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ON STABLE CONSTANT MEAN CURVATURE HYPERSURFACES

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Abstract. We study complete non-compact stable constant mean curvature hypersurfaces in a Riemannian manifold of bounded geometry, and prove that there are no nontrivial L^2 harmonic 1-forms on such hypersurfaces. We also show that any smooth map with finite energy from such a hypersurface to a compact manifold with non-positive sectional curvature is homotopic to constant on each compact set. In particular, we obtain some one-end theorems of complete non-compact weakly stable constant mean curvature hypersurfaces in the space forms.

1. Introduction. In [4], Cao, Shen and Zhu proved that a complete immersed stable minimal hypersurface M^n of $\boldsymbol{R}^{n+1}$ with $n \geq 3$ must have only one end. Its strategy was to utilize a result of Schoen and Yau [19] asserting that a complete stable minimal hypersurface of $\boldsymbol{R}^{n+1}$ can not admit a non-constant harmonic function with finite Dirichlet integral. Assuming that M^n has more than one end, they constructed in [4] a non-constant harmonic function with finite Dirichlet integral. According to the work of Li and Tam [15], Li and Wang [16] modified this proof to show that each end of a complete immersed minimal submanifold in $\boldsymbol{R}^{n+p}$ with $n \geq 3$ must be non-parabolic. Due to this connection with harmonic functions, this allows one to estimate the number of ends of the above hypersurface by estimating the dimension of the space of bounded harmonic functions with finite Dirichlet integral [15]. Following the work of Li and Wang [16], Cheng, Cheung and Zhou [8] studied the global behavior of weakly stable hypersurfaces with constant mean curvature, and proved some one-end theorems. In particular, a complete oriented weakly stable minimal hypersurface in $\boldsymbol{R}^{n+1} (n \geq 3)$ must have only one end. Since the exterior differential form of a harmonic function with finite Dirichlet integral is an L^2 harmonic 1-form, the theory of L^2 harmonic forms gives one to study submanifolds in Euclidean space [12, 16, 23]. In this direction related with stable hypersurfaces, there are some known results. For instance, if M is a complete immersed stable minimal hypersurface in $\boldsymbol{R}^{n+1}$, then there exist no nontrivial L^2 harmonic 1-forms on M [17, 18]. Also, Cheng [6] proved that a complete non-compact oriented strongly stable hypersurface M^n with constant mean curvature H in a complete oriented manifold N^{n+1} with bi-Ricci curvature b-Ric_N, satisfying along M

$$b\text{-}\mathrm{Ric}_N(u, v) = \mathrm{Ric}_N(u) + \mathrm{Ric}_N(v) - K_N(u, v) \geq n^2(n-5)H^2/4\,,$$

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admits no nontrivial L^2 harmonic 1-forms.

Let N^{n+1} be an oriented $(n+1)$-dimensional Riemannian manifold and $i: M^n \to N^{n+1}$ be an isometric immersion of a connected orientable n-dimensional manifold M with constant mean curvature H. Denote by H and A the mean curvature and the second fundamental form on M, respectively. It is convenient to introduce the trace-free second fundamental form on M, i.e., $\phi := A - HI$, where I denotes the identity. Thus $|A|^2 = |\phi|^2 + nH^2$. For N^{n+1}, we say the $(n-1)$-th Ricci curvature of N satisfies $\mathrm{Ric}_{(n-1)}(N) \geq c$ if, for all points $x \in N$ and for all n-dimensional subspaces $V \subset T_x(N)$, the curvature tensor R satisfies

$$\sum_{i=1}^{n} \langle R(e_i, v)v, e_i \rangle \geq c,\ v \in V\,,$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis for V. A manifold is called of bounded geometry if its sectional curvatures are not more than some positive constant and its injectivity radius is not less than some positive constant. Obviously, Euclidean space $\boldsymbol{R}^n$, the standard sphere $\boldsymbol{S}^n$ and the hyperbolic space $\boldsymbol{H}^n$ are of bounded geometry.

