

## CONVERGENCE OF THE SPECTRAL METHOD FOR STOCHASTIC GINZBURG-LANDAU EQUATION DRIVEN BY SPACE-TIME WHITE NOISE\*

DI LIU†

**Abstract.** In this paper, a spectral method is formulated as a numerical solution for the stochastic Ginzburg-Landau equation driven by space-time white noise. The rates of pathwise convergence and convergence in expectation in Sobolev spaces are given based on the convergence rates of the spectral approximation for the stochastic convolution. The analysis can be generalized to other spectral methods for stochastic PDEs driven by additive noises, provided the regularity condition for the noises.

**Keywords:** spectral method, Ginzburg-Landau equation, stochastic partial differential equations, space-time white noise, rate of convergence.

### 1. Introduction

In this paper, a spectral approximation is presented as a numerical method and the rates of pathwise convergence and convergence in expectation are given for the stochastic PDEs of the following form:

$$u_t = -Au + R(u) + \epsilon \dot{w}(t), \quad u(0) = u_0, \quad t \geq 0, \quad (1.1)$$

in a separable Hilbert space  $\mathbb{H}$  equipped with the inner product  $\langle \cdot, \cdot \rangle_{\mathbb{H}} \in \mathbb{R}$ .  $A$  is a self adjoint linear operator on domain  $D(A) \subset \mathbb{H}$  with eigenvalues  $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N \leq \dots$ ,  $\lim_{k \rightarrow \infty} \lambda_k = \infty$  and a complete orthonormal system of eigenvectors  $e_1, \dots, e_N, \dots$ , such that  $Ae_i = \lambda_i e_i$ .  $R$  is a nonlinear function from  $D(R) \subset \mathbb{H}$  to  $\mathbb{H}$ .  $w(t)$  is a Wiener process with values in  $\mathbb{H}$  and covariance operator  $Q$ .

A case of special interest is that in which  $w(t)$  is the space-time white noise, i.e.  $Q$  is the identity operator, which best models the fluctuations generated by microscopic effects in a homogeneous physical system. Typical situations include molecular collisions in gases and liquids, electric fluctuations in resistors, etc. ([7]) Instead of modeling each microscopic unit, which will lead to a very large complex system, people usually represent the micro effects by random perturbations in the dynamics of the macro observables. When the scale of the micro effects is very small, they should be uncorrelated jointly in time and space. After subtracting the mean intensity and putting it into the macro dynamics, we can approximate the random perturbation by the space-time white noise.

The finite difference and finite element methods for equations similar to (1.1) driven by space-time white noise have been proposed ([11], [8], [9], [4]). Uniform convergence was given for equations with Lipschitz nonlinear terms and convergence in probability was shown for equations with more general nonlinear terms. For linear equations or equations with nonlinear terms with bounded derivatives, the convergence rate of the expectation was analyzed in [2], [5] and [19]. The numerical experiments for the spectral methods for stochastic Burgers and Navier-Stokes equations were done in [16]. In this paper, a method to give the convergence rates of pathwise convergence and convergence in expectation of the spectral method for stochastic equation (1.1) driven by space-time white noise is proposed.

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†Program in Applied and Computational Mathematics, Princeton University, Princeton, NJ 08544, USA. (dliu@princeton.edu)

To illustrate our method, we choose an example equation which describes the simplest bistable system in a noisy environment. Consider the following stochastic Ginzburg-Landau (SGL) equation on  $[0, L]$  driven by space-time white noise:

$$u_t = \Delta u + \mu u - \lambda u^3 + \epsilon \dot{w}, \quad u(0) = u_0 \in \mathbb{C}_0[0, L], \quad t \geq 0, \quad (1.2)$$

with fixed parameters  $\mu > 0$  and  $\lambda > 0$  and Dirichlet boundary condition:

$$u(t, 0) = u(t, L) = 0.$$

We chose  $\{e_k(x), k \in \mathbb{N}\} = \{\frac{\sqrt{2}\sin(k\pi x/L)}{\sqrt{L}}, k \in \mathbb{N}\}$  to be an orthonormal basis of the Hilbert space  $\mathbb{L}_0^2[0, L]$ . Then

$$\Delta e_k = -\lambda_k e_k, \quad \lambda_k = (k\pi/L)^2.$$

Let  $\mathbb{H}_0^\alpha = \{u = \sum_{k \in \mathbb{N}} u_k e_k(x), |u|_{\mathbb{H}_0^\alpha} = \sum_k \lambda_k^\alpha |u_k|^2 < \infty\}$ . Define the semigroup  $e^{\Delta t}$ ,  $t \geq 0$  such that for  $\varphi_0(x)$ ,  $\varphi(t) = e^{\Delta t} \varphi_0$  is the solution of the following equation:

$$\varphi_t = \Delta \varphi(t),$$

with initial value  $\varphi_0$  and Dirichlet boundary condition. It is known that  $e^{\Delta t}$ ,  $t \geq 0$  defines a continuous semigroup on  $C_0[0, L]$  and an analytic semi-group on  $\mathbb{L}_0^2[0, L]$ . For the latter, we can write:

$$e^{\Delta t} \varphi = \sum_k e^{-\lambda_k t} \varphi_k e_k, \quad \varphi = \sum_k \varphi_k e_k.$$

And the following estimate holds on  $\mathbb{L}_0^2[0, L]$ : for any  $\alpha > 0$ ,

$$\|(-\Delta)^\alpha e^{\Delta t}\| \leq M t^{-\alpha} e^{-\delta t}, \quad t > 0, \quad (1.3)$$

for constants  $M$  and  $\delta$  which depend on  $\alpha$ .

The space-time white noise  $w$  can be interpreted in the sense of distribution as

$$w(t, x) = \sum_k w_k(t) e_k(x),$$

where  $w_k$ 's are independent standard Brownian Motions on a probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}, \mathbb{E})$ . Let

$$\Omega_T = [0, T] \times [0, L], \quad T \in [0, \infty). \quad (1.4)$$

The stochastic convolution of  $e^{\Delta t}$  and  $w(t)$  is defined to be the random field  $W$  on  $\Omega_T$ :

$$W(t, x) = \int_0^t e^{\Delta(t-s)} dw(s) = \sum_k e_k(x) \int_0^t e^{-\lambda_k(t-s)} dw_k(s).$$

It is known that given  $\gamma \in (0, \frac{1}{2})$ , there is a version of  $W(t, x)$  which is  $\alpha$ -Hölder continuous jointly in space and time for any  $\alpha \in (0, \gamma)$ . We still denote by  $W(t, x)$  this modification and denote by  $W(t)$  the function  $W(t, \cdot) \in \mathbb{C}_0[0, L]$ .

