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RIESZ TRANSFORM CHARACTERIZATION OF HARDY SPACES ASSOCIATED WITH CERTAIN LAGUERRE EXPANSIONS

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Abstract. In this paper we prove Riesz transform characterizations for Hardy spaces associated with certain systems of Laguerre functions.

1. Introduction and statement of the results. Denote the Laguerre polynomials of order $\alpha > -1$ by

$$L_n^\alpha(x) = (n!)^{-1} e^x x^{-\alpha} \left(\frac{d}{dx}\right)^n (e^{-x} x^{n+\alpha}), \quad n = 0, 1, 2, \dots .$$

In this paper we consider the following two systems of Laguerre functions on $(0, \infty)$

$$\varphi_n^\alpha(x) = \sqrt{2}\, c_{n,\alpha}\, e^{-x^2/2} x^{\alpha+1/2} L_n^\alpha(x^2), \quad n = 0, 1, 2 \dots , \tag{1}$$

$$\mathfrak{L}_n^\alpha(x) = c_{n,\alpha}\, e^{-x/2} x^{\alpha/2} L_n^\alpha(x), \quad n = 0, 1, 2, \dots , \tag{2}$$

where $c_{n,\alpha} = (\Gamma(n+1)/\Gamma(n+1+\alpha))^{1/2}$. It is well known that, for every $\alpha > -1$, each of the systems $\{\varphi_n^\alpha\}_{n=0}^\infty$ and $\{\mathfrak{L}_n^\alpha\}_{n=0}^\infty$ is complete and orthonormal on $L^2((0,\infty), dx)$. Moreover, these functions are eigenvectors, respectively, of the differential operators

$$L_\alpha = \frac{1}{2}\left(-\frac{d^2}{dy^2} + y^2 + \frac{1}{y^2}\left(\alpha^2 - \frac{1}{4}\right)\right), \quad \mathfrak{L}_\alpha = -\left(x\frac{d^2}{dx^2} + \frac{d}{dx} - \left(\frac{x}{4} + \frac{\alpha^2}{4x}\right)\right),$$

satisfying

$$L_\alpha \varphi_n^\alpha = (2n+\alpha+1)\varphi_n^\alpha \quad \text{and} \quad \mathfrak{L}_\alpha(\mathfrak{L}_n^\alpha) = (n + (\alpha+1)/2)\, \mathfrak{L}_n^\alpha .$$

As in [6, 7], the operators L_α and $\mathfrak{L}_\alpha$ can be factored as

$$L_\alpha = \frac{1}{2} D_\alpha^* D_\alpha + \alpha + 1 \quad \text{and} \quad \mathfrak{L}_\alpha = \delta_\alpha^* \delta_\alpha + \frac{\alpha+1}{2},$$

where

$$D_\alpha = \frac{d}{dx} + x - \frac{\alpha + 1/2}{x} \quad \text{and} \quad \delta_\alpha = \sqrt{x}\frac{d}{dx} + \frac{1}{2}\left(\sqrt{x} - \frac{\alpha}{\sqrt{x}}\right),$$

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and where D_α^* and δ_α^* denote, respectively, the formal adjoint operators to D_α and δ_α in $L^2((0,\infty), dx)$. Corresponding *Riesz transforms* are defined in $L^2((0,\infty), dx)$ by

$$R_\alpha = D_\alpha L_\alpha^{-1/2} \quad \text{and} \quad \mathfrak{R}_\alpha = \delta_\alpha \mathfrak{L}_\alpha^{-1/2},$$

that is, they act on the basis elements by

$$R_\alpha \varphi_n^\alpha = -\frac{2\sqrt{n}}{\sqrt{2n+\alpha+1}} \varphi_{n-1}^{\alpha+1}, \quad \mathfrak{R}_\alpha \mathfrak{L}_n^\alpha = -\frac{\sqrt{n}}{\sqrt{n+(\alpha+1)/2}} \mathfrak{L}_{n-1}^{\alpha+1}. \tag{3}$$

There exist kernels $R_\alpha(x,y)$ and $\mathfrak{R}_\alpha(x,y)$ such that

$$R_\alpha f(x) = \lim_{\varepsilon\to 0} \int_{0,\,|x-y|>\varepsilon}^\infty R_\alpha(x,y) f(y) dy, \quad \mathfrak{R}_\alpha f(x) = \lim_{\varepsilon\to 0} \int_{0,\,|x-y|>\varepsilon}^\infty \mathfrak{R}_\alpha(x,y) f(y) dy.$$

One can easily deduce from (1), (2) and (3) that these kernels are related by

$$\mathfrak{R}_\alpha(x,y) = 2^{-3/2}(xy)^{-1/4} R_\alpha(\sqrt{x}, \sqrt{y}), \quad x, y \in (0,\infty). \tag{4}$$

Riesz tranforms for Laguerre systems were defined and studied by Nowak and Stempak [7], and by Harboure, Torrea and Viviani [6], who proved that R_α for $\alpha \geq -1/2$ and $\mathfrak{R}_\alpha$ for $\alpha \geq 0$ extend as bounded linear operators on $L^p(0,\infty)$ when $1 < p < \infty$ and are of weak type (1,1). Our goal in the present paper is to characterize the spaces

$$H^1_{\mathrm{Riesz}}(L_\alpha) = \{f \in L^1(0,\infty)\ ;\ \|R_\alpha f\|_{L^1} < \infty\} \quad \text{for} \quad \alpha > -1/2,$$

and

$$H^1_{\mathrm{Riesz}}(\mathfrak{L}_\alpha) = \{f \in L^1(0,\infty)\ ;\ \|\mathfrak{R}_\alpha f\|_{L^1} < \infty\} \quad \text{for} \quad \alpha > 0.$$

In [3], the second-named author considered Hardy spaces $H^1_{\max}(L_\alpha)$ and $H^1_{\max}(\mathfrak{L}_\alpha)$ defined by means of the maximal functions associated with the semigroups generated by $-L_\alpha$ and $-\mathfrak{L}_\alpha$, respectively. To be more precise, if

$$W_t^\alpha(x,y) = \sum_{n=0}^\infty e^{-(2n+\alpha+1)t} \varphi_n^\alpha(x) \varphi_n^\alpha(y), \quad \mathfrak{W}_t^\alpha(x,y) = \sum_{n=0}^\infty e^{-t(n+(\alpha+1)/2)} \mathfrak{L}_n^\alpha(x) \mathfrak{L}_n^\alpha(y)$$

denote the integral kernels of the semigroups $\{e^{-tL_\alpha}\}_{t>0}$ and $\{e^{-t\mathfrak{L}_\alpha}\}_{t>0}$, we say that a function f in $(0,\infty)$ belongs to $H^1_{\max}(L_\alpha)$ when the maximal function

$$W_*^\alpha f(x) = \sup_{t>0} \left| \int_0^\infty W_t^\alpha(x,y) f(y) dy \right|$$

belongs to $L^1(0,\infty)$. Then we set $\|f\|_{H^1_{\max}(L_\alpha)} = \|W_*^\alpha f\|_{L^1}$. Analogously, we define the maximal function $\mathfrak{W}_*^\alpha$, the space $H^1_{\max}(\mathfrak{L}_\alpha)$ and the norm $\|\cdot\|_{H^1_{\max}(\mathfrak{L}_\alpha)}$. It was proved in [3] that the spaces $H^1_{\max}(L_\alpha)$, $\alpha > -1/2$, and $H^1_{\max}(\mathfrak{L}_\alpha)$, $\alpha > 0$, admit atomic decompositions. The notion of atom for these spaces depends on the following auxiliary functions

$$\rho_{L_\alpha}(x) = \frac{1}{8}\min(x, 1/x) \quad \text{and} \quad \rho_{\mathfrak{L}_\alpha}(x) = \frac{1}{8}\min(x,1).$$