Let $H^1(L^2(M))$ denote the space of L^2 harmonic 1-forms on M, $H_0^1(M)$ the first de Rham's cohomology group with compact support of M and Δ the Laplacian on M. Throughout this article, we always assume that M is a complete, non-compact, connected Riemannian manifold without boundary. In this case, we will simply say that M is a complete manifold.

Our main results in this paper are stated as follows:

THEOREM 1.1. *Let N^{n+1} be a Riemannian manifold of bounded geometry with* $\mathrm{Ric}_{(n-1)}(N) \geq (n-1)c$ *and M^n a complete strongly stable hypersurface with constant mean curvature H in N^{n+1}. If*

$$(n^3 - 5n^2 - 8n - 4)H^2 \leq 16nc\,,$$

then $H^1(L^2(M)) = 0$, *and M has only one non-parabolic end.*

THEOREM 1.2. *Let N^{n+1} be a Riemannian manifold of bounded geometry with* $\mathrm{Ric}_{(n-1)}(N) \geq (n-1)c$ *and M^n a complete weakly stable hypersurface with constant mean curvature H in N^{n+1}. If*

$$(n^3 - 5n^2 - 8n - 4)H^2 \leq 16nc\,,$$

then M does not admit non-constant bounded harmonic functions with finite Dirichlet integral, and M has only one non-parabolic end.

When N^{n+1} is a space form (i.e., a simply connected complete Riemannian manifold with constant sectional curvature), Theorems 1.1 and 1.2 are extensions of the main results of [6, 8] (see Remark 3.8). In Theorems 1.1 and 1.2, the case $n = 6$ is critical for $c \geq 0$ and differs from the case $n \geq 7$. The reason of the above phenomenon may be that Theorems 1.1 and 1.2 seem to be related to the generalized Bernstein's problem.

THEOREM 1.3. *Let M^n be a complete strongly stable hypersurface with constant mean curvature H in an $(n+1)$-dimensional manifold N with* $\mathrm{Ric}_{(n-1)}(N) \geq (n-1)c \geq 0$ *and W*

be a compact manifold with non-positive curvature. Let $f: M \to W$ *be a smooth map with finite energy. If*

$$(n^3 - 5n^2)H^2 \le 4(2n-1)c\,,$$

then f *is homotopic to constant on each compact set.*

REMARK 1.4. When M is a complete stable minimal hypersurface in a manifold of non-negative curvature, Theorem 1.3 corresponds to [19, Theorem 2].

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2. Preliminary. Let $i: M^n \to N^{n+1}$ be an isometric immersion of an orientable manifold M with constant mean curvature H. We denote by ν the unit normal vector field of M.

DEFINITION 2.1. The immersion i is called weakly stable if

$$\int_M [|\nabla f|^2 - (\mathrm{Ric}(\nu,\nu) + |A|^2)f^2] \ge 0 \tag{1}$$

for any $f \in C_0^\infty(M)$ satisfying $\int_M f = 0$, where ∇f is the gradient of f in the induced metric of M, while i is called strongly stable if (1) holds for any $f \in C_0^\infty(M)$. The immersion i is simply called stable if $H \neq 0$ and i is weakly stable or if $H = 0$ (i.e., M is minimal) and i is strongly stable.

Obviously, a strongly stable constant mean curvature hypersurface is weakly stable. But the converse may not be true. For example, the standard sphere $\boldsymbol{S}^n \subset \boldsymbol{R}^{n+1}$ is weakly stable, but not strongly stable [3].

For the stable hypersurfaces with constant mean curvature H, the stability inequality (1) becomes

$$\int_M [|\nabla f|^2 - (\mathrm{Ric}(\nu,\nu) + |\phi|^2 + nH^2)f^2] \ge 0\,. \tag{2}$$

In this paper, we will discuss the number of ends of hypersurfaces. Now we give some related definitions and results.

DEFINITION 2.2. Let $D \subset M$ be a compact subset of M. An end E of M with respect to D is a connected unbounded component of $M \backslash D$. When we say that E is an end, it is implicitly assumed that E is an end with respect to some compact subset $D \subset M$.

