),$$

with $\|\cdot\|$ the Euclidean norm. Assume further that $m \xrightarrow[t \rightarrow \infty]{} \infty$. Then

$$(4.21) \quad E(\|\bar{Z}^* - \bar{X}^*\|^2) = o(m^{-1}) + o(n_I^{-1}),$$

$$(4.22) \quad E(\hat{\theta}^* - \hat{\theta}_{WR}^*)^2 = o(N_I^{2\beta} m^{-1}) + o(N_I^{2\beta} n_I^{-1}).$$

PROPOSITION 4.6. Assume that the samples S_I^{WR} and S_I are selected according to Algorithm 4.1. Assume that assumptions (H1), (H2), (H5) and (H6) hold. Assume that $f_I \xrightarrow[t \rightarrow \infty]{} 0$ and $m = O(n_I)$. Then

$$(4.23) \quad \frac{V_{\{X\}}(\hat{\theta}_{WR}^*)}{V(\hat{\theta}_{WR})} \xrightarrow[\text{Pr}]{} 1 \quad \text{implies} \quad \frac{V_{\{Z\}}(\hat{\theta}^*)}{V(\hat{\theta})} \xrightarrow[\text{Pr}]{} 1,$$

with $V_{\{X\}}$ the variance conditionally on X_1, \dots, X_{n_I} , and similarly for $V_{\{Z\}}$.

The proof of Proposition 4.5 follows from the regularity assumptions on $f(\cdot)$ and from Propositions 4.2 and 4.3. Proposition 4.6 implies that the with-replacement bootstrap of PSUs provides consistent variance estimation for $\hat{\theta}$ whenever it does so for $\hat{\theta}_{WR}$. The regularity assumption (H5) is somewhat strong, and may be weakened to differentiability of $f(\cdot)$ on a compact set, under additional assumptions on the vector of interest y and on the second-stage sampling weights.

5. A simulation study. We conducted a limited simulation study to investigate on the performance of the variance estimators. We first generated 3 finite populations, each with $N_I = 2000$ PSUs. The number of SSUs inside PSUs was generated so that the average number of SSUs per PSU was approximately equal to $\bar{N} = 40$, and so that the coefficient of variation for the sizes N_i of PSUs was equal to 0 for population 1 (so that the PSUs are of equal size), approximately equal to 0.03 for population 2, and approximately equal to 0.06 for population 3.

In each population, we generated for any PSU u_i the value $\lambda_i = \lambda + \sigma v_i$ with $\lambda = 20$ and $\sigma = 2$ for each population, and the v_i 's were generated according to a normal distribution with mean 0 and variance 1. For each SSU $k \in u_i$, we generated three couples of values $(y_{1,k}, y_{2,k})$, $(y_{3,k}, y_{4,k})$ and $(y_{5,k}, y_{6,k})$ according to the model

$$(5.1) \quad y_{2h-1,k} = \lambda_i + \{\rho_h^{-1}(1 - \rho_h)\}^{0.5} \sigma(\alpha \varepsilon_k + \eta_k),$$

$$(5.2) \quad y_{2h,k} = \lambda_i + \{\rho_h^{-1}(1 - \rho_h)\}^{0.5} \sigma(\alpha \varepsilon_k + \nu_k),$$

for $h = 1, \dots, 3$, where the values ε_k , η_k and ν_k were generated according to a normal distribution with mean 0 and variance 1. In each population, the parameter ρ_h was chosen so that the intra-cluster correlation coefficient was approximately equal to 0.1 for both variables y_1 and y_2 , 0.2 for both variables y_3 and y_4 , and 0.3 for

both variables y_5 and y_6 . Also, the parameter α was chosen so that the coefficient of correlation between variables y_{2h-1} and y_{2h} , $h = 1, \dots, 3$, was approximately equal to 0.60.