A solution of equation (1.2) is defined to be the solution of the integral equation

$$u(t) = e^{\Delta t} u_0 + \int_0^t e^{\Delta(t-s)} F(u(s)) ds + W(t) , \quad (1.5)$$

where  $F(u) = \mu u - \lambda u^3$ . Based on the regularity of  $W(t, x)$  and  $u_0$ , equation (1.5) can be discussed in a pathwise way. Define

$$z = e^{\Delta t} u_0 + W(t) . \quad (1.6)$$

Then for each realization of  $z \in \mathbb{C}(\Omega_T)$ , (1.5) is an integral equation with continuous coefficients. It was shown in [6] that with probability one there is a unique solution  $u$  of (1.5) such that  $u \in \mathbb{C}(\Omega_\infty)$  and  $u$  depends continuously on  $z$ . The similar regularity for the solutions on infinite domain was given by C. R. Doering [3]. Based on this, we can extend our results in subsequent sections with little change to the case of periodic boundary conditions.

## 2. The spectral approximation for the stochastic convolution

Since the solution  $u$  of equation (1.5) is a continuous function of  $z$ , it is natural to first define the spectral approximation to  $W(t)$  and show its convergence. By the continuous dependence of  $u$  on  $z$ , the convergence of the spectral approximation for the stochastic Ginzburg-Landau equation can be analyzed. As defined above, for any  $(t, x) \in \Omega_T$ ,

$$W(t, x) = \sum_k e_k(x) \int_0^t e^{-\lambda_k(t-s)} dw_k(s) , \quad \mathbb{P} - a.s. .$$

This means  $W(t, x)$  is a Gaussian random variable for each  $(t, x)$ . Since  $W(t, x) \in \mathbb{C}(\Omega_T)$  a.s., we can write the following equality in the  $\mathbb{L}_0^2$  sense

$$W(t, x) = \sum_{k \in \mathbb{N}} \beta_k(t) e_k(x) , \quad \beta_k(t) = \int_0^L W(t, x) e_k(x) dx .$$

Here, the integral is the Riemann integral based on the continuity of  $W(t, x)$ . As a limit of mean zero Gaussian random variables,  $\beta_k(t)$   $t \in [0, \infty]$  is also a Gaussian random variable with mean zero provided the convergence of the covariance. Let

$$h = L/N, \quad x_j = j \cdot h , \quad j = 1, \dots, N - 1 .$$

Then, with probability one we have

$$\beta_k(t) = \lim_{h \rightarrow 0} h \sum_{j, l} e_k(x_j) e_l(x_j) \int_0^t e^{-\lambda_l(t-\tau)} dw_k(\tau) .$$

By the fact that  $\sum_j e_k(x_j)e_l(x_j) = 0$  if  $k - l \neq 0 \pmod N$ , we have

$$\begin{aligned} & \mathbb{E}[\beta_k(t) \cdot \beta_k(s)] \\ &= \lim_{h \rightarrow 0} \mathbb{E}\left[h \sum_{j,l} e_k(x_j)e_l(x_j) \int_0^t e^{-\lambda_l(t-\tau)} dw_k(\tau)\right] \\ & \quad \cdot \left[h \sum_{j,l} e_k(x_j)e_l(x_j) \int_0^s e^{-\lambda_l(s-\tau)} dw_k(\tau)\right] \\ &= \lim_{N \rightarrow \infty} \sum_{m=k+n \cdot N, n \geq 0} \frac{e^{-\lambda_m|t-s|} - e^{-\lambda_m(t+s)}}{2\lambda_m} \left(\sum_j e_k^2(x_j) \cdot h\right)^2 \\ &= \frac{e^{-\lambda_k|t-s|} - e^{-\lambda_k(t+s)}}{2\lambda_k}. \end{aligned}$$

The above equality implies that  $\beta_k(t)$  is an Ornstein-Uhlenbeck process with parameter  $\lambda_k$  and initial value zero. The continuity of  $\beta_k(t)$  is implied by the Hölder continuity of  $W(t, x)$ . Let  $M_k(t) = e^{\lambda_k t} \beta_k(t)$ , then

$$\mathbb{E}[(M_k(t_1) - M_k(t_2)) \cdot M_k(s)] = 0, \quad s \leq t_1, t_2. \tag{2.1}$$

Define the filtration  $(\Omega, \mathcal{F}^k, \mathcal{F}_t^k, \mathbb{P}, \mathbb{E})$ , ( $k \in \mathbb{N}$ ) such that  $\mathcal{F}_t^k = \sigma\{\beta_k(s), s \leq t\}$ . Since  $M_k(t)$  is Gaussian, (2.1) means  $M_k(t) - M_k(s)$ , ( $s \leq t$ ) is independent of  $\mathcal{F}_s^k$ . By the fact that  $M_k(t)$  is mean zero, we have

$$\mathbb{E}(M_k(t)|\mathcal{F}_s^k) = \mathbb{E}(M_k(t) - M_k(s) + M_k(s)|\mathcal{F}_s^k) = M_k(s).$$

Thus, we know that  $M_k(t)$  is a continuous martingale w.r.t.  $(\Omega, \mathcal{F}^k, \mathcal{F}_t^k, \mathbb{P}, \mathbb{E})$ .

The spectral approximation  $W_N(t, x)$  for the stochastic convolution  $W(t, x)$  is defined to be:

$$W_N(t, x) = P_N W(t, x) = \sum_{|k| \leq N} \beta_k(t) e_k(x). \tag{2.2}$$

To give the convergence rates of  $W_N(t, x)$  to  $W(t, x)$ , we need the following version of Borell inequality. The proof can be found in [1].