DEFINITION 2.3. A manifold is said to be parabolic if it does not admit a positive Green's function. Conversely, a non-parabolic manifold is one which admits a positive Green's function. An end E of a manifold is said to be non-parabolic if it admits a positive Green's function with Neumann boundary condition on ∂E. Otherwise, it is said to be parabolic.

THEOREM 2.4 ([15]). *Let* M *be a complete manifold. Let* $\mathcal{H}_D^0(M)$ *denote the space of bounded harmonic functions with finite Dirichlet integral. Then the number of non-parabolic ends of* M *is at most the dimension of* $\mathcal{H}_D^0(M)$.

THEOREM 2.5 ([16]). *Let E be an end of a complete manifold. Suppose that, for some $\nu \geq 1$, E satisfies a Sobolev type inequality of the form*

$$\left(\int_E |f|^{2\nu}\right)^{1/\nu} \leq C \int_E |\nabla f|^2 \quad \textit{for any } f \in C_0^1(E)\,.$$

Then E is of finite volume or non-parabolic.

3. Proofs of the theorems. For each $\omega \in H^1(L^2(M))$, the Bochner formula

$$\Delta|\omega|^2 = 2(|\nabla\omega|^2 + \mathrm{Ric}(\omega, \omega)) \tag{3}$$

is well-known. On the other hand, we have

$$\Delta|\omega|^2 = 2(|\omega|\Delta|\omega| + |\nabla|\omega||^2)\,. \tag{4}$$

From (3), (4) and the generalized Kato's inequality $(n/(n-1))|\nabla|\omega||^2 \leq |\nabla\omega|^2$, we obtain

$$|\omega|\Delta|\omega| \geq \mathrm{Ric}(\omega, \omega) + \frac{1}{n-1}|\nabla|\omega||^2\,. \tag{5}$$

In [21], Shiohama and Xu proved the following estimate for the Ricci curvature of a hypersurface M in a Riemannian manifold N^{n+1} with $\mathrm{Ric}_{(n-1)}(N) \geq (n-1)c$:

$$\mathrm{Ric}(M) \geq \frac{n-1}{n}\left(nc + 2nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\sqrt{|A|^2 - nH^2} - |A|^2\right).$$

Applying the above inequality to the traceless second fundamental form ϕ and using the identity $|A|^2 = |\phi|^2 + nH^2$, we get

$$\mathrm{Ric}(M) \geq (n-1)c + (n-1)H^2 - \frac{(n-2)\sqrt{n(n-1)}|\phi||H|}{n} - \frac{(n-1)|\phi|^2}{n}\,. \tag{6}$$

Combining with (5), we obtain

$$\begin{aligned} |\omega|\Delta|\omega| \geq{}& \frac{1}{n-1}|\nabla|\omega||^2 + (n-1)c|\omega|^2 \\ &- \left[\frac{(n-2)\sqrt{n(n-1)}|\phi||H|}{n} + \frac{(n-1)|\phi|^2}{n} - (n-1)H^2\right]|\omega|^2\,. \end{aligned} \tag{7}$$

Observe that if f is a harmonic function with finite Dirichlet integral then its exterior df is an L^2 harmonic 1-form. Moreover, $df = 0$ if and only if f is identically constant. Hence

$$1 \leq \dim \mathcal{H}_D^0(M) \leq \dim H^1(L^2(M)) + 1\,.$$

So we are going to prove $\dim \mathcal{H}_D^0(M) = 1$ by showing $H^1(L^2(M)) = 0$.

PROOF OF THEOREM 1.1. Let $\omega \in H^1(L^2(M))$. For a fixed point $p \in M$ and for $r > 0$, we choose a C^1 cut-off function η satisfying $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_r(p) \subset M$, $\eta \equiv 0$

on $M\backslash B_{2r}(p)$, and $|\nabla\eta| \le 1/r$ on $B_{2r}(p)\backslash B_r(p) \subset M$. Multiplying (7) by η^2 and integrating by parts over M, we get

