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LATTICES OF SOME SOLVABLE LIE GROUPS AND ACTIONS OF PRODUCTS OF AFFINE GROUPS

NOBUO TSUCHIYA AND AIKO YAMAKAWA

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Abstract. We consider solvable Lie groups which are isomorphic to unimodularizations of products of affine groups. It is shown that a lattice of such a Lie group is determined, up to commensurability, by a totally real algebraic number field. We also show that the outer automorphism group of the lattice is represented faithfully in the automorphism group of the number field. As an application, we obtain a classification of codimension one, volume preserving, locally free actions of products of affine groups.

1. Introduction. For $V \cong \boldsymbol{R}^n$, $W \cong \boldsymbol{R}^{n+1}$ and $\psi \in \mathrm{Hom}(V, SL(W))$, the semidirect product $V \ltimes_\psi W$ is the group which has the group law

$$(\boldsymbol{t}, \boldsymbol{x})(\boldsymbol{s}, \boldsymbol{y}) = (\boldsymbol{t} + \boldsymbol{s}, \boldsymbol{x} + \psi(\boldsymbol{t})(\boldsymbol{y})) \quad (\boldsymbol{t}, \boldsymbol{s} \in V,\ \boldsymbol{x}, \boldsymbol{y} \in W)\,.$$

All Lie groups of the form $V \ltimes_\psi W$ are isomorphic if the homomorphism ψ is injective and splits as a direct sum of non-equivalent 1-dimensional representations. *Throughout this paper, we denote by H such a Lie group.*

In this paper, we study lattices of H by using notions related to algebraic number fields. Here a lattice means a cocompact discrete subgroup.

Let Γ and Γ' be lattices of H. Recall that they are said to be *commensurable* if $[\Gamma : \Gamma \cap \Gamma'] < \infty$ and $[\Gamma' : \Gamma \cap \Gamma'] < \infty$ (see e.g., [12]). We say they are *weakly commensurable* if there exists an automorphism φ of H such that $\varphi(\Gamma)$ and Γ' are commensurable. Given a lattice $\Gamma \subset H$, we define a totally real algebraic number field $\boldsymbol{k}(\Gamma)$ of degree $n + 1$. The first of our main theorems is the following.

THEOREM 1.1. *The map*

$$\left\{\begin{array}{l}\text{the set of all weakly}\\ \text{commensurable classes of}\\ \text{lattices of } H\end{array}\right\} \to \left\{\begin{array}{l}\text{the set of all isomorphism}\\ \text{classes of totally real algebraic}\\ \text{number fields of degree } n+1\end{array}\right\}$$

induced from the map $\Gamma \mapsto \boldsymbol{k}(\Gamma)$ is bijective.

Secondly we define a homomorphism A_Γ from $\mathrm{Aut}(\Gamma)$ to $\mathrm{Aut}(\boldsymbol{k}(\Gamma)/\boldsymbol{Q})$ (see (3.1)). For $\varphi \in \mathrm{Aut}(\Gamma)$, the image $A_\Gamma(\varphi)$ may be regarded as a permutation of $n+1$ 1-dimensional direct summands of ψ which is induced from φ. If $A_\Gamma(\varphi) = \mathrm{id}$, then there exists $h_0 \in H$ such that

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$\varphi^2(\gamma) = h_0 \gamma h_0^{-1}$ ($\gamma \in \Gamma$) (Corollary 3.11). We show that each subgroup of $\mathrm{Aut}(\boldsymbol{k}(\Gamma)/\boldsymbol{Q})$ is realized as the image of a lattice commensurable with Γ.

THEOREM 1.2. *Let Γ be a lattice of H. For each subgroup F of* $\mathrm{Aut}(\boldsymbol{k}(\Gamma)/\boldsymbol{Q})$, *there exists a lattice $\Gamma' \subset H$ commensurable with Γ such that $A_{\Gamma'}(\mathrm{Aut}(\Gamma')) = F$.*

Finally we apply the properties of A_Γ to the classification problem of homogeneous actions, which was our original motivation of this research. Let $\mathrm{Aff}^+(\boldsymbol{R})$ denote the group of all orientation preserving affine transformations of the real line and let $G := \mathrm{Aff}^+(\boldsymbol{R})^n$. Then $\tilde{G} := G \ltimes_\Delta \boldsymbol{R}$ is unimodular, solvable and isomorphic to H where Δ is the modular function of G. In [13] the authors showed that if $n \geq 2$, then $H \cong \tilde{G}$ is the unique $(2n+1)$-dimensional, simply connected, unimodular Lie group which contains G. For a lattice Γ of H, the homogeneous manifold H/Γ admits a *homogeneous action* of G induced from the left translations of G on H. Denote by $\mathrm{Conj}(H/\Gamma)$ the set of all analytic conjugacy classes of homogeneous G actions on H/Γ. It is seen that the set of all inner conjugacy classes of subgroups of H isomorphic to G consists of $n+1$ elements (Proposition 2.3), and hence $|\mathrm{Conj}(H/\Gamma)| \leq n+1$.