LEMMA 2.1. *Let  $\{X_t\}_{t \in [0, T]}$  be a centered Gaussian process with sample paths bounded a.s. Let  $|X| = \sup_{t \in [0, T]} X_t$ . Then  $\mathbb{E}|X| < \infty$ , and for all  $\lambda > 0$*

$$\mathbb{P}\{|X| - \mathbb{E}|X| > \lambda\} \leq 2 \exp\{-\frac{1}{2} \lambda^2 / \sigma_T^2\}, \tag{2.3}$$

where  $\sigma_T^2 = \sup_{t \in [0, T]} \mathbb{E}[X_t^2]$ . By Lemma 2.1, we have for  $\lambda > \mathbb{E}|X|$ ,

$$\mathbb{P}\{|X| > \lambda\} \leq 2 \exp(-\frac{1}{2}(\lambda - \mathbb{E}|X|)^2 / \sigma_T^2).$$

Define  $\|X\| = \sup_{t \in [0, T]} |X_t|$ , then  $\mathbb{E}\|X\| > \mathbb{E}|X|$  and for  $\lambda > 0$ ,

$$\mathbb{P}\{\|X\| > \lambda\} \leq \mathbb{P}\{|X| > \lambda\} + \mathbb{P}\{-X > \lambda\}.$$

Hence, for  $\lambda > \mathbb{E}\|X\|$ ,

$$\mathbb{P}\{\|X\| > \lambda\} \leq 4 \exp(-\frac{1}{2}(\lambda - \mathbb{E}\|X\|)^2 / \sigma_T^2). \tag{2.4}$$

To apply (2.4) to the Gaussian process  $\beta_k(t)$ , we need to estimate  $\mathbb{E} [\sup_{0 \leq t \leq T} |\beta_k(t)|]$ .

LEMMA 2.2. *Let  $\|\beta_k\| = \sup_{t \in [0, T]} |\beta_k(t)|$ , then for any  $\epsilon \in (0, 1)$ , we have*

$$\mathbb{E} [\|\beta_k\|] \leq \frac{C_\epsilon}{k^{1-\epsilon}}, \tag{2.5}$$

where  $C_\epsilon$  is a constant independent of  $k$ .

*Proof.* Define the functions  $G(k, \lambda)$ , ( $k \in \mathbb{N}$ ) to be

$$G(k, \lambda) = \mathbb{P} \left\{ \|\beta_k\| = \sup_{0 \leq t \leq T} e^{-\lambda_k t} |M_k(t)| \geq \lambda \right\}.$$

Partitioning the interval  $[0, T]$  equally into  $k^2$  pieces, we have

$$G(k, \lambda) \leq \sum_{0 < m \leq k^2} \mathbb{P} \left\{ \sup_{0 \leq t \leq \frac{mT}{k^2}} |M_k(t)| \geq \lambda e^{\lambda_k \frac{(m-1)T}{k^2}} \right\}.$$

By the martingale inequality, we have the following for  $p \geq \frac{1}{2}$ :

$$\begin{aligned} \mathbb{P} \left\{ \sup_{0 \leq t \leq \frac{mT}{k^2}} |M_k(t)| \geq \lambda e^{\lambda_k \frac{(m-1)T}{k^2}} \right\} &\leq \frac{\mathbb{E} (|M_k(mT/k^2)|^{2p})}{\lambda^{2p} e^{\frac{2p\lambda_k(m-1)T}{k^2}}} = \frac{C_p (e^{\frac{2mT\pi^2}{L^2}} - 1)^p}{(2\lambda_k)^p \cdot \lambda^{2p} e^{\frac{2p(m-1)T\pi^2}{L^2}}} \\ &\leq \frac{\tilde{C}_p}{k^{2p} \lambda^{2p}}. \end{aligned}$$

This implies

$$G(k, \lambda) \leq \frac{\tilde{C}_p}{k^{2(p-1)} \lambda^{2p}}.$$

Hence, we have

$$\begin{aligned} \mathbb{E} [N_k^T] &= \int_0^\infty G(k, \lambda) d\lambda \leq k^{\epsilon-1} + \int_{k^{1-\epsilon}}^\infty \frac{\tilde{C}_p}{k^{2(p-1)} \lambda^{2p}} d\lambda \\ &\leq k^{\epsilon-1} + \frac{\tilde{C}}{(2p-1)k^{2p\epsilon-1-\epsilon}}. \end{aligned}$$

Choosing  $p$  large enough such that  $\frac{1}{\epsilon} < p$ , we have

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |\beta_k(t)| \right] \leq \frac{C_\epsilon}{k^{1-\epsilon}},$$

for some constant  $C_\epsilon$ .

The following proposition gives the pathwise convergence rates of  $W_N$  to  $W$ .

PROPOSITION 1. *Given  $r \in [0, \frac{1}{2})$  and  $A_0 > 0$ . Then, for any  $\alpha \in (0, 1 - 2r)$ , there is a  $\beta > 0$  such that for sufficiently large  $N$  and  $A \geq A_0$ :*

$$\begin{aligned} &\mathbb{P} \left\{ \sup_{t \in [0, T]} |(1 - P_M)W(t)|_{\mathbb{H}_0^r}^2 \leq \frac{1}{\alpha M^\alpha} A^2, M \geq N \right\} \\ &\geq 1 - C_\beta A^{-2([\frac{1}{2\beta}] + 1)} \exp\{-cA^2 N^{2\beta}\}, \end{aligned} \tag{2.6}$$

where  $C_\beta$  and  $c$  are positive constants independent of  $A_0$ ,  $M$  and  $N$  and  $[\frac{1}{2\beta}]$  denotes the biggest integer no larger than  $\frac{1}{2\beta}$ .