$$\begin{aligned}
0 &\le \int_M \left(\eta^2|\omega|\Delta|\omega| - \frac{1}{n-1}\eta^2|\nabla|\omega||^2 - (n-1)c\eta^2|\omega|^2\right)\\
&\quad + \int_M \eta^2\left(\frac{(n-2)\sqrt{n(n-1)}|\phi||H|}{n} + \frac{(n-1)|\phi|^2}{n} - (n-1)H^2\right)|\omega|^2\\
&= -2\int_M \eta\langle\nabla\eta, \nabla|\omega|\rangle|\omega| - \frac{n}{n-1}\int_M \eta^2|\nabla|\omega||^2 + \frac{n-1}{n}\int_M \eta^2|\phi|^2|\omega|^2\\
&\quad + \frac{(n-2)\sqrt{n(n-1)}}{n}\int_M |\phi||H|\eta^2|\omega|^2 - (n-1)\int_M (H^2+c)\eta^2|\omega|^2\\
&\le -2\int_M \eta\langle\nabla\eta, \nabla|\omega|\rangle|\omega| - \frac{n}{n-1}\int_M \eta^2|\nabla|\omega||^2 + \frac{n+1}{n}\int_M \eta^2|\phi|^2|\omega|^2 \qquad (8)\\
&\quad + \left[\frac{(n-2)^2(n-1)}{8} - (n-1)\right]\int_M H^2\eta^2|\omega|^2 - (n-1)c\int_M \eta^2|\omega|^2\\
&= -2\int_M \eta\langle\nabla\eta, \nabla|\omega|\rangle|\omega| - \frac{n}{n-1}\int_M \eta^2|\nabla|\omega||^2\\
&\quad + \frac{n+1}{n}\int_M (nc + nH^2 + |\phi|^2)\eta^2|\omega|^2\\
&\quad + \left[\frac{(n-2)^2(n-1)-16n}{8}H^2 - 2nc\right]\int_M \eta^2|\omega|^2,
\end{aligned}$$

where we use the generalized AM-GM inequality.

Choosing $f = \eta|\omega|$ in the stability inequality (2), we obtain

$$\int_M (nc + nH^2 + |\phi|^2)\eta^2|\omega|^2 \le \int_M [\mathrm{Ric}(\nu,\nu) + |\phi|^2 + nH^2]\eta^2|\omega|^2 \le \int_M |\nabla(\eta|\omega|)|^2,$$

where we use that $\mathrm{Ric}_{(n-1)}(N) \ge (n-1)c$ implies $\mathrm{Ric}(N) \ge nc$. Combining the above inequality with (8), we have

$$\begin{aligned}
0 &\le -2\int_M \eta\langle\nabla\eta, \nabla|\omega|\rangle|\omega| - \frac{n}{n-1}\int_M \eta^2|\nabla|\omega||^2 + \frac{n+1}{n}\int_M |\nabla(\eta|\omega|)|^2\\
&\quad + \left[\frac{(n-2)^2(n-1)-16n}{8}H^2 - 2nc\right]\int_M \eta^2|\omega|^2\\
&\le -2\int_M \eta\langle\nabla\eta, \nabla|\omega|\rangle|\omega| + \left[\frac{(n-2)^2(n-1)-16n}{8}H^2 - 2nc\right]\int_M \eta^2|\omega|^2\\
&\quad + \frac{n+1}{n}\int_M \left(|\omega|^2|\nabla\eta|^2 + \eta^2|\nabla|\omega||^2\right) + \frac{2(n+1)}{n}\int_M \eta\langle\nabla\eta, \nabla|\omega|\rangle|\omega| \qquad (9)\\
&\quad - \frac{n}{n-1}\int_M \eta^2|\nabla|\omega||^2\\
&\le \frac{2}{n}\int_M \eta\langle\nabla\eta, \nabla|\omega|\rangle|\omega| + \left[\frac{(n-2)^2(n-1)-16n}{8}H^2 - 2nc\right]\int_M \eta^2|\omega|^2
\end{aligned}$$

$$-\frac{1}{n(n-1)}\int_M \eta^2|\nabla|\omega||^2+\frac{n+1}{n}\int_M |\omega|^2|\nabla\eta|^2 .$$