A measurable function $b : (0, \infty) \to \boldsymbol{C}$ is said to be an $H^1(L_\alpha)$-atom if there exists a ball $B = B(y_0, R) = \{y \in (0, \infty); |y_0 - y| < R\}$ with $R \le \rho_{L_\alpha}(y_0)$ such that

$$\operatorname{supp} b \subset B\,, \quad \|b\|_\infty \le |B|^{-1} \quad \text{and}$$

$$\text{if } R \le \rho_{L_\alpha}(y_0)/2 \quad \text{then} \quad \int b(y)dy = 0\,.$$

The space $H^1_{\mathrm{at}}(L_\alpha)$ consists of all measurable functions f on $(0, \infty)$ of the form

$$f = \sum_{j=1}^{\infty} \lambda_j b_j\,,$$

where b_j are $H^1(L_\alpha)$-atoms, $\lambda_j \in \boldsymbol{C}$ and $\sum_{j=1}^\infty |\lambda_j| < \infty$. The norm in $H^1_{\mathrm{at}}(L_\alpha)$ is defined by

$$\|f\|_{H^1_{\mathrm{at}}(L_\alpha)} = \inf \sum_{j=1}^{\infty} |\lambda_j|\,,$$

where the infimum is taken over all decompositions $f = \sum_{j=1}^\infty \lambda_j b_j$, where b_j are $H^1(L_\alpha)$-atoms and $\lambda_j \in \boldsymbol{C}$. Similarly we define the space $H^1_{\mathrm{at}}(\mathfrak{L}_\alpha)$ and the norm $\|\ \|_{H^1_{\mathrm{at}}(\mathfrak{L}_\alpha)}$, the only difference being that the function $\rho_{\mathfrak{L}_\alpha}$ replaces the function ρ_{L_α} in the definition of $H^1(\mathfrak{L}_\alpha)$-atoms. The main result in [3] was to show that

$$H^1_{\max}(L_\alpha) = H^1_{\mathrm{at}}(L_\alpha) \text{ for } \alpha > -1/2 \quad \text{and} \quad H^1_{\max}(\mathfrak{L}_\alpha) = H^1_{\mathrm{at}}(\mathfrak{L}_\alpha) \text{ for } \alpha > 0\,,$$

with equivalence of the corresponding norms. Our goal in this paper is to characterize these spaces by means of the Riesz transforms R_α and $\mathfrak{R}_\alpha$. More precisely, we shall prove the following theorems.

THEOREM 1.1. *If $\alpha > -1/2$, then $H^1_{\mathrm{Riesz}}(L_\alpha) = H^1_{\mathrm{at}}(L_\alpha)$. Moreover, there exists $C > 0$ such that*

$$C^{-1}\|f\|_{H^1_{\mathrm{at}}(L_\alpha)} \le \|R_\alpha f\|_{L^1} + \|f\|_{L^1} \le C\|f\|_{H^1_{\mathrm{at}}(L_\alpha)}\,. \tag{5}$$

THEOREM 1.2. *If $\alpha > 0$, then $H^1_{\mathrm{Riesz}}(\mathfrak{L}_\alpha) = H^1_{\mathrm{at}}(\mathfrak{L}_\alpha)$. Moreover, there exists $C > 0$ such that*

$$C^{-1}\|f\|_{H^1_{\mathrm{at}}(\mathfrak{L}_\alpha)} \le \|\mathfrak{R}_\alpha f\|_{L^1} + \|f\|_{L^1} \le C\|f\|_{H^1_{\mathrm{at}}(\mathfrak{L}_\alpha)}\,. \tag{6}$$

2. Hardy spaces $H^1(L_\alpha)$ associated with Laguerre operators L_α. In the present section, we shall prove Theorem 1.1. To do this, we recall the equivalence between Riesz and atomic definitions for the Hardy space associated with the Hermite operator,

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right),$$

which were established in [4]. First we let

$$\rho_H(y) = (1 + |y|)^{-1}\,. \tag{7}$$

It is easily seen that there exist constants $C, c > 0$ such that

$$c\rho_H(x)(1+|x-y|/\rho_H(x))^{-1} \le \rho_H(y) \le C\rho_H(x)(1+|x-y|/\rho_H(x))^{1/2}. \tag{8}$$

A function $a : \boldsymbol{R} \to \boldsymbol{C}$ is an $H^1(H)$-atom if there exists a ball $B = B(y_0, R) = \{y \in \boldsymbol{R}; |y-y_0| < R\}$ with $R \le \rho_H(y_0)$ such that

$$\operatorname{supp} a \subset B, \qquad \|a\|_{L^\infty} \le |B|^{-1} \quad \text{and}$$
$$\text{if } R \le \rho_H(y_0)/2 \quad \text{then} \quad \int a(y)dy = 0.$$

The atomic Hardy space $H^1_{\mathrm{at}}(H)$ and the norm $\|\ \|_{H^1_{\mathrm{at}}(H)}$ are defined in the standard way. On the other hand, a Riesz transform R^H can be defined in $L^2(\boldsymbol{R})$ by

$$R^H = \left(\frac{d}{dx} + x\right)H^{-1/2},$$

motivated by the factorization of the Hermite operator

$$H = -\frac{1}{4}\left[\left(\frac{d}{dx}+x\right)\left(\frac{d}{dx}-x\right) + \left(\frac{d}{dx}-x\right)\left(\frac{d}{dx}+x\right)\right].$$

To obtain a kernel expression for R^H, recall first the Mehler formula for Hermite functions (cf. [10, Lemma 1.1.1]), which asserts that the integral kernel $W_t^H(x,y)$ of the Hermite semigroup $\{e^{-tH}\}_{t>0}$ is given by

$$W_t^H(x,y) = \left[\frac{e^{-t}}{\pi(1-e^{-2t})}\right]^{1/2} \exp\left(-\frac{1}{2}\left(\frac{1+e^{-2t}}{1-e^{-2t}}\right)(x^2+y^2) + 2xy\frac{e^{-t}}{1-e^{-2t}}\right) \tag{9}$$

when $t > 0$ and $x, y \in \boldsymbol{R}$. Using the formula $H^{-1/2} = \pi^{-1/2}\int_0^\infty e^{-tH}t^{-1/2}dt$, we can express the Riesz transform R^H as a principal value singular integral operator of the form

$$R^H(f)(x) = \lim_{\varepsilon\to 0}\int_{y\in\boldsymbol{R}\,:\,|x-y|>\varepsilon} R^H(x,y)f(y)dy,$$

with the kernel given by

$$\begin{aligned} R^H(x,y) &= \frac{1}{\sqrt{\pi}}\int_0^\infty \left(\frac{d}{dx}+x\right)W_t^H(x,y)\frac{dt}{\sqrt{t}} \\ &= \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{d}{dx}W_t^H(x,y)\frac{dt}{\sqrt{t}} + \frac{1}{\sqrt{\pi}}\int_0^\infty xW_t^H(x,y)\frac{dt}{\sqrt{t}} \\ &= R_1^H(x,y) + R_2^H(x,y). \end{aligned} \tag{10}$$

It is not difficult to prove using (9) and (10) that

$$\sup_{y\in\boldsymbol{R}}\int_{-\infty}^\infty |R_2^H(x,y)|dx < \infty, \quad \sup_{x\in\boldsymbol{R}}\int_{-\infty}^\infty |R_2^H(x,y)|\,dy < \infty \tag{11}$$

(see Section 4). Therefore, denoting $R_2^H = xH^{-1/2}$, we have

$$\|R_2^H f\|_{L^1(\boldsymbol{R})} \le C\|f\|_{L^1(\boldsymbol{R})} \tag{12}$$

(see also [2, Theorem 4.5]). It was proved by Thangavelu [9] that the operator R^H is bounded on $L^p(\boldsymbol{R})$ for $1 < p < \infty$. Moreover, Theorem 1.2 of Zhong [11] asserts that the operator $R_1^H = (d/dx)H^{-1/2}$ is a Calderón-Zygmund operator, hence it is of weak type (1,1) (see also [8] for a proof based on analysis of the Melher kernel). The above facts could also be deduced from the following lemma.