*Proof.* By definition, we have

$$|(1 - P_M)W(t, x)|_{\mathbb{H}_0^r}^2 = \sum_{k > M} (\lambda_k^{r/2} \beta_k(t))^2.$$

Define the events  $A_k^\gamma$  ( $\frac{1}{2} < \gamma < 1 - r$ ), ( $k \in \mathbb{N}$ ) to be

$$A_k^\gamma = \left\{ \sup_{0 \leq t \leq T} |\beta_k(t)| \leq \frac{A}{\lambda_k^{(\gamma+r)/2}} \right\}.$$

Choosing  $\epsilon$  small enough in (2.5), we have for sufficiently large  $N$  and  $A \geq A_0$ ,

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |\beta_k(t)| \right] = \frac{C_\epsilon}{k^{1-\epsilon}} < \frac{1}{4} \frac{A}{\lambda_k^{(\gamma+r)/2}}, \quad k \geq N.$$

Hence, by (2.4), for  $N$  big enough,

$$\mathbb{P}\{A_k^\gamma\} \geq 1 - 4 \exp\left\{-\frac{A^2}{2} \lambda_k^{(1-r-\gamma)}\right\}, \quad k \geq N.$$

Since  $\frac{1}{2} < \gamma < 1 - r$ , we have

$$0 < \alpha = 2\gamma - 1 < 1 - 2r, \quad 0 < \beta = 1 - r - \gamma < \frac{1}{2}.$$

We also have

$$\sum_{k > M} \frac{1}{k^{2\gamma}} = \sum_{k > M} \frac{1}{k^{\alpha+1}} \leq \int_M^\infty \frac{1}{x^{\alpha+1}} dx \leq \frac{1}{\alpha M^\alpha}.$$

And for arbitrary constant  $\bar{c} > 0$ , we have

$$\begin{aligned} \sum_{k > N} \exp\{-\bar{c}k^{2\beta}\} &\leq \int_N^\infty \exp\{-\bar{c}x^{2\beta}\} dx \stackrel{y = x^{2\beta}}{=} \frac{1}{2\beta} \int_{N^{2\beta}}^\infty y^{\frac{1}{2\beta}-1} \exp\{-\bar{c}y\} dy \\ &\leq \frac{1}{2\beta} \int_{N^{2\beta}}^\infty y^{[\frac{1}{2\beta}]} \exp\{-\bar{c}y\} dy \\ &= \frac{1}{2\beta} \frac{[\frac{1}{2\beta}]!}{\bar{c}^{[\frac{1}{2\beta}] + 1}} \exp\{-\bar{c}N^{2\beta}\}, \end{aligned}$$

Hence

$$\begin{aligned} &\mathbb{P}\left\{ \sup_{t \in [0, T]} |(1 - P_M)W(t)|_{\mathbb{H}_0^r}^2 \leq \frac{1}{\alpha M^\alpha} A^2, M \geq N \right\} \\ &\geq \mathbb{P}\left\{ \bigcap_{k > N} A_k^\gamma \right\} \geq 1 - \sum_{k > N} \exp\left\{-\frac{A^2}{2} \lambda_k^\beta\right\} \\ &\geq 1 - C_\beta A^{-2([\frac{1}{2\beta}] + 1)} \exp\{-cA^2 N^{2\beta}\}, \end{aligned}$$

where  $C_\beta = \frac{1}{2\beta} \frac{[\frac{1}{2\beta}]!}{\bar{c}^{[\frac{1}{2\beta}] + 1}}$  and  $c = \frac{\pi^2}{2L^2}$ .

By the estimates of the probability in Proposition 1, we can give the following convergence in expectation.

PROPOSITION 2. *Given  $r \in [0, \frac{1}{2})$  and  $1 \leq p < \infty$ . Then for any  $\alpha \in (0, 1 - 2r)$ ,*

$$\left[ \mathbb{E} \sup_{t \in [0, T]} |(1 - P_N)W(t)|_{\mathbb{H}_0^r}^p \right]^{\frac{1}{p}} \leq \frac{C}{N^{\alpha/2}}, \tag{2.7}$$

where  $C$  is a constant independent of  $N$ .

*Proof.* Let

$$Y_T = \sup_{t \in [0, T]} |(1 - P_N)W(t)|_{\mathbb{H}_0^r}.$$

By Proposition 1, we have for  $A \geq A_0$  and  $N$  big enough,

$$\mathbb{P}\{Y_T \geq (\frac{1}{\alpha N^\alpha})^{\frac{1}{2}} A\} \leq C_\beta A^{-2(\lfloor \frac{1}{2\beta} \rfloor + 1)} \exp\{-cA^2 N^\beta\}.$$

And by definition, we have

$$\begin{aligned} \mathbb{E}[Y_T(t)^p] &= \int_0^\infty \mathbb{P}\{Y_k(t) \geq \lambda\} p \lambda^{p-1} d\lambda \\ &\leq \int_0^{\frac{1}{\sqrt{\alpha N^\alpha}}} \mathbb{P}\{Y_k(t) \geq \lambda\} p \lambda^{p-1} d\lambda + \int_{\frac{1}{\sqrt{\alpha N^\alpha}}}^\infty \mathbb{P}\{Y_k(t) \geq \lambda\} p \lambda^{p-1} d\lambda. \end{aligned}$$

Letting  $A_0 = 1$  in (2.6) and  $\lambda = \frac{1}{\sqrt{\alpha N^\alpha}} A$ , we have

$$\begin{aligned} \mathbb{E}[Y_T(t)^p] &\leq \left(\frac{1}{\alpha N^\alpha}\right)^{\frac{p}{2}} \left(1 + \int_1^\infty \mathbb{P}\{Y_k(t)^2 \geq \frac{1}{\alpha N^\alpha} A^2\} p A^{p-1} dA\right) \\ &\leq \left(\frac{1}{\alpha N^\alpha}\right)^{\frac{p}{2}} \left(1 + \int_1^\infty C_\beta p A^{-2(\lfloor \frac{1}{2\beta} \rfloor + 1) + p - 1} \exp\{-cA^2 N^{2\beta}\} dA\right) \\ &\leq \frac{C}{N^{\alpha p/2}}, \end{aligned}$$

for some constant  $C$ .

Borell inequality implies that  $\mathbb{E}\left[\sup_{t \in [0, T]} |\beta_k(t)|^{2p}\right] < \infty$ . Hence, we have

COROLLARY 2.3. *Given  $r \in [0, \frac{1}{2})$  and  $1 \leq p < \infty$ , then*

$$\mathbb{E} \sup_{t \in [0, T]} |W(t)|_{\mathbb{H}_0^r}^p < \infty. \tag{2.8}$$

### 3. The spectral approximation for the SGL equation

Let

$$P_N = \{\varphi(x) : \varphi = \sum_{k \leq N} c_k e_k, c_k \in \mathbb{R}\}.$$

Denote also by  $P_N$  the projection from  $L_0^2$  onto  $P_N$ . The following lemma can be found in [18].