Using Schwarz inequality, we get

$$2\left|\int_M \eta\langle\nabla\eta,\nabla|\omega|\rangle|\omega|\right|\le \varepsilon\int_M \eta^2|\nabla|\omega||^2+\frac{1}{\varepsilon}\int_M |\omega|^2|\nabla\eta|^2 . \tag{10}$$

From (9), (10) and the inequality $(n^3-5n^2-8n-4)H^2\le 16nc$, we obtain

$$\begin{aligned}\frac{1-(n-1)\varepsilon}{n(n-1)}\int_M \eta^2|\nabla|\omega||^2 &\le \frac{(n^3-5n^2-8n-4)H^2-16nc}{8}\int_M \eta^2|\omega|^2\\ &\quad+\frac{1+(n+1)\varepsilon}{n\varepsilon}\int_M |\omega|^2|\nabla\eta|^2\\ &\le \frac{1+(n+1)\varepsilon}{n\varepsilon}\int_M |\omega|^2|\nabla\eta|^2\\ &\le \frac{1+(n+1)\varepsilon}{n\varepsilon}\frac{1}{r^2}\int_{B_{2r}(p)} |\omega|^2 .\end{aligned}$$

Choosing $\varepsilon>0$ sufficiently small and letting $r\to\infty$, from the above inequality, we get $\nabla|\omega|=0$ on M, i.e., $|\omega|$ is constant. Since $\int_M|\omega|^2<\infty$ and the volume of M is infinite by [8, Proposition 2.1] (that is, each end of M has infinite volume), we have $\omega=0$. Hence $H^1(L^2(M))=0$, and $\dim\mathcal{H}^0_D(M)=1$. Due to Theorem 2.4, we conclude that M has only one non-parabolic end. □

PROOF OF THEOREM 1.2. Suppose that $f\colon M\to \boldsymbol{R}$ is a non-constant bounded harmonic function with finite Dirichlet integral. From the proof of [8, Theorem 3.1], we know $\int_M|\nabla f|=\infty$ and can choose a compactly supported piecewise smooth function $\varphi=\psi(t_0,a,R)|\nabla f|$ satisfying $\int_M\varphi=0$, where

$$\psi(t_0,a,R)=\begin{cases}1 & \text{on } B_p(a)\,,\\ (a+R-x)/R & \text{on } B_p(a+R)\backslash B_p(a)\,,\\ t_0(a+R-x)/R & \text{on } B_p(a+2R)\backslash B_p(a+R)\,,\\ -t_0 & \text{on } B_p(a+2R+b)\backslash B_p(a+2R)\,,\\ t_0(x-(a+3R+b))/R & \text{on } B_p(a+3R+b)\backslash B_p(a+2R+b)\,,\\ 0 & \text{on } M\backslash B_p(a+3R+b)\,,\end{cases}$$

a, b are positive constants and t_0 is a constant satisfying $0\le t_0\le 1$.

For any harmonic function f, we have

$$|\nabla f|\Delta|\nabla f|\ge \mathrm{Ric}(\nabla f,\nabla f)+\frac{1}{n-1}|\nabla|\nabla f||^2 . \tag{11}$$

Multiplying (11) by $\psi(t_0,a,R)^2$ and using the same argument as Theorem 1.1, one can show that M does not admit non-constant bounded harmonic functions with finite Dirichlet integral, and that M has only one non-parabolic end. □

REMARK 3.1. The assertions of Theorems 1.1 and 1.2 still hold when N^{n+1} is a Riemannian manifold with $\mathrm{Ric}_{(n-1)}(N)\ge(n-1)c\ge 0$ and the assumption of bounded geometry in Theorems 1.1 and 1.2 is removed. In the proofs of Theorems 1.1 and 1.2, the assumption of bounded geometry is used to guarantee that the volume of M is infinite. The condition $c \geq 0$ implies that the volume of M is infinite, i.e., $\mathrm{vol}(M) = \infty$. In fact, choosing $f = \eta$ in the stability inequality (2), we obtain

$$\int_M |A|^2\eta^2 \leq \int_M [\mathrm{Ric}(\nu, \nu) + |A|^2]\eta^2 \leq \int_M |\nabla\eta|^2 ,$$

which gives

$$\int_M |A|^2 \leq \frac{\mathrm{vol}(M)}{r^2} . \tag{12}$$

If $\mathrm{vol}(M) < \infty$, by letting $r \to \infty$, we obtain that M is totally geodesic. Thus the Ricci curvatures of M are non-negative. Since every complete non-compact Riemannian manifold with non-negative Ricci curvature has infinite volume [22, Theorem 7], we get $\mathrm{vol}(M) = \infty$, a contraction.