THEOREM 1.3. *Let Γ be a lattice of H. Then we have*

$$|\mathrm{Conj}(H/\Gamma)| = \frac{n+1}{|A_\Gamma(\mathrm{Aut}(\Gamma))|}\,.$$

Furthermore, for each subgroup F of $\mathrm{Aut}(\boldsymbol{k}(\Gamma)/\boldsymbol{Q})$, *there exists a lattice $\Gamma' \subset H$ commensurable with Γ such that*

$$|\mathrm{Conj}(H/\Gamma')| = \frac{n+1}{|F|}\,.$$

In [13, Theorem 1], the authors showed that a codimension one locally free C^∞-action of G on a closed manifold is C^∞-conjugate to a homogeneous action if it preserves a volume form (see also [2]). Thus the above Theorem 1.3 gives a classification of volume preserving, codimension one, locally free actions of G on closed manifolds (see [11] for a survey of codimension one locally free actions). In Theorem 1.3 we do not have to specify the class of differentiability of the conjugacy because an equivariant C^r-diffeomorphism ($r = 0, 1, 2, \dots$) between two homogeneous G actions turns out to be analytic (Proposition 5.1).

In §2 we give cohomological descriptions of subgroups, automorphisms and lattices of H. In §3 we define the algebraic number field $\boldsymbol{k}(\Gamma)$ associated with a lattice Γ and give a proof of Theorem 1.1. In §4 we first define the homomorphism A_Γ and then give a proof of Theorem 1.2. In the last section §5, we prove Theorem 1.3.

2. Subgroups, automorphisms and lattices of H.

2.1. Subgroups of H and their cohomological descriptions. Recall that H is a Lie group of the form $H_1 \ltimes_\psi H_0$ where H_0 (resp. H_1) denotes $[H, H]$ (resp. H/H_0). Let K be a closed subgroup of H. The group structure of K is described by using continuous cohomology. Put $K_0 := K \cap H_0$ and $K_1 := K/K_0$. Then K_1 is a closed subgroup of H_1 and K_0 is a K_1-invariant closed subgroup of H_0.

Let A be a topological K_1-module by a homomorphism $\rho : K_1 \to \mathrm{Aut}(A)$. For each non-negative integer p, let $C_N^p(K_1, A)$ denote the module consisting of all normalized continuous maps from K_1^p to A, where a cochain ξ is said to be normalized if $\xi(\boldsymbol{t}_1, \boldsymbol{t}_2, \dots, \boldsymbol{t}_p) = 0$ whenever some $\boldsymbol{t}_i$ is zero (cf. [4, p. 19]). Then the cochain complex $\{C_N^*(K_1, A), \delta\}$ defines the cohomology group $H^*(K_1, A)$ in the usual way. Recall that the definitions of the coboundary operator $\delta : C_N^{p-1}(K_1, A) \to C_N^p(K_1, A)$ and a cochain homotopy $D_{\boldsymbol{t}_0} : C_N^p(K_1, A) \to C_N^{p-1}(K_1, A)$ for $\boldsymbol{t}_0 \in K_1$ ($\xi \in C_N^{p-1}(K_1, A), \eta \in C_N^p(K_1, A), \boldsymbol{t}_i \in K_1, i = 1, 2, \dots, p$) are as follows:

$$
\begin{aligned}
(\delta\xi)(\boldsymbol{t}_1, \boldsymbol{t}_2, \dots, \boldsymbol{t}_p) &= \rho(\boldsymbol{t}_1)\big(\xi(\boldsymbol{t}_2, \dots, \boldsymbol{t}_p)\big) \\
&\quad + \sum_{i=1}^{p-1} (-1)^i \xi(\boldsymbol{t}_1, \dots, \boldsymbol{t}_{i-1}, \boldsymbol{t}_i + \boldsymbol{t}_{i+1}, \boldsymbol{t}_{i+2}, \dots, \boldsymbol{t}_p) + (-1)^p \xi(\boldsymbol{t}_1, \boldsymbol{t}_2, \dots, \boldsymbol{t}_{p-1}) .
\end{aligned}
$$

$$
(D_{\boldsymbol{t}_0}(\eta))(\boldsymbol{t}_1, \boldsymbol{t}_2, \dots, \boldsymbol{t}_{p-1}) = \sum_{i=1}^{p} (-1)^{i+1} \eta(\boldsymbol{t}_1, \dots, \boldsymbol{t}_{i-1}, \boldsymbol{t}_0, \boldsymbol{t}_i, \dots, \boldsymbol{t}_{p-1}) .
$$

By direct calculations, we obtain (1) of the following lemma. The remaining assertions follow from (1).