LEMMA 3.1. For  $1 < p < \infty$ , the operator  $P_N$  from  $\mathbb{L}^p$  to  $\mathbb{L}^p$  is uniformly bounded, i.e.  $\exists C_p$  s.t.

$$\|P_N\| \leq C_p, \quad \forall N \geq 1.$$

The spectral approximation, or the semi-discrete Fourier-Galerkin approximation, for the SGL equation has the following form:

$$\begin{cases} u_N(t) \in P_N \cap \mathbb{C}([0, T] \times [0, L]) \\ u_N(t) = e^{\Delta t} P_N u_0 + \int_0^t e^{\Delta(t-s)} F_N(u_N(s)) ds + W_N(t) \\ u_N(0) = P_N u_0 \end{cases} \quad (3.1)$$

where  $F_N(\cdot) = P_N F(\cdot)$  and  $F(u) = \mu u - \lambda u^3$ . By the regularity of  $W_N(t)$  and standard fixed point analysis, it easy to show that (3.1) has a unique continuous solution.

Taking derivatives of both sides of (3.1), we can get

$$\begin{cases} u_N(t) = \Delta u_N(t) + F_N(u_N(t)) - \Delta W_N(t) + dW_N(t)/dt \\ u_N(0) = P_N u_0. \end{cases}$$

Define

$$B_N(t) = W_N(t) - \int_0^t \Delta W_N(s) ds, \quad B_N^k(t) = \langle B_N(t), e_k \rangle_{\mathbb{L}^2}.$$

Direct computation can show that

$$\begin{aligned} \mathbb{E}\{(B_N^k(t_1) - B_N^k(t_2)) \cdot B_N(s)\} &= 0, & s \leq t_1, t_2, \\ \mathbb{E}\{(B_N^k(t) \cdot B_N^j(t))\} &= \delta_{kj} t, & 1 \leq k, j, \leq N. \end{aligned}$$

Then by Levy's characterization of Brownian Motion, we know  $B_N^k(t)$ 's are independent standard Brownian Motions. Then we give the differential form of (3.1).

$$\begin{cases} u_N(t) = \Delta u_N(t) + F_N(u_N(t)) + dB(t)/dt \\ u_N(0) = P_N u_0. \end{cases} \quad (3.2)$$

The analysis of numerical numerical methods for (3.2) can be found in [15] and [14]. Since the coefficients are smooth, numerical solutions of arbitrary orders of accuracy can be given.

**3.1. A priori estimates** Let

$$z(t, x) = e^{\Delta t} u_0(x) + W(t, x). \quad (3.3)$$

Define the integral operator on  $\mathbb{L}^q(\Omega_T)$ ,  $1 \leq q \leq \infty$  such that

$$G(v) = \int_0^t e^{\Delta(t-s)} v(s) ds. \quad (3.4)$$

Then the solution of the SGL equation is a fixed point on space  $\mathbb{C}(\Omega_\infty)$ , such that

$$u = G(F(u)) + z. \quad (3.5)$$

The following lemma shows the smoothing effect of the semi-group  $e^{\Delta t}$ . The proof can be found in ([6]).

LEMMA 3.2. *Suppose  $1 \leq r < 3$ ,  $1 \leq p, q \leq \infty$  with*

$$\frac{1}{p} = \frac{1}{r} + \frac{1}{q} - 1, \tag{3.6}$$

*then  $G$  is a bounded operator from  $\mathbb{L}^q(\Omega_T)$  to  $\mathbb{L}^p(\Omega_T)$ . It is shown in ([6]) that due to the positivity of the coefficients of the nonlinear terms of the SGL equation, for some constant  $C$ ,*

$$|u|_{\mathbb{L}^4} \leq C(|z|_{\mathbb{L}^4} + 1). \tag{3.7}$$

By the same analysis, we can show for some constant  $\tilde{C}$  independent of  $N$

$$|u_N|_{\mathbb{L}^4} \leq \tilde{C}(|z_N|_{\mathbb{L}^4} + 1), \tag{3.8}$$

where

$$z_N(t, x) = W_N(t, x) + e^{\Delta t} P_N u_0(x). \tag{3.9}$$

The following Proposition gives the estimates for higher order norms.

PROPOSITION 3. *Suppose  $4 \leq p < \infty$ , then there exists a polynomial  $P(x)$  such that with probability one,*

$$|u|_{\mathbb{L}^p(\Omega_T)} \leq P(|z|_{\mathbb{L}^p(\Omega_T)}), \tag{3.10}$$

and

$$|u_N|_{\mathbb{L}^p(\Omega_T)} \leq P(|z_N|_{\mathbb{L}^p(\Omega_T)}). \tag{3.11}$$

*Proof.* By Lemma 3.2 and the fact that  $|\mu x - \lambda x^3| \leq \lambda|x|^3 + \eta$  for some constant  $\eta > 0$ , we can get the following estimate for the solution  $u$  of the SGL equation:

$$|u|_{\mathbb{L}^p((\Omega_T))} \leq \tilde{C}(|u|_{\mathbb{L}^{3q}((\Omega_T))}^3 + |z|_{\mathbb{L}^p((\Omega_T))} + 1), \tag{3.12}$$

where  $p$  and  $q$  satisfies condition (3.6) for some  $r$  and  $\tilde{C}$  is a constant which depends on  $p$  and  $q$ . Fix  $r = \frac{8}{3}$  in (3.6), then  $\frac{1}{p} = \frac{1}{q} - \frac{5}{8}$ . Since  $4 \leq p < \infty$ , we have

$$p > \frac{27}{5p+8}p = \frac{27}{24} \cdot 3q.$$

By (3.7) and the fact that  $|u|_{\mathbb{L}^{p1}} \leq C|u|_{\mathbb{L}^{p2}}$  ( $p1 \leq p2$ ), using (3.12) recursively, we get (3.10). By Lemma 3.1 and (3.8), we can repeat the analysis for  $u_N$  and obtain (3.11). Due to the uniformity of Lemma 3.1, the polynomial in (3.11) is independent of  $N$ . Then we can choose an appropriate  $P(\cdot)$  such that both (3.10) and (3.11) are satisfied.