COROLLARY 3.2. *Let $M^n (n \leq 6)$ be a complete strongly stable constant mean curvature hypersurface in the hyperbolic space $\boldsymbol{H}^{n+1}$. If*

$$H^2 \geq \frac{16n}{5n^2 + 8n + 4 - n^3} ,$$

then $H^1(L^2(M)) = 0$, and M has only one end. Moreover $H_0^1(M) = 0$.

PROOF. When $3 \leq n \leq 6$, for each end E of M, we have

$$0 < C' \int_E f^2 \leq \int_E (-n + |\phi|^2 + nH^2) f^2 \leq \int_E |\nabla f|^2 ,$$

i.e.,

$$\int_E f^2 \leq C \int_E |\nabla f|^2.$$

By Theorem 2.5 and [8, Proposition 2.1], E must be non-parabolic. According to Theorem 1.1, M has only one end. When $n = 2$, from [9, Theorem 1.4], we see that M has only one end. Hence $H_0^1(M) = 0$ by [5, Lemma 2.3]. □

REMARK 3.3. Corollary 3.2 extends the result of [8] that any complete non-compact weakly stable hypersurface with constant mean curvature H in the hyperbolic space $\boldsymbol{H}^{n+1}$, $n = 3, 4$, with $H^2 \geq 10/9, 7/4$, respectively, has only one end.

COROLLARY 3.4. *If M is a complete stable minimal immersed hypersurface in $\boldsymbol{R}^{n+1}$, then $H^1(L^2(M)) = 0$, and M has only one end. Moreover $H_0^1(M) = 0$.*

PROOF. When $n \geq 3$, by using the Sobolev inequality of [13], we have

$$\left(\int_E f^{2n/(n-2)}\right)^{(n-2)/n} \leq C \int_E |\nabla f|^2 .$$

By Theorem 2.5 and [8, Proposition 2.1], E must be non-parabolic. According to Theorem 1.1, M has only one end. When $n = 2$, from [10, Theorem 1.2], M has only one end. Hence $H_0^1(M) = 0$ by [5, Lemma 2.3]. $\square$

Note that Palmer [18] proved the first part of Corollary 3.4. Applying the arguments as in the proofs of Corollaries 3.2 and 3.4, we get the following corollary.

COROLLARY 3.5. *If $M^n (n \leq 6)$ is a complete strongly stable constant mean curvature hypersurface in $\boldsymbol{R}^{n+1}$, then $H^1(L^2(M)) = 0$, and M has only one end. Moreover $H_0^1(M) = 0$.*

REMARK 3.6. It is known that there is no complete non-compact weakly stable hypersurfaces with nonzero constant mean curvature in $\boldsymbol{R}^{n+1}$ for $n \leq 4$ [7, 9].

It is known that any 3- and 4-dimensional complete weakly stable hypersurface with constant mean curvature in a manifold of non-negative sectional curvature must be compact [7, 8]. Hence by Theorem 1.1, we obtain the following corollary.

COROLLARY 3.7. *Let M^n be a complete strongly stable hypersurface with constant mean curvature H in the standard sphere $\boldsymbol{S}^{n+1}$. If*

(1) $5 \leq n \leq 6$, *or*

(2) $n \geq 7$ *and* $H^2 \leq 16n/(n^3 - 5n^2 - 8n - 4)$,

then $H^1(L^2(M)) = 0$, and M has only one end. Moreover $H_0^1(M) = 0$.