LEMMA 2.1. (1) *For each* $\xi \in C_N^p(K_1, A)$ $(p \geq 0)$ *and* $\boldsymbol{t}_0 \in K_1$, *the equation* $D_{\boldsymbol{t}_0} \circ \delta(\xi) + \delta \circ D_{\boldsymbol{t}_0}(\xi) = \rho(\boldsymbol{t}_0) \circ \xi - \xi$ *holds.*

(2) *If there exists* $\boldsymbol{t}_0 \in K_1$ *such that the homomorphism* $\rho(\boldsymbol{t}_0) - \mathrm{id} : A \to A$ *is invertible, then* $H^p(K_1, A) = \{0\}$ *for any* $p \geq 0$.

(3) *If there exists* $\boldsymbol{t}_0 \in K_1$ *such that the homomorphism* $\psi(\boldsymbol{t}_0) - \mathrm{id} : H_0 \to H_0$ *is invertible, then* $H^p(K_1, H_0/K_0)$ *is naturally isomorphic to* $H^{p+1}(K_1, K_0)$ *for any* $p \geq 0$.

Choose a cochain $\eta \in C_N^1(K_1, H_0)$ such that $(\boldsymbol{t}, \eta(\boldsymbol{t})) \in K$ for any $\boldsymbol{t} \in K_1$. Then the subset K and its multiplication law are expressed as follows:

$$
\begin{aligned}
K &= \{(\boldsymbol{t}, \eta(\boldsymbol{t}) + \boldsymbol{x}) \;;\; \boldsymbol{t} \in K_1,\ \boldsymbol{x} \in K_0\} \subset H_1 \ltimes H_0 = H , \\
(\boldsymbol{t}, \eta(\boldsymbol{t}) + \boldsymbol{x})\,(\boldsymbol{s}, \eta(\boldsymbol{s}) + \boldsymbol{y}) &= (\boldsymbol{t} + \boldsymbol{s}, \eta(\boldsymbol{t} + \boldsymbol{s}) + \boldsymbol{x} + \psi(\boldsymbol{t})(\boldsymbol{y}) + \delta\eta(\boldsymbol{t}, \boldsymbol{s})) .
\end{aligned}
\tag{2.1}
$$

We say a cochain $\xi \in C_N^1(K_1, H_0)$ is a *mod* K_0 *relative cocycle* (or a *relative cocycle* in short) if $\delta\xi \in C_N^2(K_1, K_0)$. From (2.1), the cochain η is a relative cocycle.

Conversely, given a closed subgroup K_1 of H_1, a $\psi(K_1)$-invariant closed subgroup K_0 of H_0 and a mod K_0 relative cocycle η, we can construct a subgroup K of H by following the rules in (2.1). We denote by $\mathrm{Gr}(K_0, K_1, \eta)$ the subgroup obtained in this way. The proof of the following lemma is easy and is omitted.

LEMMA 2.2. (1) *The equality* $\mathrm{Gr}(K_0, K_1, \eta) = \mathrm{Gr}(K_0', K_1', \eta')$ *holds if and only if* $K_i = K_i'$ $(i = 0, 1)$ *and* $\eta' - \eta \in C_N^1(K_1, K_0)$.

(2) *The group* $\mathrm{Gr}(K_0, K_1, \eta)$ *is a semidirect product of* K_0 *and* K_1 *if and only if there exists a cocycle* $\xi \in C^1_N(K_1, H_0)$ *such that* $\xi - \eta \in C^1_N(K_1, K_0)$.

Let $H_0 = \bigoplus_{i=1}^{n+1} W_i$ be the splitting of the H_1-module H_0, and, for each i, put $W(i) := \bigoplus_{j \neq i} W_j$ and $G(i) := H_1 \ltimes_\psi W(i) \subset H$. The following proposition will become a basic fact in classifying codimension one homogeneous actions of $\mathrm{Aff}^+(\boldsymbol{R})^n$ in §5. For $h_0 \in H$, let $\mathrm{Ad}(h_0) : H \to H$ be the automorphism defined by $\mathrm{Ad}(h_0)(h) = h_0 h h_0^{-1}$ $(h \in H)$.

PROPOSITION 2.3. *Let* K *be a subgroup of* H *which is isomorphic to* $G = \mathrm{Aff}^+(\boldsymbol{R})^n$. *Then there exist an integer* i $(1 \le i \le n+1)$ *and an element* $h_0 \in H$ *such that* $K = \mathrm{Ad}(h_0)(G(i))$.

PROOF. Put $K = \mathrm{Gr}(K_0, K_1, \eta)$. Then $K_1 = H_1$ and there exists i $(1 \le i \le n+1)$ such that $K_0 = W(i)$. By Lemma 2.2, we may assume $\delta\eta = 0$. By Lemma 2.1 (2), $H^1(H_1, H_0) = 0$ and hence there exists $\boldsymbol{x}_0 \in H_0$ such that $\eta = \delta\boldsymbol{x}_0$. Thus we get $K = \mathrm{Gr}(W(i), H_1, \delta\boldsymbol{x}_0) = \mathrm{Ad}((\boldsymbol{0}, -\boldsymbol{x}_0))(G(i))$. □