From each population, we selected $B = 1000$ two-stage samples. The sample S_I of PSUs was selected by means of SI sampling of size $n_I = 20, 40, 100$ or 200 . Inside each $u_i \in S_I$, the sample S_i of SSUs was selected by means of systematic sampling of size $n_0 = 5$ or 10 . Note that, due to the systematic sampling at the second stage, the variance may not be unbiasedly estimated. Our objective is to estimate the variance of the Horvitz–Thompson estimator of the totals of the variables y_1, y_3 and y_5 , by using the simplified variance estimator $v_{\text{SIMP}}(\hat{Y})$ in (3.19) or the with-replacement bootstrap of PSUs. Also, our objective is to estimate the variance of the substitution estimator for the ratios

$$(5.3) \quad R_h = (\mu_{y,2h})^{-1} \mu_{y,2h-1}$$

with $\mu_{y,2h-1} = N^{-1} \sum_{k \in U} y_{2h-1,k}$ and $\mu_{y,2h} = N^{-1} \sum_{k \in U} y_{2h,k}$, and the variance for the substitution estimator for the coefficient of correlations

$$(5.4) \quad r_h = \frac{\sum_{k \in U} (y_{2h-1,k} - \mu_{y,2h-1})(y_{2h,k} - \mu_{y,2h})}{\{\sum_{k \in U} (y_{2h-1,k} - \mu_{y,2h-1})^2 \sum_{k \in U} (y_{2h,k} - \mu_{y,2h})^2\}^{0.5}},$$

for $h = 1, \dots, 3$ by using the with-replacement bootstrap of PSUs. The true variance was approximated from a separate simulation run of $C = 20,000$ samples.

As a measure of bias of a point estimator $\hat{\theta}$ of a parameter θ , we used the Monte Carlo percent relative bias (RB) given by

$$\text{RB}_{\text{MC}}(\hat{\theta}) = 100 \frac{B^{-1} \sum_{b=1}^B \hat{\theta}_{(b)} - \theta}{\theta},$$

where $\hat{\theta}_{(b)}$ gives the value of the estimator for the b th sample. As a measure of variance of an estimator $\hat{\theta}$, we used the Monte Carlo percent relative stability (RS) given by

$$\text{RS}_{\text{MC}}(\hat{\theta}) = 100 \frac{\{B^{-1} \sum_{b=1}^B (\hat{\theta}_{(b)} - \theta)^2\}^{0.5}}{\theta}.$$

When the simplified variance estimator in (3.19) is used, we also assess the coverage of confidence intervals based on asymptotic normality. When the with-replacement bootstrap of PSUs is used, we assess the coverage of confidence intervals obtained by means of the percentile method. We used a nominal one-tailed error rate of 2.5% in each tail.

The results obtained for population 3 are presented in Tables 1 and 2. We observed no qualitative difference on populations 1 and 2, and the results for these two populations are therefore presented in the supplement [7] for brevity. We first consider the results of variance estimation for a total with the simplified variance estimator $v_{\text{SIMP}}(\hat{Y})$ and with the bootstrap of PSUs, which are presented in Table 1.

TABLE 1
Relative bias, relative stability and nominal one-tailed error rates for the simplified variance estimator of the Horvitz–Thompson estimator, and for the bootstrap for the estimation of a total for population 3