REMARK 3.8. If "strongly stable" is replaced by "weakly stable" in Corollaries 3.2, 3.5 and 3.7, then M does not admit non-constant bounded harmonic functions with finite Dirichlet integral, and M has only one end. Corollaries 3.2, 3.5 and 3.7 can be considered as generalizations of some main results in [6, 8, 17, 18].

From the main theorem in [20], we see that if M is an oriented complete hypersurface with constant mean curvature and finite total curvature in $\boldsymbol{R}^{n+1}$, then M must be minimal. A theorem due to Anderson [1] says that the n-dimensional complete minimal submanifold with only one end and finite total curvature in $\boldsymbol{R}^{n+p}$ for $n \geq 3$ is an affine space. Hence by the above results, [9, Theorem 1.3] and Remark 3.8, we have the following corollary.

COROLLARY 3.9. *If M^n is a complete weakly stable constant mean curvature hypersurface in $\boldsymbol{R}^{n+1}$ with finite total curvature, then M is a hyperplane.*

THEOREM 3.10. *Let M^n be a complete strongly stable hypersurface with constant mean curvature H in an $(n+1)$-dimensional manifold N of bounded geometry with* $\mathrm{Ric}_{(n-1)}(N) \geq (n-1)c$. *If*

$$(n^3 - 5n^2)H^2 \leq 4(2n-1)c\,,$$

then any harmonic map with finite energy from M to a manifold with non-positive curvature is constant.

PROOF. Let W be an m-dimensional Riemannian manifold and let $f: M \to W$ be a harmonic map. Take local orthonormal frames $\{e_i\}_{i=1}^n$ and $\{\bar{e}_\alpha\}_{\alpha=1}^m$ of M and W, respectively, and denote by $\{\omega_i\}_{i=1}^n$ and $\{\theta_\alpha\}_{\alpha=1}^m$ the corresponding dual frames and by $\{\omega_{ij}\}_{i,j=1}^n$ and $\{\theta_{\alpha\beta}\}_{\alpha,\beta=1}^m$ the corresponding connection forms, respectively. Then define $f_{\alpha i}$ by $f^*(\theta_\alpha) = \sum_i f_{\alpha i}\omega_i$ and the energy density $e(f)$ by $e(f) = \sum_{\alpha,i} f_{\alpha i}^2$. By the Bochner type formula for harmonic maps between Riemannian manifolds [11] and the non-positivity of the sectional curvature of W, we have

$$\frac{1}{2}\Delta e \geq \sum_{\alpha,i,j} f_{\alpha ij}^2 + \mathrm{Ric}(M)e\,, \tag{13}$$

where $f_{\alpha ij}$ is defined by $\sum_j f_{\alpha ij}\omega_j = df_{\alpha i} + \sum_\beta f_{\beta i} f^*(\theta_{\beta\alpha}) + \sum_j f_{\alpha j}\omega_{ji}$. Schoen and Yau [19] gave the following estimate:

$$\sum_{\alpha,i,j} f_{\alpha ij}^2 \geq \left(1 + \frac{1}{2nm}\right)|\nabla\sqrt{e}|^2\,. \tag{14}$$

It follows from (13) and (14) that

$$\frac{1}{2}\Delta e \geq \left(1 + \frac{1}{2nm}\right)|\nabla\sqrt{e}|^2 + \mathrm{Ric}(M)e. \tag{15}$$

Multiplying (15) by η^2 and integrating by parts over M, we get

$$\left(1 + \frac{1}{2nm}\right)\int_M |\nabla\sqrt{e}|^2\eta^2 + \int_M \mathrm{Ric}(M)e\eta^2 \leq -2\int_M \sqrt{e}\eta\langle\nabla\sqrt{e}, \nabla\eta\rangle\,.$$

Combining with (6) and using the same argument as Theorem 1.1, one can prove that any harmonic map with finite energy from M to a manifold with non-positive curvature is constant. □

REMARK 3.11. By Remark 3.1, the assertion of Theorem 3.10 still holds when N^{n+1} is a Riemannian manifold with $\mathrm{Ric}_{(n-1)}(N) \geq (n-1)c \geq 0$.

Combining the existence theorem of harmonic map [19] with Remark 3.11, we obtain Theorem 1.3.

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