2.2. Automorphisms of H. Let $K = \mathrm{Gr}(K_0, K_1, \eta)$ and $K' = \mathrm{Gr}(K'_0, K'_1, \eta')$ be two isomorphic closed subgroups of H and $\phi : K \to K'$ an isomorphism such that $\phi(K_0) = K'_0$. Then ϕ is described as

$$(\boldsymbol{t}, \eta(\boldsymbol{t}) + \boldsymbol{x}) \mapsto (\phi_1(\boldsymbol{t}), \eta'(\phi_1(\boldsymbol{t})) + \phi_0(\boldsymbol{x}) + \xi(\phi_1(\boldsymbol{t}))) \quad (\boldsymbol{t} \in K_1,\ \boldsymbol{x} \in K_0) \tag{2.2}$$

where ϕ_i $(i = 0, 1)$ are the induced isomorphisms from K_i to K'_i and $\xi \in C^1_N(K'_1, K'_0)$. Because ϕ is a homomorphism, these maps satisfy the following two conditions:

$$\psi(\phi_1(\boldsymbol{t}))(\phi_0(\boldsymbol{x})) = \phi_0(\psi(\boldsymbol{t})(\boldsymbol{x})) \quad (\boldsymbol{x} \in K_0, \boldsymbol{t} \in K_1)\,, \tag{2.3}$$

$$\phi_0 \circ \delta\eta = (\delta\xi + \delta\eta') \circ (\phi_1 \times \phi_1)\,. \tag{2.4}$$

We call (2.3) the *compatibility condition* for (ϕ_0, ϕ_1) and denote this isomorphism ϕ by $\mathrm{hom}(\phi_0, \phi_1, \xi)$. In the case where $K = H$, we may put $\eta = \eta' = 0$ in (2.2) and obtain the following lemma from (2.4) and Lemma 2.1.

LEMMA 2.4. *Let* φ *be an automorphism of* H. *Then there exists an element* $\boldsymbol{x}_0 \in H_0$ *such that* $\varphi = \mathrm{hom}(\varphi_0, \varphi_1, \delta\boldsymbol{x}_0)$. *In particular, for* $h_0 = (\boldsymbol{t}_0, \boldsymbol{x}_0) \in H$, *we have* $\mathrm{Ad}(h_0) = \mathrm{hom}(\psi(\boldsymbol{t}_0), \mathrm{id}, -\delta\boldsymbol{x}_0)$.

2.3. Lattices of H. First we review a known result.

PROPOSITION 2.5 ([13, Lemma 2.4]). *A closed subgroup* $\Gamma = \mathrm{Gr}(\Gamma_0, \Gamma_1, \eta)$ *of* H *is a lattice if and only if* Γ_i *is a lattice of* H_i $(i = 0, 1)$.

By Lemma 2.1(3) and Proposition 2.5, for a lattice Γ, we have $H^1(\Gamma_1, H_0/\Gamma_0) \cong H^2(\Gamma_1, \Gamma_0)$. The set of equivalence classes of group extensions of Γ_1 by Γ_0 corresponds bijectively to $H^2(\Gamma_1, \Gamma_0)$ ([5, p. 162]), and hence to $H^1(\Gamma_1, H_0/\Gamma_0)$. We show that this set is finite.

PROPOSITION 2.6. *The group* $H^1(\Gamma_1, H_0/\Gamma_0)$ *is finite.*

PROOF. Choose a basis $\{\boldsymbol{t}_1, \boldsymbol{t}_2, \dots, \boldsymbol{t}_n\}$ of $\Gamma_1 \cong \boldsymbol{Z}^n$ such that $\det(\psi(\boldsymbol{t}_1) - \mathrm{id}) \neq 0$. Let $(H_0/\Gamma_0)^{\boldsymbol{t}_1}$ denote the finite group $\{\boldsymbol{x} \in H_0/\Gamma_0\,;\ (\psi(\boldsymbol{t}_1) - \mathrm{id})(\boldsymbol{x}) = \boldsymbol{0}\}$. Then by Lemma 2.1 (1) the short exact sequence of Γ_1-modules

$$0 \to (H_0/\Gamma_0)^{\boldsymbol{t}_1} \to H_0/\Gamma_0 \overset{\psi(\boldsymbol{t}_1)-\mathrm{id}}{\longrightarrow} H_0/\Gamma_0 \to 0$$

induces a short exact sequence

$$0 \to H^0(\Gamma_1, H_0/\Gamma_0) \to H^1(\Gamma_1, (H_0/\Gamma_0)^{\boldsymbol{t}_1}) \to H^1(\Gamma_1, H_0/\Gamma_0) \to 0\,. \tag{2.5}$$

If a Γ_1-module A is finite, then the group $H^1(\Gamma_1, A)$ is known to be finite ([9, p. 189]). Thus $H^1(\Gamma_1, (H_0/\Gamma_0)^{\boldsymbol{t}_1})$ is finite. By (2.5), the group $H^1(\Gamma_1, H_0/\Gamma_0)$ is also finite. □

REMARK 2.7. (1) When $n = 1$, it is easy to see that $H^1(\Gamma_1, H_0/\Gamma_0) = \{0\}$ for every lattice $\Gamma \subset H$. It follows that Γ is inner conjugate to $\mathrm{Gr}(\Gamma_0, \Gamma_1, 0)$.