		Simplified variance estimator $v_{\text{SIMP}}(\hat{Y})$ for Y_{2h-1}							
n_0		5				10			
	n_I	20	40	100	200	20	40	100	200
$\rho = 0.1$	RB	-0.02	0.01	-0.03	-0.04	0.00	0.01	-0.01	-0.03
	RS	0.31	0.23	0.14	0.10	0.31	0.22	0.14	0.10
	L	2.9	3.0	2.0	2.3	2.4	2.5	2.4	2.3
	U	3.7	3.0	2.3	2.1	3.0	2.9	2.8	3.9
	L + U	6.6	6.0	4.3	4.4	5.4	5.4	5.2	6.2
$\rho = 0.2$	RB	-0.04	0.00	-0.01	-0.04	-0.01	0.01	-0.01	-0.01
	RS	0.31	0.21	0.14	0.10	0.32	0.23	0.14	0.09
	L	3.6	3.3	2.4	2.1	3.2	3.7	1.9	3.0
	U	3.6	3.3	2.0	1.9	2.7	3.1	2.1	2.6
	L + U	7.2	6.6	4.4	4.0	5.9	6.8	4.0	5.6
$\rho = 0.3$	RB	-0.03	0.01	-0.01	-0.02	0.00	0.02	0.00	-0.02
	RS	0.31	0.22	0.13	0.09	0.33	0.22	0.14	0.09
	L	3.0	2.9	2.0	3.1	4.3	3.0	2.5	2.6
	U	3.1	2.6	2.1	1.9	3.1	3.5	2.2	3.7
	L+U	6.1	5.5	4.1	5.0	7.4	6.5	4.7	6.3
		Bootstrap of PSUs for Y_{2h-1}							
$\rho = 0.1$	RB	-0.01	0.03	0.02	0.06	0.02	0.03	0.04	0.08
	RS	0.31	0.24	0.15	0.13	0.32	0.23	0.16	0.14
	L	3.3	2.7	2.4	1.7	2.7	2.4	2.2	1.8
	U	3.8	2.9	2.3	1.6	3.1	2.9	2.4	2.6
	L + U	7.1	5.6	4.7	3.3	5.8	5.3	4.6	4.4
$\rho = 0.2$	RB	-0.03	0.02	0.04	0.07	0.01	0.03	0.04	0.10
	RS	0.32	0.22	0.16	0.13	0.33	0.24	0.16	0.15
	L	3.6	3.5	2.1	1.8	3.1	3.3	2.1	2.4
	U	3.5	3.3	1.9	1.4	2.8	3.0	1.8	1.8
	L + U	7.1	6.8	4.0	3.2	5.9	6.3	3.9	4.2
$\rho = 0.3$	RB	-0.02	0.02	0.04	0.09	0.02	0.04	0.06	0.09
	RS	0.32	0.23	0.15	0.14	0.34	0.24	0.16	0.15
	L	2.9	3.1	1.8	2.2	4.3	2.9	2.4	2.3
	U	3.3	2.9	1.8	1.8	3.4	3.5	2.4	2.4
	L + U	6.2	6.0	3.6	4.0	7.7	6.4	4.8	4.7

We note that both variance estimators are approximately unbiased, with absolute relative biases no greater than 10%. As expected, $v_{\text{SIMP}}(\hat{Y})$ is slightly negatively biased while the bootstrap variance estimator is slightly positively biased. The ab-

TABLE 2
Relative bias, relative stability and nominal one-tailed error rates for the bootstrap for the estimation of a ratio and a coefficient of correlation for population 3

		Bootstrap of PSUs for R_h							
n_0		5				10			
n_I		20	40	100	200	20	40	100	200
$\rho = 0.1$	RB	0.03	0.01	0.03	0.01	0.01	0.00	0.02	0.03
	RS	0.33	0.24	0.16	0.11	0.34	0.24	0.16	0.12
	L	2.6	3.5	2.5	2.6	3.8	2.9	3.0	4.0
	U	3.3	3.7	3.3	3.1	2.6	3.1	2.5	2.5
	L + U	5.9	7.2	5.8	5.7	6.4	6.0	5.5	6.5
$\rho = 0.1$	RB	0.01	0.01	0.00	0.02	0.01	0.01	0.01	0.03
	RS	0.34	0.23	0.15	0.12	0.34	0.23	0.14	0.12
	L	3.2	2.4	2.7	2.3	2.6	2.4	2.2	2.9
	U	3.0	2.4	2.2	2.2	4.4	4.0	1.8	2.5
	L + U	6.2	4.8	4.9	4.5	7.0	6.4	4.0	5.4
$\rho = 0.1$	RB	-0.02	0.02	0.02	0.03	0.01	0.02	0.02	0.02
	RS	0.32	0.24	0.15	0.12	0.35	0.24	0.15	0.12
	L	3.5	2.8	2.7	3.6	2.2	2.4	2.9	3.2
	U	3.8	3.5	1.9	2.4	3.3	2.2	2.4	2.4
	L + U	7.3	6.3	4.6	6.0	5.5	4.6	5.3	5.6
		Bootstrap of PSUs for r_h							
$\rho = 0.1$	RB	0.02	0.00	-0.03	0.01	0.02	0.00	0.02	0.03
	RS	0.44	0.30	0.19	0.14	0.38	0.27	0.18	0.13
	L	3.6	2.7	2.8	2.3	3.5	3.0	1.7	1.9
	U	2.3	3.1	2.5	2.6	3.4	3.7	1.7	2.7
	L + U	5.9	5.8	5.3	4.9	6.9	6.7	3.4	4.6
$\rho = 0.2$	RB	-0.01	0.00	0.00	-0.01	0.00	0.01	0.03	0.04
	RS	0.41	0.32	0.20	0.14	0.37	0.28	0.18	0.14
	L	2.6	3.5	2.7	3.4	2.0	2.7	1.2	2.9
	U	3.0	2.5	2.7	2.5	3.6	3.5	2.8	3.2
	L + U	5.6	6.0	5.4	5.9	5.6	6.2	4.0	6.1
$\rho = 0.3$	RB	-0.01	0.01	0.02	0.01	-0.01	0.00	0.00	0.03
	RS	0.43	0.32	0.20	0.15	0.38	0.28	0.18	0.14
	L	3.7	3.8	2.0	3.1	2.8	2.8	2.6	2.0
	U	4.0	3.2	2.8	3.0	5.0	5.0	2.9	2.6
	L + U	7.7	7.0	4.8	6.1	7.8	7.8	5.5	4.6