LEMMA 2.1. *Let $\psi \in C_c^\infty(-2^{-4}, 2^{-4})$ be such that $\psi(x) = 1$ for $|x| < 2^{-5}$. Then there exists a constant $c_0 \neq 0$ and a kernel $h(x, y)$ such that*

$$R^H(x, y) = \frac{c_0}{x-y}\psi\left(\frac{x-y}{\rho_H(x)}\right) + h(x, y)\,, \tag{13}$$

$$\sup_{y\in\boldsymbol{R}}\int_{-\infty}^{\infty}|h(x, y)|dx + \sup_{x\in\boldsymbol{R}}\int_{-\infty}^{\infty}|h(x, y)|dy < \infty\,. \tag{14}$$

This lemma is known, but a self-contained proof based on analysis of the Mehler kernel will be presented in Section 4. We set

$$H^1_{\mathrm{Riesz}}(H) = \{f \in L^1(\boldsymbol{R})\ ;\ \|R^H f\|_{L^1(\boldsymbol{R})} < \infty\}\,.$$

In view of (12), an L^1-function f belongs to $H^1_{\mathrm{Riesz}}(H)$ if and only if $(d/dx)H^{-1/2}f$ belongs to $L^1(\boldsymbol{R})$. From this remark and the results in [4], it follows that

$$H^1_{\mathrm{Riesz}}(H) = H^1_{\mathrm{at}}(H)$$

and there exists a constant $C > 0$ such that

$$C^{-1}\|f\|_{H^1_{\mathrm{at}}(H)} \le \|R^H f\|_{L^1} + \|f\|_{L^1} \le C\|f\|_{H^1_{\mathrm{at}}(H)}\,. \tag{15}$$

Having established the Riesz and atomic characterizations of the Hardy space associated with the Hermite operator, we continue our preparation for the proof of Theorem 1.1.

For a function f defined on $(0, \infty)$, we denote $R^H_{\mathrm{loc}}f = R^H_{1,\mathrm{loc}}f + R^H_{2,\mathrm{loc}}f$, where

$$R^H_{j,\mathrm{loc}}f(x) = \lim_{\varepsilon\to 0}\int_{x/2,\ |x-y|>\varepsilon}^{2x} R^H_j(x, y)f(y)dy, \quad x > 0,\ j = 1, 2\,.$$

PROPOSITION 2.2. *For $f \in L^1(0, \infty)$, let f_o denote its odd extension. Then $R_1^H f_o \in L^1(\boldsymbol{R})$ if and only if $R^H_{1,\mathrm{loc}}f$ is in $L^1(0, \infty)$. Moreover, there exists $C > 0$ such that*

$$\|R_1^H f_o - R^H_{1,\mathrm{loc}}f\|_{L^1(0,\infty)} \le C\|f\|_{L^1(0,\infty)}\,.$$

PROOF. Set $r = r(t) = e^{-t} \in (0, 1)$. According to (9) and (10), we have

$$\begin{aligned} R_1^H(x, y) =& \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{\sqrt{r}\,(2ry - (1+r^2)x)}{(1-r^2)^{3/2}} \\ &\times \exp\left(-\frac{1+r^2}{2(1-r^2)}(x^2+y^2) + \frac{2r}{1-r^2}xy\right)\frac{dt}{\sqrt{t}}\,. \end{aligned} \tag{16}$$

Note that $\|R_1^H f_o\|_{L^1(\boldsymbol{R})} = 2\|R_1^H f_o\|_{L^1(0,\infty)}$, because $R_1^H f_o$ is an even function. Moreover,

$$R_1^H f_o(x) = \lim_{\varepsilon\to 0}\int_{0,\,|x-y|>\varepsilon}^{\infty} \big(R_1^H(x,y) - R_1^H(x,-y)\big) f(y)dy\,, \quad \text{a.e. } x \in (0,\infty)\,.$$

Further,

$$\begin{aligned} R_1^H f_o(x) - R_{1,\mathrm{loc}}^H f(x) &= \int_0^{x/2} (R_1^H(x,y) - R_1^H(x,-y)) f(y)dy \\ &\quad + \int_{2x}^{\infty} (R_1^H(x,y) - R_1^H(x,-y)) f(y)dy \\ &\quad - \int_{x/2}^{2x} R_1^H(x,-y) f(y)dy \\ &= \sum_{j=1}^{3} T_j(f)(x), \;\; a.e. \;\; x \in (0,\infty)\,. \end{aligned} \tag{17}$$

It suffices to show that the operators $T_j,\ j = 1,2,3$, are bounded on $L^1((0,\infty),dx)$. To deal with T_1 and T_2, we estimate the difference $D^H(x,y) = |R_1^H(x,y) - R_1^H(x,-y)|$ for $x, y > 0$. By (16)

$$\begin{aligned} D^H(x,y) &\le C\int_0^{\infty} \frac{\sqrt{r}\,x}{(1-r^2)^{3/2}}\left(\exp\left(\frac{2r}{1-r^2}xy\right) - \exp\left(-\frac{2r}{1-r^2}xy\right)\right) \\ &\qquad \times \exp\left(-\frac{1+r^2}{2(1-r^2)}(x^2+y^2)\right)\frac{dt}{\sqrt{t}} \\ &\quad + C\int_0^{\infty} \frac{\sqrt{r}\,y}{(1-r^2)^{3/2}} \exp\left(-\frac{1+r^2}{2(1-r^2)}(x^2+y^2)\right) \\ &\qquad \times \exp\left(\frac{2r}{1-r^2}xy\right)\frac{dt}{\sqrt{t}}\,. \end{aligned} \tag{18}$$

Applying the mean value theorem in the first integral, we can assert that

$$\begin{aligned} D^H(x,y) &\le C\int_0^{\infty} \frac{\sqrt{r}}{(1-r^2)^{3/2}}\left(\frac{rx^2y}{1-r^2} + y\right)\exp\left(-\frac{1+r^2}{2(1-r^2)}(x^2+y^2)\right) \\ &\qquad \times \exp\left(\frac{2r}{1-r^2}xy\right)\frac{dt}{\sqrt{t}} \\ &= C\int_0^{\infty} \frac{\sqrt{r}}{(1-r^2)^{3/2}}\left(\frac{rx^2y}{1-r^2} + y\right)\exp\left(-\frac{1+r^2}{2(1-r^2)}(x-y)^2\right) \\ &\qquad \times \exp\left(-\frac{1-r}{1+r}xy\right)\frac{dt}{\sqrt{t}}\,. \end{aligned} \tag{19}$$

It is now not difficult to verify using (19) that

$$D^H(x,y) \le \begin{cases} Cyx^{-2} & \text{for } x > 2y\,, \\ Cy^{-1} & \text{for } 2x < y\,. \end{cases} \tag{20}$$

The estimate (20) easily implies $\|T_1 f\|_{L^1(0,\infty)} + \|T_2 f\|_{L^1(0,\infty)} \le C\|f\|_{L^1(0,\infty)}$. Moreover, from (16), we conclude

$$|R_1^H(x,-y)| \le C\left(xe^{-cx^2}\int_1^\infty e^{-t}dt + x\int_0^1 \frac{1}{t^2}e^{-cx^2/t}dt\right) \le \frac{C}{y} \quad \text{for } x/2 < y < 2x\,.$$