(2) When $n \geq 2$, there exists a lattice $\Gamma \subset H$ such that $H^1(\Gamma_1, H_0/\Gamma_0) \neq \{0\}$. For example, take an injective homomorphism $\psi_1 : \boldsymbol{R}^2 \to SL(3, \boldsymbol{R})$ given by

$$\psi_1\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}1 & -2 & 1\\ -2 & 5 & -2\\ 4 & -10 & 5\end{pmatrix} \quad \text{and} \quad \psi_1\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}-7 & 10 & 6\\ -12 & 17 & 10\\ -20 & 28 & 17\end{pmatrix}.$$

Then, for $\Gamma := \boldsymbol{Z}^2 \ltimes_{\psi_1} \boldsymbol{Z}^3 \subset \boldsymbol{R}^2 \ltimes_{\psi_1} \boldsymbol{R}^3 \cong H$, one can show that $H^1(\Gamma_1, H_0/\Gamma_0) \cong \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$.

By Remark 2.7, a lattice is not always a semidirect product. However the following proposition shows that every lattice of H contains a sublattice of finite index which has a semidirect product structure.

PROPOSITION 2.8. *Let $\Gamma = \mathrm{Gr}(\Gamma_0, \Gamma_1, \eta)$ be a lattice of H. Then there exists a subgroup Γ' (resp. Γ_1') of finite index of Γ (resp. Γ_1) such that Γ' is conjugate to* $\mathrm{Gr}(\Gamma_0, \Gamma_1', 0)$.

PROOF. By Proposition 2.6 there exist a natural number p and $\boldsymbol{y}_0 \in H_0$ such that $\zeta := \eta - \delta\boldsymbol{y}_0 \in C^1_N(\Gamma_1, (1/p)\Gamma_0)$. Put $\Gamma_1' := \{\boldsymbol{t} \in \Gamma_1\,;\ \zeta(\boldsymbol{t}) \in \Gamma_0\}$. The subset Γ_1' is a subgroup of Γ_1 from the fact

$$\delta\zeta(\boldsymbol{t}, \boldsymbol{s}) = \zeta(\boldsymbol{t}) - \zeta(\boldsymbol{t} + \boldsymbol{s}) + \psi(\boldsymbol{t})(\zeta(\boldsymbol{s})) = \delta\eta(\boldsymbol{t}, \boldsymbol{s}) \in \Gamma_0 \quad (\boldsymbol{t}, \boldsymbol{s} \in \Gamma_1)\,.$$

Let $\boldsymbol{s} \in \Gamma_1$. Then there exist integers l_1, l_2 such that $0 < l_1 < l_2 \leq p^{n+1} + 1$ and $\zeta(l_2\boldsymbol{s}) - \zeta(l_1\boldsymbol{s}) \in \Gamma_0$. Because $\zeta(l_1\boldsymbol{s}) - \zeta(l_2\boldsymbol{s}) + \psi(l_1\boldsymbol{s})(\zeta((l_2 - l_1)\boldsymbol{s})) \in \Gamma_0$, we have $\zeta((l_2 - l_1)\boldsymbol{s}) \in \Gamma_0$ and hence $p^{n+1}!\,\boldsymbol{s} \in \Gamma_1'$. We have proved $[\Gamma_1 : \Gamma_1'] < \infty$.

Put $\Gamma' := \mathrm{Gr}(\Gamma_0, \Gamma_1', \eta|_{\Gamma_1'})$. Then, from Lemmas 2.2 and 2.4, we obtain

$$\Gamma' = \mathrm{Gr}(\Gamma_0, \Gamma_1', \delta\boldsymbol{y}_0) = \mathrm{Ad}((\boldsymbol{0}, -\boldsymbol{y}_0))(\mathrm{Gr}(\Gamma_0, \Gamma_1', 0))\,. \qquad \square$$

Although the following rigidity theorem is a special case of a general result of Saito ([10, Theorem 5]), we give a proof here because it is simple and because we will need a description of $\phi \in \mathrm{Aut}(\Gamma)$ shown in the proof.

PROPOSITION 2.9. *Let* $\Gamma = \mathrm{Gr}(\Gamma_0, \Gamma_1, \eta)$ *and* $\Gamma' = \mathrm{Gr}(\Gamma'_0, \Gamma'_1, \eta')$ *be lattices of* H *and* $\phi : \Gamma \to \Gamma'$ *an isomorphism. Then* ϕ *can be extended uniquely to an automorphism of* H.