By the Sobolev Embedding theorem, for  $2 \leq p < \infty$ ,  $\mathbb{H}^{\frac{1}{2} - \frac{1}{p}} \subset \mathbb{L}^p$ . By Corollary 2.3 we have the following corollary:

COROLLARY 3.3. *For all  $1 \leq p, q < \infty$ , we have*

$$\mathbb{E} \left[ |u|_{\mathbb{L}^q(\Omega_T)}^p \right] < \infty, \tag{3.13}$$

and

$$\sup_N \mathbb{E} \left[ |u_N|_{\mathbb{L}^q(\Omega_T)}^p \right] < \infty. \tag{3.14}$$

**3.2. Convergence of the spectral approximation** To give the order of the pathwise convergence and convergence in expectation for the spectral approximation of the SGL equation, we need the following version of Gronwall’s inequality. Many inequalities of this type can be found in [10]. The proof for the version to be used can be found in [17].

LEMMA 3.4. *Let  $T, \alpha, \beta, \nu$  be positive constants.  $0 < \nu < 1$ . Then for any continuous function  $f: [0, T] \rightarrow [0, \infty)$  satisfying*

$$f(t) \leq \alpha + \beta \int_0^t (t - s)^{-\nu} f(s) ds, \quad 0 \leq t \leq T, \tag{3.15}$$

we have

$$f(t) \leq C\alpha \cdot \exp\{C\beta^{\frac{1}{1-\nu}} t\}, \tag{3.16}$$

with a positive constant  $C$  which depends on  $\nu$ .

Let  $c > 0$  be the same positive constant as in Proposition 1. We say a random variable  $\xi$  is of finite moments if

$$\mathbb{E}[\xi^n] < \infty, \quad \forall n \in \mathbb{N}. \tag{3.17}$$

The following theorem gives the convergence rates for pathwise convergence and convergence in expectation of the spectral approximation for the SGL equation.

THEOREM 3.5. *Given  $r \in [0, 1/2)$ , for any  $\alpha \in (0, 1 - 2r)$ , there are positive constants  $\beta$  and  $C$  and a random variable  $\xi$  such that*

1. *for sufficiently large  $N$ ,*

$$\begin{aligned} & \mathbb{P}\left\{ \sup_{t \in [0, T]} |u(t) - u_M(t)|_{\mathbb{H}_r^0} \leq \xi \left( |u_0 - u_M(0)|_{\mathbb{H}_r^0} + \frac{1}{M^{\alpha/2}} \right), M \geq N \right\} \\ & \geq 1 - Ce^{-cN^{2\beta}}, \end{aligned} \tag{3.18}$$

2. *for any  $p \geq 1$ ,*

$$\left[ \mathbb{E} \sup_{t \in [0, T]} |u(t) - u_N(t)|_{\mathbb{H}_r^0}^p \right]^{\frac{1}{p}} \leq C \left( |u - u_0|_{\mathbb{H}_r^0} + \frac{1}{N^{\alpha/2}} \right). \tag{3.19}$$

*Proof.* Let  $e_M(t) = u(t) - u_M(t)$ , we have

$$\begin{aligned} e_M(t) &= e^{\Delta t}(u_0 - P_M u_0) + \int_0^t e^{\Delta(t-s)}(F(u(s)) - F_M(u_M(s)))ds + (1 - P_M)W(t) \\ &= e^{\Delta t}e_M(0) + \int_0^t e^{\Delta(t-s)}(F(u(s)) - F_M(u(s)))ds \\ &\quad + \int_0^t e^{\Delta(t-s)}(F_M(u(s)) - F_M(u_M(s)))ds + (1 - P_M)W(t) \\ &\stackrel{\text{def}}{=} I_1 + I_2 + I_3 + I_4. \end{aligned}$$

By definition,

$$|I_1(t)|_{\mathbb{H}_0^r} \leq e^{-\lambda_{M+1}t} |e_M(0)|_{\mathbb{H}_0^r} . \quad (3.20)$$

By the semi-group property of  $e^{\Delta t}$  and (1.3), we have for  $0 < \gamma < 1$ ,

$$\begin{aligned} \|(-\Delta)^\alpha e^{\Delta t}(1 - P_M)\| &\leq \|(-\Delta)^\alpha e^{\Delta(1-\gamma)t}\| \cdot \|e^{\Delta\gamma t}(1 - P_M)\| \\ &\leq M t^{-\alpha} e^{-\delta(1-\gamma)t} \cdot e^{-\lambda_{M+1}\gamma t} . \end{aligned}$$

Let  $\kappa = \delta(1 - \gamma) + \lambda_{M+1}\gamma$ , we have

$$\begin{aligned} |I_2(t)|_{\mathbb{H}_0^r} &\leq \int_0^t \left| (-\Delta)^{r/2} e^{\Delta(t-s)}(1 - P_M)F(u(s)) \right|_{\mathbb{L}^2} ds \\ &\leq \int_0^t M(t-s)^{-r/2} e^{-\kappa(t-s)} |F(u(s))|_{\mathbb{L}^2} ds \\ &\leq M \left( \int_0^\infty \frac{e^{-2\kappa s}}{s^r} ds \right)^{\frac{1}{2}} \left( \int_0^T |F(u(s))|_{\mathbb{L}^2}^2 ds \right)^{\frac{1}{2}} . \end{aligned}$$

And we have for some constant  $C'_2$ ,

$$\int_0^\infty \frac{e^{-2\kappa s}}{s^r} ds \leq \frac{\int_0^\infty s^{-r} e^{-2s} ds}{\kappa^{1-r}} \leq \frac{C'_2}{(M+1)^{2-2r}} .$$

By (3.13) we have for any  $p \geq 1$ , there is some constant  $C''_2$  such that

$$\mathbb{E} \left[ \left( \int_0^T |F(u(s))|_{\mathbb{L}^2}^2 ds \right)^p \right] \leq C \mathbb{E} \left[ (|u|_{L^6(\Omega_T)}^6 + 1)^p \right] \leq C''_2 .$$

This implies that there is a random variable  $\bar{\xi}_2$  of finite moments such that

$$|I_2(t)|_{\mathbb{H}_0^r} \leq \frac{\bar{\xi}_2}{(M+1)^{1-r}} , \quad a.s. \quad (3.21)$$

Thus, for any  $p \geq 1$ , we have some constant  $\bar{C}_2$  such that

$$\mathbb{E} \sup_{t \in [0, T]} |I_2(t)|_{\mathbb{H}_0^r}^p \leq \frac{\bar{C}_2}{(M+1)^{p(1-r)}} . \quad (3.22)$$