Hence T_3 is a bounded operator from $L^1(0,\infty)$ into itself. □

PROPOSITION 2.3. *Let $\alpha > -1/2$, $f \in L^1(0,\infty)$ and f_o be the odd extension of f to $\boldsymbol{R}$. Then $R_\alpha f$ is in $L^1(0,\infty)$ if and only if $R^H f_o$ is in $L^1(\boldsymbol{R})$. Moreover, there exists $C > 0$ such that*

$$C^{-1}(\|f_o\|_{L^1(\boldsymbol{R})} + \|R^H f_o\|_{L^1(\boldsymbol{R})}) \le \|f\|_{L^1(0,\infty)} + \|R_\alpha f\|_{L^1(0,\infty)}$$

and

$$\|f\|_{L^1(0,\infty)} + \|R_\alpha f\|_{L^1(0,\infty)} \le C(\|f_o\|_{L^1(\boldsymbol{R})} + \|R^H f_o\|_{L^1(\boldsymbol{R})})\,.$$

PROOF. According to [1, Lemma 2.13], we have

$$\begin{aligned} &|R_\alpha(x,y)| \le Cx^{\alpha+3/2}y^{-(\alpha+5/2)} \quad \text{for } 0 < 2x < y < \infty\,, \\ &|R_\alpha(x,y)| \le Cy^{\alpha+1/2}x^{-(\alpha+3/2)} \quad \text{for } 0 < y < x/2\,, \end{aligned} \tag{21}$$

and

$$|R_\alpha(x,y) - R^H(x,y)| \le \frac{C}{y}\left(1 + \frac{(xy)^{1/4}}{|x-y|^{1/2}}\right) \quad \text{for } 0 < x/2 < y < 2x\,.$$

Each of the Hardy operators

$$H_\alpha(g)(x) = x^{-\alpha-3/2}\int_0^x y^{\alpha+1/2}g(y)dy\,, \quad x > 0$$

and

$$H^\alpha(g)(x) = x^{\alpha+1/2}\int_x^\infty y^{-\alpha-3/2}g(y)dy\,, \quad x > 0$$

are bounded on $L^1(0,\infty)$ when $\alpha > -1/2$. Moreover, the operator N defined by

$$Nf(x) = \int_{x/2}^{2x} \frac{1}{y}\left(1 + \frac{(xy)^{1/4}}{|x-y|^{1/2}}\right)f(y)dy$$

is also bounded in $L^1(0,\infty)$. Hence, by (21), (11) and Proposition 2.2, we obtain

$$\begin{aligned} &\|R_\alpha f - R^H f_o\|_{L^1(0,\infty)} \\ &\quad \le \|R_\alpha f - R^H_{\mathrm{loc}} f\|_{L^1(0,\infty)} + \|R^H_{\mathrm{loc}} f - R^H f_o\|_{L^1(0,\infty)} \\ &\quad \le C(\|N|f|\,\|_{L^1(0,\infty)} + \|H^{\alpha+1}|f|\,\|_{L^1(0,\infty)} + \|H_\alpha|f|\,\|_{L^1(0,\infty)}) \\ &\qquad + \|R^H_{1,\mathrm{loc}} f - R^H_1 f_o\|_{L^1(0,\infty)} + \|R^H_{2,\mathrm{loc}} f\|_{L^1(0,\infty)} + \|R^H_2 f_o\|_{L^1(0,\infty)} \\ &\quad \le C\|f\|_{L^1(0,\infty)}\,. \end{aligned}$$ □

The next elementary lemma will be used below.

LEMMA 2.4. *Let $b : (0,\infty) \to \boldsymbol{C}$ be an $H^1(L_\alpha)$-atom. Then, its odd extension b_o satisfies*

$$\|b_o\|_{H^1_{\mathrm{at}}(H)} \le 36 .$$

PROOF. Let $B = B(y, R) \subset (0,\infty)$ be a ball associated with b, that is, $R \le \rho_{L_\alpha}(y)$, $\operatorname{supp} b \subset B$ and $\|b\|_\infty \le |B|^{-1}$. Moreover, $\int b(y)dy = 0$ if $R \le \rho_{L_\alpha}(y)/2$. In this last case, since $\rho_{L_\alpha}(y) \le \rho_H(y)/2$, the function $b(x)$ (extended as 0 when $x \le 0$) is an $H^1(H)$-atom, and hence so is $-b(-x)$. Thus $\|b_o\|_{H^1_{\mathrm{at}}(H)} \le 2$.

Suppose now that $\rho_{L_\alpha}(y)/2 < R \le \rho_{L_\alpha}(y)$. We distinguish two cases. If $y \in (0, 8/9)$ then

$$\operatorname{supp} b_o \subset B(0, y+R) \subset B(0, 9y/8) \equiv B_o .$$

Since $\int_{\boldsymbol{R}} b_o = 0$ and $\|b_o\|_\infty \le \rho_{L_\alpha}(y)^{-1} = 18/|B_o|$, it follows that $b_o/18$ is an $H^1(H)$-atom associated with the ball B_o, and hence $\|b_o\|_{H^1_{\mathrm{at}}(H)} \le 18$. In the second case, i.e. $y > 8/9$, we may regard $b/18$ as an $H^1(H)$-atom associated with the ball $B(y, \rho_H(y))$, since

$$\operatorname{supp} b \subset B(y, \rho_H(y)) \quad \text{and} \quad \|b\|_\infty \le (2R)^{-1} \le 18|B(y, \rho_H(y))|^{-1} .$$

Similarly, $b(-x)/18$ is an $H^1(H)$-atom associated with the ball $B(-y, \rho_H(-y))$. We conclude that $\|b_o\|_{H^1_{\mathrm{at}}(H)} \le 36$, establishing the lemma. □

PROOF OF THEOREM 1.1. Assume that f is in $H^1_{\mathrm{at}}(L_\alpha)$. Then f can be written as $\sum_j c_j b_j$, where b_j are $H^1(L_\alpha)$-atoms and $\sum_j |c_j| \sim \|f\|_{H^1_{\mathrm{at}}(L_\alpha)}$. By the previous lemma, the odd extension f_o of f belongs to $H^1_{\mathrm{at}}(H)$ and $\|f_o\|_{H^1_{\mathrm{at}}(H)} \le 36\|f\|_{H^1_{\mathrm{at}}(L_\alpha)}$. Applying Proposition 2.3 and using (15), we obtain

$$\|R_\alpha f\|_{L^1(0,\infty)} \le C(\|f_o\|_{L^1(\boldsymbol{R})} + \|R^H f_o\|_{L^1(\boldsymbol{R})}) \le C'\|f_o\|_{H^1_{\mathrm{at}}(H)} \le C''\|f\|_{H^1_{\mathrm{at}}(L_\alpha)} .$$

To prove the converse, assume that f is in $H^1_{\mathrm{Riesz}}(L_\alpha)$. Again, using Proposition 2.3 combined with (15), we obtain $f_o \in H^1_{\mathrm{Riesz}}(H) = H^1_{\mathrm{at}}(H)$ and

$$\|f_o\|_{H^1_{\mathrm{at}}(H)} \le C(\|f_o\|_{L^1(\boldsymbol{R})} + \|R^H f_o\|_{L^1(\boldsymbol{R})}) \le C(\|f\|_{L^1(0,\infty)} + \|R_\alpha f\|_{L^1(0,\infty)}) .$$

Hence $f_o(x) = \sum_j c_j a_j(x)$, where a_j are $H^1(H)$-atoms and $\sum_j |c_j| \sim \|f_o\|_{H^1_{\mathrm{at}}(H)}$. Letting $b_j = a_j\big|_{(0,\infty)}$, one easily verifies the inequality $\|b_j\|_{H^1_{\mathrm{at}}(L_\alpha)} \le C$. Thus f is in $H^1_{\mathrm{at}}(L_\alpha)$ and $\|f\|_{H^1_{\mathrm{at}}(L_\alpha)} \le C'(\|f\|_{L^1(0,\infty)} + \|R_\alpha f\|_{L^1(0,\infty)})$. □

REMARK 2.5. Using a similar analysis based on a comparison of the kernels $W_t^\alpha(x, y)$ and $W_t^H(x, y)$ (see [1, Lemma 2.11]), one can prove that $W_*^H f_o$ belongs to $L^1(\boldsymbol{R})$ if and only if $W_*^\alpha f$ belongs to $L^1(0,\infty)$ and $\|f_o\|_{L^1(\boldsymbol{R})} + \|W_*^H f_o\|_{L^1(\boldsymbol{R})} \sim \|f\|_{L^1(0,\infty)} + \|W_*^\alpha f\|_{L^1(0,\infty)}$.