solute bias tends to increase with n_I , that is, when the sampling fraction becomes nonnegligible. The simplified variance estimator is slightly more stable in all scenarios, while the bootstrap performs slightly better in terms of coverage rates. We now consider the results obtained for the bootstrap of PSUs when estimating a

ratio and a correlation coefficient, which are presented in Table 2. The bootstrap variance estimator is almost unbiased, with absolute relative biases no greater than 4%. The coverage rates are well respected in all cases.

6. Application on the panel for urban policy. We illustrate the proposed methods in the context of the Panel for Urban Policy (PUP), which was conducted by the French General Secretariat of the Inter-ministerial Committee for Cities (SGCIV). The PUP is a panel survey in four waves conducted between 2011 and 2014, which focuses on individuals in the Sensitive Urban Zones (ZUS), and which collects information on various aspects including security, employment, precariousness, schooling and health. In this paper, we focus on the 2011 edition. It involved two stages of sampling, with the selection of districts as PSUs, and of households as SSUs. All the individuals within the selected households were surveyed.

For the purpose of illustration, we consider a subset of districts as our population U_I of interest. At the first stage, the population U_I is partitioned into $L = 11$ strata U_{Il} according to the district. In each stratum U_{Il} of size N_{Il} , a SI sample S_{Il} of n_{Il} households is selected and all the individuals within the households $u_i \in S_{Il}$ are surveyed. In summary, our data set consists in a sample of 576 individuals obtained by stratified SI cluster sampling of households. The first-stage sampling rates $f_{Il} = N_{Il}^{-1}n_{Il}$ inside the L strata range from 0.002 to 0.017, which can be considered as negligible.

We are interested in four variables related to health. The variable y_1 gives the perceived health status (very good, good, fair, poor). The variable y_2 is an indicator of chronic disease (with, without). The variable y_3 indicates if the individual is limited by his health status in his usual activities (very limited, limited, not limited). The variable y_4 indicates if the individual benefits from a free universal health care (yes, no). For any possible characteristic c of some variable y , we are interested in the proportion

$$(6.1) \quad p_c = \frac{\sum_{l=1}^L \sum_{u_i \in U_{Il}} Y_{ic}}{\sum_{l=1}^L \sum_{u_i \in U_{Il}} N_i} \quad \text{with } Y_{ic} = \sum_{k \in u_i} 1(y_k = c),$$

which is estimated by its substitution estimator

$$(6.2) \quad \hat{p}_c = \frac{\sum_{l=1}^L N_{Il} n_{Il}^{-1} \sum_{u_i \in S_{Il}} Y_{ic}}{\hat{N}} \quad \text{with } \hat{N} \equiv \sum_{l=1}^L \frac{N_{Il}}{n_{Il}} \sum_{u_i \in S_{Il}} N_i.$$