PROOF. From Proposition 2.5, each of the groups Γ'_0 and Γ_0 is non-trivial. Then Γ'_0 and $\phi(\Gamma_0)$ coincide because both are maximal normal nilpotent subgroups of Γ' and a polycyclic group admits a unique maximal normal nilpotent subgroup. Hence we can write $\phi = \mathrm{hom}(\phi_0, \phi_1, \xi)$ where $\phi_i : \Gamma_i \to \Gamma'_i$ $(i = 0, 1)$ are the induced isomorphisms and $\xi \in C^1_N(\Gamma'_1, \Gamma'_0)$. Let $\varphi_i \in \mathrm{Aut}(H_i)$ be the linear isomorphism of H_i extending ϕ_i $(i = 0, 1)$. By (2.4) and Lemma 2.1 (3) there exists $\boldsymbol{x}_0 \in H_0$ such that $\xi + \eta' - \phi_0 \circ \eta \circ \phi_1^{-1} = \delta \boldsymbol{x}_0$. It is easy to see that the isomorphism $\varphi := \mathrm{hom}(\varphi_0, \varphi_1, \delta\boldsymbol{x}_0)$ is the unique automorphism of H which extends ϕ. □

3. Arithmeticity of lattices of H.

3.1. Algebraic number field associated with Γ. Let W_i (resp. $W(i)$) be the 1-dimensional subspace (resp. hyperplane) of H_0 defined in §2.1. For $\boldsymbol{t} \in H_1$ and $1 \le i \le n+1$, let $\lambda_i(\boldsymbol{t})$ denote the eigenvalue of $\psi(\boldsymbol{t})|_{W_i}$. Consider the map $l_\psi : H_1 \to \boldsymbol{R}^{n+1}$ defined by

$$l_\psi(\boldsymbol{t}) = (\log \lambda_1(\boldsymbol{t}), \log \lambda_2(\boldsymbol{t}), \dots, \log \lambda_{n+1}(\boldsymbol{t}))\,.$$

The map l_ψ is a linear isomorphism from H_1 to the hyperplane $V := \{(\mu_1, \mu_2, \dots, \mu_{n+1})\,;\ \sum_{i=1}^{n+1} \mu_i = 0\} \subset \boldsymbol{R}^{n+1}$. For a subset $S \subset \{1, 2, \dots, n+1\}$, let $\Sigma_S := \{(\mu_1, \mu_2, \dots, \mu_{n+1})\,;\ \sum_{i \in S} \mu_i = 0\} \subset V$ and $\Sigma := \bigcup_{\emptyset \ne S \subsetneq \{1,2,\dots,n+1\}} \Sigma_S$. Put ${H_1}^0 := (l_\psi)^{-1}(V \setminus \Sigma) \subset H_1$.

Take a lattice $\Gamma = \mathrm{Gr}(\Gamma_0, \Gamma_1, \eta)$ of H. By Proposition 2.5, there exist isomorphisms $(\Gamma_0, H_0) \cong (\boldsymbol{Z}^{n+1}, \boldsymbol{R}^{n+1})$ and $(\Gamma_1, H_1) \cong (\boldsymbol{Z}^n, \boldsymbol{R}^n)$. Under these identifications, the representation $\psi : H_1 \to GL(H_0)$ is a homomorphism from $\boldsymbol{R}^n$ to $SL(n+1, \boldsymbol{R})$ such that $\psi(\boldsymbol{Z}^n) \subset SL(n+1, \boldsymbol{Z})$. For a vector $w \in \Gamma_0 \otimes \boldsymbol{Q}$ (resp. a $\boldsymbol{Q}$-linear isomorphism $B \in GL(\Gamma_0 \otimes \boldsymbol{Q})$), let $\bar{w} \in \boldsymbol{Q}^{n+1}$ (resp. $\bar{B} \in GL(n+1, \boldsymbol{Q})$) denote the corresponding vector (resp. matrix). Choose $\boldsymbol{t}_0 \in {H_1}^0 \cap \Gamma_1$ and let $A := \psi(\boldsymbol{t}_0) \in \mathrm{Aut}(\Gamma_0) \subset GL(H_0)$. We show that the characteristic polynomial $\chi_A(x)$ of A is $\boldsymbol{Z}$-irreducible. Suppose there exists an integral polynomial $f(x)$ of degree $m < n+1$ which divides $\chi_A(x)$. Let $\{\lambda_j\,;\ 1 \le j \le m\}$ be the roots of $f(x)$. Then $f(x)$ is monic, $f(0) = (-1)^m$ and each λ_j $(1 \le j \le m)$ is positive because $\chi_A(x)$ is monic, $\chi_A(0) = (-1)^{n+1}$ and each $\lambda_i(\boldsymbol{t}_0)$ $(1 \le i \le n+1)$ is positive. It follows that $\prod_{j=1}^m \lambda_j = 1$ and hence $l_\psi(\boldsymbol{t}_0)$ is in Σ, which contradicts the hypothesis. Thus $\chi_A(x)$ is $\boldsymbol{Z}$-irreducible, hence is $\boldsymbol{Q}$-irreducible and has distinct roots.

Let $\boldsymbol{Q}[\bar{A}]$ denote the polynomial ring $\{g(\bar{A})\,;\ g(X) \in \boldsymbol{Q}[X]\} \subset M(n+1, \boldsymbol{Q})$.