3. Hardy spaces $H^1(\mathfrak{L}_\alpha)$ associated with Laguerre operators $\mathfrak{L}_\alpha$. In this section we prove Theorem 1.2. The proof is based on the following estimates for the kernel $\mathfrak{R}_\alpha(x, y)$.

PROPOSITION 3.1. *Let ψ be as in Lemma* 2.1. *Then, for every $\alpha > 0$, there exists a kernel $K(x, y)$ such that*

$$\mathfrak{R}_\alpha(x, y) = \frac{c_0}{\sqrt{2}(x - y)}\psi\left(\frac{x - y}{\rho_{\mathfrak{L}_\alpha}(x)}\right) + K(x, y), \quad x, y \in (0, \infty)\,, \tag{22}$$

$$\sup_{y>0}\int_0^\infty |K(x, y)|dx < \infty\,, \tag{23}$$

where c_0 is the constant from (13).

PROOF. Set

$$K(x, y) = \mathfrak{R}_\alpha(x, y) - \frac{c_0}{\sqrt{2}(x - y)}\psi\left(\frac{x - y}{\rho_{\mathfrak{L}_\alpha}(x)}\right). \tag{24}$$

If $x < y/4$ or $y < x/4$, then $K(x, y) = \mathfrak{R}_\alpha(x, y)$. From (4) and (21), we conclude

$$|K(x, y)| \le \begin{cases} Cx^{(\alpha+1)/2}y^{-(\alpha+3)/2} & \text{if } 4x < y < \infty\,, \\ Cy^{\alpha/2}x^{-(\alpha+2)/2} & \text{if } 0 < y < x/4\,. \end{cases} \tag{25}$$

Hence

$$\sup_{y>0}\left(\int_0^{y/4} |K(x, y)|dx + \int_{4y}^\infty |K(x, y)|dx\right) < \infty\,. \tag{26}$$

In order to deal with the kernel $K(x, y)$ in the local part $y/4 \le x \le 4y$, we set

$$E(x, y) = \mathfrak{R}_\alpha(x, y) - 2^{-3/2}(xy)^{-1/4}R^H(\sqrt{x}, \sqrt{y})\,,$$

$$G(x, y) = 2^{-3/2}\left((xy)^{-1/4}\frac{c_0}{\sqrt{x} - \sqrt{y}}\psi\left(\frac{\sqrt{x} - \sqrt{y}}{\rho_H(\sqrt{x})}\right) - \frac{2c_0}{x - y}\psi\left(\frac{x - y}{\rho_{\mathfrak{L}_\alpha}(x)}\right)\right).$$

Then, by (4) and Lemma 2.1, we have

$$K(x, y) = E(x, y) + 2^{-3/2}(xy)^{-1/4}h(\sqrt{x}, \sqrt{y}) + G(x, y)\,. \tag{27}$$

According to (21), we get

$$|E(x, y)| \le C\frac{(xy)^{-1/4}}{\sqrt{y}}\left(1 + \frac{(xy)^{1/8}}{|\sqrt{x} - \sqrt{y}|^{1/2}}\right) \le C\frac{1}{y}\left(1 + \frac{\sqrt{x}}{|x - y|^{1/2}}\right) \tag{28}$$

for $y/4 \le x \le 4y$. Trivially, using (28) and (14), we obtain

$$\int_{y/4}^{4y} (|E(x, y)| + (xy)^{-1/4}|h(\sqrt{x}, \sqrt{y})|)dx \le C\,. \tag{29}$$

The proof will be complete if we show the inequality

$$\int_{y/4}^{4y} |G(x, y)|dx \le C\,. \tag{30}$$

Let us note that

$$G(x, y) = \frac{2^{-3/2}c_0}{x - y}\left[\frac{\sqrt{x} + \sqrt{y}}{(xy)^{1/4}}\psi\left(\frac{x - y}{(\sqrt{x} + \sqrt{y})\rho_H(\sqrt{x})}\right) - 2\psi\left(\frac{x - y}{\rho_{\mathfrak{L}_a}(x)}\right)\right]. \tag{31}$$

If $y > 10$, $y/4 \le x \le 4y$ and $|x-y| > 1$, then $G(x,y) = 0$. If $y > 10$, $y/4 < x < 4y$ and $|x-y| \le 1$, then, by the mean value theorem, $|G(x,y)| \le C$. Thus (30) is satisfied for $y > 10$. If $0 < y \le 10$ and $y/4 \le x \le 4y$, then applying the mean value theorem we deduce $|G(x,y)| \le Cy^{-1}$ and, consequently, (30) holds. □

Before we turn to the proof of Theorem 1.2, we state some results from the theory of local Hardy spaces [5]. Fix $l > 0$. We say that a function b is an atom for the local Hardy space $\mathbf{h}^1_l(\boldsymbol{R})$ if there exists a ball $B(y_0, R)$ with $R < l$ such that $\operatorname{supp} b \subset B(y_0, R)$, $\|b\|_\infty \le (2R)^{-1}$, and if $R \le l/2$, then $\int b(y)\,dy = 0$. A function f belongs to the space $\mathbf{h}^1_l$ if there exist a sequence b_j of $\mathbf{h}^1_l$-atoms and $\lambda_j \in \boldsymbol{C}$ with $\sum_j |\lambda_j| < \infty$ such that

$$f = \sum_j \lambda_j b_j \,. \tag{32}$$

The atomic norm in $\mathbf{h}^1_l$ is defined in a standard way, that is, $\|f\|_{\mathbf{h}^1_l} = \inf \sum_j |\lambda_j|$, where the infimum is taken over all decompositions (32). Moreover, if $f \in \mathbf{h}^1_l$ and $\operatorname{supp} f \subset B(y_0, l)$, then there exists a decomposition of f as in (32) such that $\operatorname{supp} b_j \subset B(y_0, 10l/9)$ and $\sum_j |\lambda_j| \le C\|f\|_{\mathbf{h}^1_l}$. We define a local Hilbert transform

$$\mathcal{H}_l f(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{c_0}{\sqrt{2}(x-y)} \psi\left(\frac{x-y}{l}\right) f(y)dy\,,$$

where c_0 and ψ are as in Lemma 2.1. The following result was actually proved in [5]. There exists a constant $C > 0$ independent of l such that

$$C^{-1}\|f\|_{\mathbf{h}^1_l} \le \|\mathcal{H}_l f\|_{L^1} + \|f\|_{L^1} \le C\|f\|_{\mathbf{h}^1_l}\,. \tag{33}$$

PROOF OF THEOREM 1.2. Since $\mathfrak{R}_\alpha$ maps continuously $L^1(0,\infty)$ into the space of distributions, to prove the second inequality in (6), it suffices to verify that there exists a constant $C > 0$ such that, for every $H^1(\mathfrak{L}_\alpha)$-atom b, one has

$$\|\mathfrak{R}_\alpha b\|_{L^1} \le C\,. \tag{34}$$

Let b be an $H^1(\mathfrak{L}_\alpha)$-atom with associated ball $B(y_0, R)$. Clearly, letting $l = \rho_{\mathfrak{L}_\alpha}(y_0)$, we see that b is also an $\mathbf{h}^1_l$-atom. By Proposition 3.1,

$$\begin{aligned}\mathfrak{R}_\alpha b(x) &= \int K(x,y)b(y)dy + \mathcal{H}_l b(x) \\ &\quad + \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \frac{c_0}{\sqrt{2}(x-y)} \left(\psi\left(\frac{x-y}{\rho_{\mathfrak{L}_\alpha}(x)}\right) - \psi\left(\frac{x-y}{l}\right)\right) \chi_{B(y_0,l)}(y) b(y)dy\,.\end{aligned} \tag{35}$$

The kernel

$$U(x,y) = \frac{c_0}{\sqrt{2}(x-y)} \left(\psi\left(\frac{x-y}{\rho_{\mathfrak{L}_\alpha}(x)}\right) - \psi\left(\frac{x-y}{l}\right)\right) \chi_{B(y_0,l)}(y)\,,$$

as a function of (x,y), is supported by $B(y_0, 3l) \times B(y_0, l)$. Moreover, $|U(x,y)| \le Cl^{-1}$, which implies $\sup_{y>0} \int |U(x,y)|dx < \infty$. Therefore, (34) holds by applying (23) and (33).