For each proportion, we give the normality-based confidence interval. For that purpose, we adapt the simplified variance estimator in (3.22) to the stratified context and make use of the linearized variable of p_c . This leads to the variance estimator

$$(6.3) \quad v_{STWR}(\hat{p}_c) = \sum_{l=1}^L \frac{N_{Il}^2}{n_{Il}} s_{El}^2 \quad \text{with } s_{El}^2 = \frac{1}{n_{Il} - 1} \sum_{u_i \in S_{Il}} (E_i - \bar{E}_l)^2$$

for \hat{p}_c , with

$$E_i = \frac{1}{\hat{N}}(Y_{ic} - \hat{p}_c) \quad \text{and}$$

$$\bar{E}_l = \frac{1}{n_{ll}} \sum_{u_i \in S_{ll}} E_i.$$

For each proportion, we also give the percentile bootstrap and the Studentized bootstrap confidence intervals, using the with-replacement bootstrap of PSUs with $D = 1000$ resamples. The results with a nominal one-tailed error rate of 2.5% are presented in Table 3. The three confidence intervals are very similar in any case, though the normality-based confidence intervals tend to be slightly larger.

TABLE 3
Substitution estimator of the marginal proportions, normality-based confidence interval (CI), Percentile bootstrap confidence interval and Studentized bootstrap confidence interval for four variables

	Perceived health status			
	Very good	Good	Fair	Poor
Estimator \hat{p}_c	0.19	0.43	0.23	0.15
Normality-based CI	[0.15, 0.24]	[0.38, 0.49]	[0.18, 0.28]	[0.10, 0.19]
Percentile bootstrap CI	[0.15, 0.23]	[0.39, 0.48]	[0.19, 0.27]	[0.10, 0.20]
Studentized bootstrap CI	[0.16, 0.24]	[0.39, 0.48]	[0.19, 0.28]	[0.11, 0.21]
	Indicator of chronic disease			
	With	Without		
Estimator \hat{p}_c	0.28	0.72		
Normality-based CI	[0.23, 0.33]	[0.65, 0.79]		
Percentile bootstrap CI	[0.24, 0.33]	[0.67, 0.76]		
Studentized bootstrap CI	[0.24, 0.33]	[0.68, 0.77]		
	Limitation in usual activities			
	Very limited	Limited	Not limited	
Estimator \hat{p}_c	0.09	0.14	0.77	
Normality-based CI	[0.05, 0.13]	[0.11, 0.18]	[0.70, 0.84]	
Percentile bootstrap CI	[0.06, 0.14]	[0.11, 0.18]	[0.71, 0.81]	
Studentized bootstrap CI	[0.06, 0.15]	[0.11, 0.18]	[0.72, 0.82]	
	Recipient from a free universal health care			
	Yes	No		
Estimator \hat{p}_c	0.13	0.87		
Normality-based CI	[0.08, 0.18]	[0.80, 0.94]		
Percentile bootstrap CI	[0.08, 0.18]	[0.82, 0.92]		
Studentized bootstrap CI	[0.09, 0.19]	[0.83, 0.92]		

7. Proofs of results.

7.1. *Proof of Theorem 3.1.* We note $\hat{Y}_{iB} \equiv N_I n_I^{-1} I_i^B \hat{Y}_i$. Under (H2), we have $\sum_{u_i \in U_i} E|\hat{Y}_{iB} - Y_i|^{2+\delta} = O(N_I^{2+\delta} n_I^{-1-\delta})$. We obtain

$$(7.1) \quad \frac{\sum_{u_i \in U_i} E|\hat{Y}_{iB} - Y_i|^{2+\delta}}{V(\hat{Y}_B)^{1+\delta/2}} = O(n_I^{-\delta/2})$$