LEMMA 3.1. *The centralizer of* $\bar{A}$ *in* $GL(n+1, \boldsymbol{Q})$ *coincides with* $\boldsymbol{Q}[\bar{A}] \cap GL(n+1, \boldsymbol{Q})$.

PROOF. Choose $v_0 \in \big(H_0 \setminus \bigcup_{i=1}^{n+1} W(i)\big) \cap \Gamma_0$. Let $\bar{B} \in GL(n+1, \boldsymbol{Q})$ such that $\bar{B}\bar{A} = \bar{A}\bar{B}$. Then, because the vectors $\{\bar{v}_0, \bar{A}\bar{v}_0, \bar{A}^2\bar{v}_0, \dots, \bar{A}^n\bar{v}_0\}$ are independent, the vector $\bar{B}\bar{v}_0 \in \boldsymbol{Q}^{n+1}$ can be written as $\bar{B}\bar{v}_0 = b_0\bar{v}_0 + b_1\bar{A}\bar{v}_0 + \cdots + b_n\bar{A}^n\bar{v}_0$ $(b_i \in \boldsymbol{Q},\ 0 \le i \le n)$.

Thus we obtain $\bar{B} = b_0 E + b_1 \bar{A} + \cdots + b_n \bar{A}^n$ since $\bar{B}(\bar{A}^i \bar{v}_0) = \bar{A}^i(\bar{B}\bar{v}_0) = (b_0 E + b_1 \bar{A} + \cdots + b_n \bar{A}^n)\bar{A}^i \bar{v}_0$ for any i $(0 \le i \le n)$. □

Let $\mathrm{Hom}_{\boldsymbol{Q}}(\Gamma_0 \otimes \boldsymbol{Q})$ denote the ring of all $\boldsymbol{Q}$-linear homomorphisms of $\Gamma_0 \otimes \boldsymbol{Q}$.

COROLLARY 3.2. $\boldsymbol{Q}[A] = \boldsymbol{Q}[\{\psi(\boldsymbol{t})\,;\ \boldsymbol{t} \in \Gamma_1\}] \subset \mathrm{Hom}_{\boldsymbol{Q}}(\Gamma_0 \otimes \boldsymbol{Q})$.

Put $\alpha := \lambda_1(\boldsymbol{t}_0)$. As mentioned above, the number α is an algebraic integer of degree $n+1$ and is totally positive in the sense that each conjugate number $\alpha_i := \lambda_i(\boldsymbol{t}_0)$ $(1 \le i \le n+1)$ of α is positive. The field $\boldsymbol{Q}(\alpha)$ is isomorphic to the field

$$\boldsymbol{Q}(A) := \left\{ \frac{g(A)}{f(A)}\,;\ f(X), g(X) \in \boldsymbol{Q}[X],\ f(A) \neq 0 \right\} \subset \mathrm{Hom}_{\boldsymbol{Q}}(\Gamma_0 \otimes \boldsymbol{Q})$$

and $\boldsymbol{Q}[A] = \boldsymbol{Q}(A)$. By Corollary 3.2, the field $\boldsymbol{Q}(A)$ does not depend on the choice of $\boldsymbol{t}_0 \in {H_1}^0 \cap \Gamma_1$. We denote by $\boldsymbol{k}(\Gamma)$ the field $\boldsymbol{Q}(A) \subset \mathrm{Hom}_{\boldsymbol{Q}}(\Gamma_0 \otimes \boldsymbol{Q})$ and call it the *algebraic number field associated with* Γ. Obviously the field $\boldsymbol{k}(\Gamma) \cong \boldsymbol{Q}(\alpha)$ is totally real.

For an isomorphism $\varphi = \mathrm{hom}(\varphi_0, \varphi_1, \delta\boldsymbol{x}_0) \in \mathrm{Aut}(H)$ and a lattice Γ of H, let $\varphi_* : \mathrm{Hom}_{\boldsymbol{Q}}(\Gamma_0 \otimes \boldsymbol{Q}) \to \mathrm{Hom}_{\boldsymbol{Q}}(\varphi_0(\Gamma_0) \otimes \boldsymbol{Q})$ be the natural homomorphism defined by $\varphi_*(f) = \varphi_0 \circ f \circ \varphi_0^{-1}$ $(f \in \mathrm{Hom}_{\boldsymbol{Q}}(\Gamma_0 \otimes \boldsymbol{Q}))$.

LEMMA 3.3. *Let Γ and Γ' be lattices of H. Suppose Γ is weakly commensurable with Γ' by $\varphi \in \mathrm{Aut}(H)$. Then $\boldsymbol{k}(\Gamma') = \varphi_*(\boldsymbol{k}(\Gamma))$.*

PROOF. Since $\varphi(\Gamma)$ and Γ' are commensurable, the lemma follows from Corollary 3.2 and the compatibility condition (2.3). □

Note in particular that the field $\boldsymbol{k}(\Gamma)$ depends only on the commensurability class of Γ.