We now turn to prove the first inequality in (6). We define the intervals $\{I_j\}_{j\in\mathbf{Z}}$, $I_j = (\beta_j, \beta_{j+1})$, $\beta_j = (9/8)^j$ for $j \le 1$, and $\beta_j = 1 + j/8$ for $j \ge 1$. Set $l_j = \rho_{\mathfrak{L}_\alpha}(\beta_j)$. Let η_j be a family of smooth functions such that

$$0 \le \eta_j \le 1\,, \quad \operatorname{supp} \eta_j \subset I_j^*, \quad \Big|\frac{d}{dx}\eta_j(x)\Big| \le C l_j^{-1}\,, \quad \sum_j \eta_j(x) = 1 \ \text{ for } x > 0\,, \tag{36}$$

where $I_j^* = [\beta_{j-1}, \beta_{j+2}]$. Set $I_j^{**} = [\beta_{j-2}, \beta_{j+3}]$. Then $\sum_j \chi_{I_j^{**}} \le 5$. Fix $f \in L^1(0,\infty)$ such that $\|\mathfrak{R}_\alpha f\|_{L^1} < \infty$. We shall verify that

$$\sum_j \|\mathcal{H}_{l_j}(\eta_j f)\|_{L^1} \le C(\|\mathfrak{R}_\alpha f\|_{L^1} + \|f\|_{L^1}) \tag{37}$$

with a constant $C > 0$ independent of f. To this end, note that

$$\begin{aligned} \mathcal{H}_{l_j}(\eta_j f)(x) &= \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} (\eta_j(y) - \eta_j(x)) \frac{c_0}{\sqrt{2}(x-y)} \psi\Big(\frac{x-y}{l_j}\Big) f(y)dy \\ &\quad + \eta_j(x)\mathcal{H}_{l_j} f(x) \\ &= \Xi_j f(x) + \eta_j(x)\mathcal{H}_{l_j} f(x)\,. \end{aligned} \tag{38}$$

Observe that the kernel

$$\Big|(\eta_j(y) - \eta_j(x)) \frac{c_0}{\sqrt{2}(x-y)} \psi\Big(\frac{x-y}{l_j}\Big)\Big|\,,$$

as a function of (x, y), is supported by $I_j^{**} \times I_j^{**}$ and bounded by Cl_j^{-1}. Since each $y > 0$ belongs to at most 5 intervals I_j^{**}, and $|I_j^{**}| \sim l_j$, we can easily obtain

$$\sum_j \int |\Xi_j f(x)| dx \le C\|f\|_{L^1}\,. \tag{39}$$

Now we shall deal with $\eta_j(x)\mathcal{H}_{l_j} f(x)$, defined by

$$\begin{aligned} \eta_j(x)\mathcal{H}_{l_j} f(x) &= \int \eta_j(x)\Big[\psi\Big(\frac{x-y}{l_j}\Big) - \psi\Big(\frac{x-y}{\rho_{\mathfrak{L}_\alpha(x)}}\Big)\Big]\frac{c_0}{\sqrt{2}(x-y)} f(y)dy \\ &\quad + \eta_j(x)\mathfrak{R}_\alpha f(x) - \eta_j(x)\int K(x,y) f(y)\,dy\,. \end{aligned} \tag{40}$$

The integral kernel

$$\Big|\eta_j(x)\Big[\psi\Big(\frac{x-y}{l_j}\Big) - \psi\Big(\frac{x-y}{\rho_{\mathfrak{L}_\alpha}(x)}\Big)\Big]\frac{c_0}{\sqrt{2}(x-y)}\Big|\,,$$

as a function of (x, y), is supported by $I_j^* \times I_j^{**}$ and bounded by Cl_j^{-1}. Hence

$$\sup_{y>0} \int_0^\infty \sum_j \Big|\eta_j(x)\Big(\psi\Big(\frac{x-y}{l_j}\Big) - \psi\Big(\frac{x-y}{\rho_{\mathfrak{L}_\alpha}(x)}\Big)\Big)\frac{c_0}{\sqrt{2}(x-y)}\Big| dx < \infty\,. \tag{41}$$

Using (40) and (41), we obtain

$$\sum_j \|\eta_j \mathcal{H}_{l_j} f\|_{L^1} \le C(\|f\|_{L^1} + \|\mathfrak{R}_\alpha f\|_{L^1})\,, \tag{42}$$

which combined with (38), (39) and (36) gives (37). Having (37) already proved, we are in a position to complete the proof of the first inequality in (6). Applying (37) together with the results from the theory of local Hardy spaces stated in this section, we have

$$f = \sum_j (\eta_j f) = \sum_j \Big(\sum_i \lambda_{ij} a_{ij} \Big), \tag{43}$$

where a_{ij} are $\mathbf{h}^1_{l_j}$-atoms supported by I_j^{**}, and $\sum_{ij} |\lambda_{ij}| \le C(\|\mathfrak{R}_\alpha f\|_{L^1} + \|f\|_{L^1})$. The proof will be complete once we observe that each of these atoms is either an $H^1(\mathfrak{L}_\alpha)$-atom, or can be written as a sum of at most 20 such atoms. Indeed, fix an $h^1_{l_j}$-atom a supported in I_j^{**}. Then, for some $0 < R_0 < l_j$ and $y_0 \in I_j^{**}$ we have $\operatorname{supp} a \subset B(y_0, R_0) \subset I_j^{**}$, $\|a\|_\infty \le (2R_0)^{-1}$, and if $R_0 \le l_j/2$ then also $\int a(x)dx = 0$. Notice that, by construction,

$$\rho_{\mathfrak{L}_\alpha}(y) \le 2\rho_{\mathfrak{L}_\alpha}(y'), \quad \text{for all} \quad y, y' \in I_j^{**} = [\beta_{j-2}, \beta_{j+3}].$$

If $R_0 \le l_j/2 = \rho_{\mathfrak{L}_\alpha}(\beta_j)/2$ then $\int a = 0$ and $R_0 \le \rho_{\mathfrak{L}_\alpha}(y_0)$, and therefore a is also an $H^1(\mathfrak{L}_\alpha)$-atom. If $R_0 > l_j/2$, then

$$I_j^{**} = \bigcup_{k=0}^{4} I_{j-2+k} \quad \text{with} \quad |I_{j-2+k}| = \rho_{\mathfrak{L}_\alpha}(\beta_{j-2+k}),$$

and using again $\rho_{\mathfrak{L}_\alpha}(\beta_{j+2}) \le 2\rho_{\mathfrak{L}_\alpha}(\beta_j)$ we see that

$$\|a\chi_{I_{j-2+k}}\|_\infty \le (2R_0)^{-1} \le \rho_{\mathfrak{L}_\alpha}(\beta_j)^{-1} \le 2|I_{j-2+k}|^{-1}.$$