so that the Lyapunov condition is satisfied and (3.5) follows from the central limit theorem for triangular arrays. Noting $\Delta = N_I^{-2} n_I \{v_B(\hat{Y}_B) - V(\hat{Y}_B)\}$, we have

$$(7.2) \quad E_{\{n_I^B > 0\}}(\Delta^2) = \frac{1}{1 - (1 - f_I)^{N_I}} \sum_{k=1}^{n_I} \Pr(n_I^B = k) E_{\{n_I^B = k\}}(\Delta^2),$$

where $\Pr(n_I^B = k) = C_{N_I}^k f_I^k (1 - f_I)^{N_I - k}$. Using the fact that conditionally on n_I^B , S_I^B may be seen as a simple random sample of size n_I^B from U_I , we have after some algebra that there exists some constant C_5 such that $E_{\{n_I^B = k\}}(\Delta^2) \leq C_5/k$ for any $k > 0$. This leads to

$$(7.3) \quad E_{\{n_I^B > 0\}}(\Delta^2) \leq \frac{C_5}{1 - (1 - f_I)^{N_I}} \sum_{k=1}^{N_I} \frac{C_{N_I}^k f_I^k (1 - f_I)^{N_I - k}}{k}.$$

The term in the right-hand side of (7.3) tends to 0 (see Lemma 1.1 in the supplement [7]), which leads to (3.6). To prove (3.7), it suffices to notice that under (H2) there exists some constant C_6 such that $E_{\{n_I^B = 0\}}(\Delta^2) = \{N_I^{-2} n_I V(\hat{Y}_B)\}^2 \leq C_6$, and that $\Pr(n_I^B = 0) = (1 - f_I)^{N_I}$ tends to 0.

7.2. *Proof of Theorem 3.2.*

LEMMA 7.1. *Let X_t and Z_t denote two random variables such that $E(X_t) = E(Z_t)$. Assume that $E(X_t - Z_t)^2 = o\{V(X_t)\}$ and that $V(X_t) \xrightarrow[t \rightarrow \infty]{} \infty$. Then*

$$(7.4) \quad \{V(X_t)\}^{-1} V(Z_t) \xrightarrow[t \rightarrow \infty]{} 1.$$

Also, if for some distribution \mathcal{L}_0

$$(7.5) \quad \{V(X_t)\}^{-0.5} \{X_t - E(X_t)\} \xrightarrow[\mathcal{L}]{} \mathcal{L}_0,$$

then $\{V(Z_t)\}^{-0.5} \{Z_t - E(Z_t)\} \xrightarrow[\mathcal{L}]{} \mathcal{L}_0$.

The proof of Lemma 7.1 is omitted. We take $X_t = \sum_{u_i \in S_I^B} (\hat{Y}_i - \mu_Y)$ and $Z_t = \sum_{u_i \in S_I} (\hat{Y}_i - \mu_Y)$. Under assumptions (H1) and (H2), Proposition 3.1 implies that the assumptions of Lemma 7.1 are satisfied. Using the same proof as for (3.5) in Theorem 3.1, it is easily shown that (7.5) holds with \mathcal{L}_0 replaced with the standard normal distribution. This completes the proof.

7.3. *Proof of Theorem 4.1.* Since \hat{Y}_{WR} is a sum of independent and identically distributed random variables, (4.6) follows from the classical central limit theorem for triangular arrays in the i.i.d. case. After some algebra, we have

$$V(s_{\bar{X}}^2) = \frac{1}{n_I} \left[E(X_j - \mu_Y)^4 - \frac{n_I - 3}{n_I - 1} \{E(X_j - \mu_Y)^2\}^2 \right].$$

From (H2), there exists C_{10} such that $V(s_{\bar{X}}^2) \leq C_{10}n_I^{-1}$, so that (4.7) follows.

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SUPPLEMENTARY MATERIAL

Supplement to “Coupling methods for multistage sampling” (DOI: [10.1214/15-AOS1348SUPP](https://doi.org/10.1214/15-AOS1348SUPP); pdf). The supplement [7] contains additional proofs of Propositions in Section 1, and additional simulation results in Section 2.

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