For  $I_3$ , let  $\eta = r/2 + 1/2$ , we have

$$|I_3(t)|_{\mathbb{H}_0^r} \leq \int_0^t \left| (-\Delta)^\eta e^{\Delta(t-s)} (-\Delta)^{-1/2} (F_M(u(s)) - F_M(u_M(s))) \right|_{\mathbb{L}^2} ds .$$

Again by (1.3), we have

$$\|(-\Delta)^\eta e^{\Delta(t-s)}\| \leq M'(t-s)^{-\eta} e^{-\delta'(t-s)} .$$

By the Sobolev inequality, we have

$$\begin{aligned} & \left| (-\Delta)^{-1/2} (F_M(u(s)) - F_M(u_M(s))) \right|_{\mathbb{L}^2} \\ &= \sup_{|w|_{\mathbb{L}^2}=1} \langle (-\Delta)^{-1/2} (F(u(s)) - F(u_M(s))), w \rangle_{\mathbb{L}^2} \\ &= \sup_{|w|_{\mathbb{L}^2}=1} \langle F(u(s)) - F(u_M(s)), (-\Delta)^{-1/2} w \rangle_{\mathbb{L}^2} \\ &\leq C' \cdot |F(u(s)) - F(u_M(s))|_{\mathbb{L}^1} \cdot |w|_{\mathbb{L}^2} \\ &\leq C'' (|u(s)|_{\mathbb{L}^4}^2 + |u_M(s)|_{\mathbb{L}^4}^2) \cdot |e_M(s)|_{\mathbb{L}^2} . \end{aligned}$$

For  $1 < p < \infty$ , let  $\frac{1}{p} + \frac{1}{q} = 1$ . The Hölder inequality implies that

$$\begin{aligned} |I_3(t)|_{\mathbb{H}_0^r} &\leq C''' \left( \int_0^t (t-s)^\eta e^{-\delta'(t-s)ds} (|u(s)|_{\mathbb{L}^4}^2 + |u_M(s)|_{\mathbb{L}^4}^2)^q ds \right)^{\frac{1}{q}} \\ &\quad \left( \int_0^t (t-s)^{-\eta} e^{-\delta'(t-s)} |e_M(s)|_{\mathbb{H}_0^r}^p ds \right)^{\frac{1}{p}} . \end{aligned}$$

For  $1 < p' < \frac{1}{\eta}$  and  $\frac{1}{p'} + \frac{1}{q'} = 1$ , we have

$$\begin{aligned} & \int_0^t (t-s)^\eta e^{-\delta'(t-s)} (|u(s)|_{\mathbb{L}^4}^2 + |u_M(s)|_{\mathbb{L}^4}^2)^q ds \\ &\leq \left( \int_0^t (t-s)^{-p'\eta} e^{-p'\delta'(t-s)} ds \right)^{\frac{1}{p'}} \left( \int_0^t (|u(s)|_{\mathbb{L}^4}^2 + |u_M(s)|_{\mathbb{L}^4}^2)^{qq'} ds \right)^{\frac{1}{q'}} \\ &\leq C_3' \left( |u(s)|_{\mathbb{L}^P(\Omega_T)}^P + |u_M(s)|_{\mathbb{L}^P(\Omega_T)}^P + 1 \right)^{\frac{1}{q'}} , \end{aligned}$$

where  $P = \max\{4, 2qq'\}$ . Again by (3.13) and (3.14), we have for some random variable  $\bar{\xi}_3$  of finite moments such that

$$|I_3(t)|_{\mathbb{H}_0^r}^p \leq (\bar{\xi}_3)^p \int_0^t (t-s)^{-\eta} e^{-\delta'(t-s)} |e_M(s)|_{\mathbb{H}_0^r}^p ds . \tag{3.23}$$

Define

$$\begin{aligned} I(s) &= \int_0^s (s-\tau)^{-\eta} e^{-\delta'(s-\tau)} \mathbb{E} \sup_{\mu \in [0, \tau]} |e_M(\mu)|_{\mathbb{H}_0^r}^p d\tau \\ &= \int_0^s \tau^{-\eta} e^{-\delta'\tau} \mathbb{E} \sup_{\mu \in [0, s-\tau]} |e_M(\mu)|_{\mathbb{H}_0^r}^p d\tau , \end{aligned}$$

then  $I(s)$  is monotonically increasing. Hence, we have

$$\begin{aligned} \mathbb{E} \sup_{s \in [0, t]} |I_3(s)|_{\mathbb{H}_0^r}^p &\leq \bar{C}_3 \sup_{s \in [0, t]} \int_0^s (s-\tau)^{-\eta} e^{-\delta'(s-\tau)} \mathbb{E} \sup_{\mu \in [0, \tau]} |e_M(\mu)|_{\mathbb{H}_0^r}^p d\tau \tag{3.24} \\ &\leq \bar{C}_3 \int_0^t (t-s)^{-\eta} e^{-\delta'(t-s)} \mathbb{E} \sup_{\tau \in [0, s]} |e_M(\tau)|_{\mathbb{H}_0^r}^p ds . \end{aligned}$$

By Proposition 1 and Proposition 2, for any  $\alpha \in (0, 1 - 2r)$  and  $N$  large enough, there is a  $\beta$  such that

$$\mathbb{P} \left\{ \sup_{t \in [0, T]} |I_4(t)|_{\mathbb{H}_0^r} \leq \frac{1}{\sqrt{\alpha} M^{\alpha/2}} , M \geq N \right\} \geq 1 - C_\beta \exp\{-cN^{2\beta}\} ,$$

and

$$\left[ \mathbb{E} \sup_{t \in [0, T]} |I_4(t)|_{\mathbb{H}_0^r}^p \right]^{\frac{1}{p}} \leq \frac{C}{N^{\alpha/2}} .$$

Thus, with probability no less than  $1 - C_\beta \exp\{-cN^{2\beta}\}$ , for  $M \geq N$ ,

$$\begin{aligned} |e_M(t)|_{\mathbb{H}_0^r} &\leq e^{-\lambda_{M+1}t} |e_M(0)|_{\mathbb{H}_0^r} + \frac{\bar{\xi}_2}{(M+1)^{1-r}} + \frac{1}{\sqrt{\alpha}M^{\alpha/2}} \\ &\quad + \bar{\xi}_3 \int_0^t (t-s)^{-\eta} e^{-\delta'(t-s)} |e_M(s)|_{\mathbb{H}_0^r} ds . \end{aligned}$$