Hence, each piece $a\chi_{I_{j-2+k}}/4$ is an $H^1(\mathfrak{L}_\alpha)$-atom for the ball $B(\beta_{j-2+k}, \rho_{\mathfrak{L}_\alpha}(\beta_{j-2+k}))$ and, consequently, $\|a\|_{H^1_{\mathrm{at}}(\mathfrak{L}_\alpha)} \le 20$. □

4. Proof of (11) and Lemma 2.1. During the proof we set $r = e^{-t} \in (0, 1)$. We can rewrite (9) as

$$W_t^H(x, y) = \frac{\sqrt{r}}{\sqrt{\pi(1-r^2)}} \exp\Big(-\frac{1}{2}\Big(\frac{1+r^2}{1-r^2}\Big)|x-y|^2\Big) \exp\Big(-\frac{1-r}{1+r}xy\Big), \tag{44}$$

for all $x, y \in \boldsymbol{R}$. A simple computation using (44) or (9) gives

$$W_t^H(x, y) \le \frac{\sqrt{r}}{\sqrt{\pi(1-r^2)}} \exp\Big(-\frac{1}{4}\Big(\frac{1+r^2}{1-r^2}\Big)|x-y|^2\Big). \tag{45}$$

Let us note that, for every $N > 0$, there exists a constant C_N such that

$$W_t^H(x, y) \le C_N \frac{e^{-t/3}}{\sqrt{1-r^2}} \Big(1 + \frac{t}{\rho_H(x)^2}\Big)^{-N}. \tag{46}$$

Indeed, if $|x - y| > |x|/2$, then

$$\begin{aligned} W_t^H(x, y) &\le \frac{e^{-t/2}}{\sqrt{\pi(1-r^2)}} \exp\Big(-\frac{1}{8}\Big(\frac{1+r^2}{1-r^2}\Big)x^2\Big) \\ &\le C_N \frac{e^{-t/3}}{\sqrt{1-r^2}} \Big(1 + \frac{t}{\rho_H(x)^2}\Big)^{-N}. \end{aligned} \tag{47}$$

If $|x-y| \leq |x|/2$, then $xy \sim x^2$ and, using (44), we get

$$W_t^H(x,y) \leq C \frac{e^{-t/2}}{\sqrt{1-r^2}} \exp(-c(1-r)x^2) \leq C_N \frac{e^{-t/3}}{\sqrt{1-r^2}} \left(1+\frac{t}{\rho_H(x)^2}\right)^{-N}. \tag{48}$$

Applying (45) and (46) combined with the fact that $W_t^H(x,y) = W_t^H(y,x)$, we obtain

$$W_t^H(x,y) \leq C_N \frac{e^{-t/3}}{\sqrt{1-e^{-2t}}} \exp\left(-\frac{|x-y|^2}{12(1-e^{-2t})}\right)\left(1+\frac{t}{\rho(x)^2}\right)^{-N}\left(1+\frac{t}{\rho(y)^2}\right)^{-N}. \tag{49}$$

We are now in a position to prove (11). If $|x-y| \leq C\rho_H(y)$, then by (10) and (49) we have

$$\begin{aligned} |R_2^H(x,y)| &\leq C_N \bigg(\int_0^{|x-y|^2} |x| \left(\frac{t}{|x-y|^2}\right)^N \frac{dt}{t} + \int_{|x-y|^2}^{C^2\rho_H(y)^2} |x| \frac{dt}{t} \\ &\qquad + \int_{C^2\rho_H(y)^2}^{\infty} |x| \left(\frac{\rho_H(y)^2}{t}\right)^N \frac{dt}{t}\bigg) \\ &\leq C_N \left(|x| + |x| \ln\left(\frac{C\rho_H(y)}{|x-y|}\right)\right). \end{aligned} \tag{50}$$

If $|x-y| \geq C\rho_H(y)$, then we use again (49) and get

$$\begin{aligned} |R_2^H(x,y)| &\leq C_N \bigg(\int_0^{C^2\rho_H(y)^2} |x| \left(\frac{t}{|x-y|^2}\right)^N \frac{dt}{t} \\ &\qquad + \int_{C^2\rho_H(y)^2}^{|x-y|^2} |x| \left(\frac{t}{|x-y|^2}\right)^N \left(\frac{t}{\rho_H(y)^2}\right)^{-2N} \frac{dt}{t} \\ &\qquad + \int_{|x-y|^2}^{\infty} |x| \left(\frac{\rho_H(y)^2}{t}\right)^N \frac{dt}{t}\bigg) \\ &\leq C_N \frac{|x|\rho_H(y)^{2N}}{|x-y|^{2N}} \\ &\leq C_N \left(\frac{|x-y|\rho_H(y)^{2N}}{|x-y|^{2N}} + \frac{|y|\rho_H(y)^{2N}}{|x-y|^{2N}}\right). \end{aligned} \tag{51}$$

Now the first inequality in (11) is a consequence of (50) and (51). Similarly to (50) and (51), we also conclude that

$$|R_2^H(x,y)| \leq \begin{cases} C(|x| + |x| \ln(C\rho_H(x)/|x-y|)) & \text{for } |x-y| \leq C\rho_H(x), \\ C_N|x|\rho_H(x)^N/|x-y|^N & \text{for } |x-y| > C\rho_H(x), \end{cases} \tag{52}$$

from which we easily obtain the second inequality in (11).

Having (11) already established, we now turn to prove Lemma 2.1. By (44),

$$\begin{aligned}\frac{\partial}{\partial x}W_t^H(x,y) = &-\frac{\sqrt{r}}{\sqrt{\pi(1-r^2)}}\frac{1+r^2}{1-r^2}(x-y)\exp\Big(-\frac{1}{2}\Big(\frac{1+r^2}{1-r^2}\Big)|x-y|^2\Big)\\ &\times\exp\Big(-\frac{1-r}{1+r}xy\Big)\\ &-y\frac{1-r}{1+r}\frac{\sqrt{r}}{\sqrt{\pi(1-r^2)}}\exp\Big(-\frac{1}{2}\Big(\frac{1+r^2}{1-r^2}\Big)|x-y|^2\Big)\\ &\times\exp\Big(-\frac{1-r}{1+r}xy\Big).\end{aligned}\tag{53}$$

From (53) we deduce that, for $|x-y| > C\rho_H(y)$, we have

$$\begin{aligned}\Big|\frac{\partial}{\partial x}W_t^H(x,y)\Big| \le\, & C_N\Big(\frac{1}{|x-y|}+|y|(1-r)\Big)\frac{e^{-t/3}}{\sqrt{1-r^2}}\\ &\times\exp\Big(-\frac{|x-y|^2}{12(1-r^2)}\Big)\Big(1+\frac{t}{\rho_H(y)^2}\Big)^{-N}.\end{aligned}\tag{54}$$

Proceeding as in (51), we obtain

$$\Big|\int_0^\infty \frac{\partial}{\partial x}W_t^H(x,y)\frac{dt}{\sqrt{t}}\Big| \le C_N\Big(\frac{1}{|x-y|}+|y|\Big)\frac{\rho_H(y)^{2N}}{|x-y|^{2N}} \quad \text{for } |x-y| > C\rho_H(y)\,,\tag{55}$$

which leads to

$$\sup_{y\in\boldsymbol{R}}\int_{|x-y|>C\rho_H(y)}|R_1^H(x,y)|dx \le C\,.\tag{56}$$

Our next step is to estimate $R_1^H(x,y)$ for $|x-y|\le C\rho_H(y)$. Note that (53) implies

$$\begin{aligned}\Big|\frac{\partial}{\partial x}W_t(x,y)\Big| \le\, & C_N\frac{e^{-t/3}}{\sqrt{1-r^2}}\Big(\frac{1+r^2}{1-r^2}\Big)|x-y|\\ &\times\exp\Big(-\frac{|x-y|^2}{12(1-r^2)}\Big)\Big(1+\frac{t}{\rho_H(y)^2}\Big)^{-N-1}\\ &+C_N\frac{e^{-t/3}}{\sqrt{1-r^2}}|y|(1-r)\exp\Big(-\frac{|x-y|^2}{12(1-r^2)}\Big)\Big(1+\frac{t}{\rho_H(y)^2}\Big)^{-N-1}\\ \le\, & C_N\frac{e^{-t/4}}{1-r^2}\Big(1+\frac{t}{\rho_H(y)^2}\Big)^{-N}.\end{aligned}\tag{57}$$

Consequently, using (57) we get

$$\int_{\rho_H(y)^2}^\infty \Big|\frac{\partial}{\partial x}W_t(x,y)\Big|\frac{dt}{\sqrt{t}} \le C\rho_H(y)^{-1}\,.\tag{58}$$

In order to investigate the integral

$$\int_0^{\rho_H(y)^2}\frac{\partial}{\partial x}W_t(x,y)\frac{dt}{\sqrt{t}}\,,$$

we study first the difference

$$Q(x,y)=\int_0^{\rho_H(y)^2}\frac{\partial}{\partial x}(W_t^H(x,y)-P_t(x-y))\frac{dt}{\sqrt{t}}\,,$$

where $P_t(x)=(2\pi t)^{-1/2}\exp(-x^2/2t)$ is the classical Gauss-Weierstrass kernel. The perturbation formula asserts that

$$Q(x,y)=-\frac{1}{2}\int_0^{\rho_H(y)^2}\int_0^t\int_{-\infty}^{\infty}\frac{\partial}{\partial x}P_{t-s}(x-z)z^2W_s^H(z,y)dzds\,\frac{dt}{\sqrt{t}}\,.$$

Therefore,

$$\begin{aligned}J&=\int_{|x-y|<C\rho_H(y)}|Q(x,y)|dx\\&\le C\int_{|x-y|\le C\rho_H(y)}\int_0^{\rho_H(y)^2}\int_0^t\int_{-\infty}^{\infty}\frac{|x-z|}{t-s}P_{t-s}(x-z)(|z-x|^2+x^2)\\&\qquad\times W_s^H(z,y)dz\,ds\,\frac{dt}{\sqrt{t}}dx\,.\end{aligned}\tag{59}$$

Observe that $x^2\le C\rho_H(y)^{-2}$ for $|x-y|\le C\rho_H(y)$. Substituting this inequality inside the above integral and then integrating with respect to dx and dz, we conclude

$$J\le C\int_0^{\rho_H(y)^2}\int_0^t\left((t-s)^{1/2}+\frac{1}{(t-s)^{1/2}\rho_H(y)^2}\right)ds\,\frac{dt}{\sqrt{t}}\le C\rho_H(y)^4+C\le C\,.\tag{60}$$

Proceeding as in (55), we also get

$$|R_1^H(x,y)|\le C_N\rho_H(x)^{-1}\frac{\rho_H(x)^N}{|x-y|^N}\ \text{ for } |x-y|>C\rho_H(x)\,,$$

and consequently,

$$\sup_{x\in\boldsymbol{R}}\int_{|x-y|>C\rho_H(x)}|R_1^H(x,y)|dy<\infty\,.\tag{61}$$

A similar procedure to that employed to estimate J gives

$$\sup_{x\in\boldsymbol{R}}\int_{|x-y|\le C\rho_H(x)}|Q(x,y)|dy\le C\,.\tag{62}$$

Finally, our analysis of the kernel $R_1^H(x,y)$ is reduced to the integral

$$\begin{aligned}\int_0^{\rho_H(y)^2}\frac{\partial}{\partial x}P_t(x-y)\frac{dt}{\sqrt{t}}&=-\int_0^{\rho_H(y)^2}\frac{x-y}{t}\frac{1}{\sqrt{2\pi t}}\exp(-|x-y|^2/2t)\frac{dt}{\sqrt{t}}\\&=-\frac{2}{\sqrt{2\pi}(x-y)}\exp\left(-\frac{|x-y|^2}{2\rho_H(y)^2}\right).\end{aligned}\tag{63}$$

Taking into account (10), (55), (58), (60), (61), (62) and (63), we get

$$R_1^H(x,y)=-\frac{\sqrt{2}}{\pi(x-y)}\exp\left(-\frac{|x-y|^2}{2\rho_H(y)^2}\right)+h_1(x,y)\tag{64}$$

with

$$\sup_{y\in \boldsymbol{R}} \int_{-\infty}^{\infty} |h_1(x,y)|dx + \sup_{x\in \boldsymbol{R}} \int_{-\infty}^{\infty} |h_1(x,y)|dy < \infty . \tag{65}$$

To complete the proof, take any $\psi \in C_c^\infty(\boldsymbol{R})$ as in the statement of Lemma 2.1. Define a function $h_2(x,y)$ by

$$h_2(x,y) = \frac{\sqrt{2}}{\pi(x-y)}\psi\Big(\frac{x-y}{\rho_H(x)}\Big) - \frac{\sqrt{2}}{\pi(x-y)}\exp\Big(-\frac{|x-y|^2}{2\rho_H(y)^2}\Big), \quad x,y \in \boldsymbol{R} .$$

By (10), (64), (65) and (11), the lemma will be established once we show that, for some $C > 0$ we have

$$\sup_{x\in\boldsymbol{R}} \int |h_2(x,y)|dy \le C \quad \text{and} \quad \sup_{y\in\boldsymbol{R}} \int |h_2(x,y)|dx \le C . \tag{66}$$

Set $A = \{(x,y) \in \boldsymbol{R}^2; |x-y| > \rho_H(x)\}$, $B = \{(x,y) \in \boldsymbol{R}^2; |x-y| \le \rho_H(x)\}$. Then

$$|h_2(x,y)| \le \frac{C}{|x-y|}\exp\Big(-\frac{|x-y|^2}{2\rho_H(y)^2}\Big)\chi_A(x,y) + C\Big(\frac{1}{\rho_H(x)} + \frac{|x-y|}{\rho_H(y)^2}\Big)\chi_B(x,y) , \tag{67}$$

where the last summand is obtained by applying the mean value theorem. Using (8), we see that $\rho_H(y)^2 \le c\rho_H(x)|x-y|$ when $(x,y) \in A$, and therefore

$$\begin{aligned} &\int \frac{1}{|x-y|}\exp\Big(-\frac{|x-y|^2}{2\rho_H(y)^2}\Big)\chi_A(x,y)dy \\ &\quad\le \int \frac{1}{|x-y|}\exp\Big(-c\frac{|x-y|}{\rho_H(x)}\Big)\chi_A(x,y)dy \\ &\quad\le \int_{|u|>1} \exp(-c|u|)\frac{du}{|u|} \le C . \end{aligned} \tag{68}$$

On the other hand, $\rho_H(x) \sim \rho_H(y)$ when $(x,y) \in B$ (again by (8)), so we have

$$\int \Big(\frac{1}{\rho_H(x)} + \frac{|x-y|}{\rho_H(y)^2}\Big)\chi_B(x,y)dy \le C ,$$

which together with (68) implies the first inequality in (66). From (8) we also see that $A \subset \widetilde{A} = \{(x,y) \in \boldsymbol{R}^2; |x-y| > \varepsilon\rho_H(y)\}$ and $B \subset \widetilde{B} = \{(x,y) \in \boldsymbol{R}^2; |x-y| \le \rho_H(y)/\varepsilon\}$ for some $\varepsilon > 0$. Using this fact, the second inequality in (66) follows by similar arguments. This completes the proof of Lemma 2.1. □

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