And

$$\begin{aligned} \mathbb{E} \sup_{s \in [0, t]} |e_N(s)|_{\mathbb{H}_0^r}^p &\leq 3^{p-1} |e_M(0)|_{\mathbb{H}_0^r}^p + 3^{p-1} \left( \frac{\tilde{C}_2}{(N+1)^{p(1-r)}} + \frac{C^p}{N^{\alpha p/2}} \right) \\ &\quad + 3^{p-1} \tilde{C}_3 \int_0^t (t-s)^{-\eta} e^{-\delta'(t-s)} \mathbb{E} \sup_{\tau \in [0, s]} |e_N(\tau)|_{\mathbb{H}_0^r}^p ds . \end{aligned}$$

By Lemma 3.4, we have with probability no less than  $1 - C_2 \exp\{-cN^{2\beta}\}$ , for some random variable  $\xi$ ,

$$|e_M(t)|_{\mathbb{H}_0^r} \leq \xi \left( |e_M(0)|_{\mathbb{H}_0^r} + \frac{1}{M^{\alpha/2}} \right) \quad M \geq N . \tag{3.25}$$

And for some constant  $C'$ ,

$$\mathbb{E} \sup_{t \in [0, T]} |e_N(t)|_{\mathbb{H}_0^r}^p \leq C' \left( |e_N(0)|_{\mathbb{H}_0^r}^p + \frac{1}{N^{\alpha p/2}} \right) . \tag{3.26}$$

Hence

$$\left[ \mathbb{E} \sup_{t \in [0, T]} |e_N(t)|_{\mathbb{H}_0^r}^p \right]^{\frac{1}{p}} \leq C \left( |e_N(0)|_{\mathbb{H}_0^r} + \frac{1}{N^{\alpha/2}} \right) . \tag{3.27}$$

By the Sobolev Embedding theorem, we have the following corollary:

**COROLLARY 3.6.** *Given  $q \in [2, \infty)$ , let  $r = \frac{1}{2} - \frac{1}{q}$ . Then for any  $\alpha \in (0, 2/q)$ , there are positive constants  $\beta$  and  $C$  and a random variable  $\xi$  such that*

1. *for sufficiently large  $N$ ,*

$$\begin{aligned} &\mathbb{P} \left\{ \sup_{t \in [0, T]} |u(t) - u_M(t)|_{\mathbb{L}^q} \leq \xi \left( |u_0 - u_M(0)|_{\mathbb{H}_0^r} + \frac{1}{M^{\alpha/2}} \right) , M \geq N \right\} \\ &\geq 1 - C e^{-cN^{2\beta}} , \end{aligned} \tag{3.28}$$

2. *for any  $p \geq 1$*

$$\left[ \mathbb{E} \sup_{t \in [0, T]} |u(t) - u_N(t)|_{\mathbb{L}^q}^p \right]^{\frac{1}{p}} \leq C \left( |u - u_0|_{\mathbb{H}_0^r} + \frac{1}{N^{\alpha/2}} \right) . \tag{3.29}$$

**4. Generalizations**

Based on the spectral representation of the finite difference method, we believe the analysis here can be extended to the finite difference method for the SGL equation to show its convergence. To give the convergence rate, we need the regularity more than continuity. This can be done by giving the Hölder continuity of the solution  $u$  through the Hölder continuity of the heat kernel and the stochastic convolution.

The same method we have developed for the SGL equation can be generalized to general stochastic PDEs with additive noise. We rewrite (1.1) in the integral form:

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}F(u(s))ds + \int_0^t e^{-A(t-s)}dw(s) . \tag{4.1}$$

Let  $z = e^{-At}u_0 + \int_0^t e^{-A(t-s)}dw(s)$ . Then (4.1) becomes

$$u(t) = \int_0^t e^{-A(t-s)}F(u(s))ds + z(t) . \tag{4.2}$$

Hence, we can discuss the equation as a pathwise integral equation. The same approximation in section 2 for the stochastic convolution can be used for more general noise and the convergence rate can be estimated similarly. We can define the spectral approximation for (4.1):

$$\begin{cases} u_N(t) \in P_N\mathbb{H} \\ u_N(t) = e^{-At}P_Nu_0 + \int_0^t e^{-A(t-s)}F_N(u_N(s))ds + W_N(t) \\ u_N(0) = P_Nu_0 \end{cases} \tag{4.3}$$

where  $F_N(\cdot) = P_NF(\cdot)$  and  $P_N$  is the projection operator from  $\mathbb{H}$  to the space spanned by the first  $N$  eigenvalues of  $A$ .

The interest here is to extend our convergence result to a variety of equations (e.g. Navier-Stokes, Burgers) and more computationally efficient spectral methods (eg. Fourier-Collocation, Legendre, Chebyshev). Since the equation is pathwise defined, we can use the same technique to show the convergence and give the convergence rate of the approximation for the stochastic convolution and hence the convergence of the approximation for the equation.

The difficulty lies in the regularity of the solutions. Usually, the regularity of the solution  $u$  of (4.1) depends on the regularity of the noise  $w$ . Smoother noises imply smoother solutions. Although considered to be the most physical, the space-time white noise sometimes reduces the regularity of the solution into a very unsatisfactory situation compared with the original unperturbed PDEs. For example, consider the following stochastic Ginzburg-Landau equation in two dimension with Dirichlet boundary condition and space-time noise:

$$u_t = \Delta u - u + :u^3: + \dot{w} .$$

Here, to make sense out of the equation, we have to replace  $u^3$  by the wick ordering  $:u^3:$  (see [12] for definition). It can be shown ([12]) that  $u \in \mathbb{H}^{-1}$ . On the other hand, the convergence of the spectral methods needs sufficient regularity of the solutions. For example, for the Fourier collocation  $P_c u$  of  $u \in \mathbb{H}^m$ ,  $m > 1/2$  we have

$$|u - P_c u|_{\mathbb{H}_c^r} \leq CN^{r-m}|u|_{\mathbb{H}^m} .$$

The requirement  $m > 1/2$  is not satisfied by the SGL equation here and most other interesting cases of stochastic PDEs. Hence, future work such as new efficient numerical methods or new interpretation of the equations for stochastic PDEs needs to be done.

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