3.2. Lattices defined from an algebraic number field. Let $\boldsymbol{k}$ be a totally real algebraic number field of degree $n+1$ over $\boldsymbol{Q}$. (Such a field exists for each $n \ge 1$. See Lemma 2.6 in [13] for a concrete example.) Let $\mathcal{O}(\boldsymbol{k})$ denote the subring of all algebraic integers in $\boldsymbol{k}$ and $\mathcal{E}(\boldsymbol{k})$ the unit group of $\mathcal{O}(\boldsymbol{k})$. It is well-known that $\mathcal{O}(\boldsymbol{k})$ is isomorphic to $\boldsymbol{Z}^{n+1}$ as an abelian group.

Let $\mathrm{Imb}(\boldsymbol{k}) = \{f^{(1)}, f^{(2)}, \ldots, f^{(n+1)}\}$ be the set of all imbeddings of the field $\boldsymbol{k}$ into $\boldsymbol{R}$. Let $\mathcal{E}^+(\boldsymbol{k}) := \{\varepsilon \in \mathcal{E}(\boldsymbol{k})\,;\ f^{(i)}(\varepsilon) > 0\ (1 \le i \le n+1)\}$ and $\mathcal{E}^+_{n+1}(\boldsymbol{k}) := \{\varepsilon \in \mathcal{E}^+(\boldsymbol{k})\,;\ \deg(\varepsilon) = n+1\}$. Define an injective map $l_{\boldsymbol{k}} : \mathcal{E}^+(\boldsymbol{k}) \to \boldsymbol{R}^{n+1}$ by

$$l_{\boldsymbol{k}}(\varepsilon) = (\log(f^{(1)}(\varepsilon)), \log(f^{(2)}(\varepsilon)), \ldots, \log(f^{(n+1)}(\varepsilon)))\,.$$

The image of $l_{\boldsymbol{k}}$ is contained in the hyperplane $V \subset \boldsymbol{R}^{n+1}$ defined in §3.1. Dirichlet's unit theorem asserts that the image $l_{\boldsymbol{k}}(\mathcal{E}^+(\boldsymbol{k}))$ is a lattice of the vector group V (see e.g., [3]). Obviously $\mathcal{E}^+_{n+1}(\boldsymbol{k}) \supset l_{\boldsymbol{k}}^{-1}(V \setminus \Sigma) \cap \mathcal{E}^+(\boldsymbol{k})$ and is not empty.

Let $\mathrm{Aut}(\mathcal{O}(\boldsymbol{k}))$ denote the group of all automorphisms of the abelian group $\mathcal{O}(\boldsymbol{k})$, and $GL(\boldsymbol{k})$ (resp. $GL(\boldsymbol{k} \otimes \boldsymbol{R})$) the group of all isomorphisms of the $\boldsymbol{Q}$-vector space $\boldsymbol{k}$ (resp. $\boldsymbol{R}$-vector space $\boldsymbol{k} \otimes \boldsymbol{R}$). We sometimes identify an element of $\mathrm{Aut}(\mathcal{O}(\boldsymbol{k}))$ with an element of $GL(\boldsymbol{k})$ or $GL(\boldsymbol{k} \otimes \boldsymbol{R})$. Let $\iota_{\boldsymbol{k}} : \boldsymbol{k} \to \mathrm{Hom}_{\boldsymbol{Q}}(\boldsymbol{k})$ be the tautological map given by $\iota_{\boldsymbol{k}}(\beta)(\gamma) = \beta\gamma$ $(\beta, \gamma \in \boldsymbol{k})$ where $\mathrm{Hom}_{\boldsymbol{Q}}(\boldsymbol{k})$ denotes the ring of all $\boldsymbol{Q}$-linear maps from $\boldsymbol{k}$ to $\boldsymbol{k}$.

Let $\psi_{\boldsymbol{k}} : l_{\boldsymbol{k}}(\mathcal{E}^+(\boldsymbol{k})) \to \mathrm{Aut}(\mathcal{O}(\boldsymbol{k}))$ be the homomorphism given by $\psi_{\boldsymbol{k}} \circ l_{\boldsymbol{k}} = \iota_{\boldsymbol{k}}$. Take a basis of $\mathcal{O}(\boldsymbol{k})$ and an element $\alpha \in \mathcal{E}^+_{n+1}(\boldsymbol{k})$. We identify $\iota_{\boldsymbol{k}}(\alpha) \in \mathrm{Aut}(\mathcal{O}(\boldsymbol{k}))$ with a matrix $\bar{A} \in GL(n+1, \boldsymbol{Z})$ with respect to the basis chosen above. Then the matrix $\bar{A}$ is diagonalizable over $\boldsymbol{R}$ and the set of the eigenvalues of $\bar{A}$ coincides with $\{f^{(i)}(\alpha)\,;\ 1 \le i \le n+1\}$. In particular, $\bar{A}$ is in $SL(n+1, \boldsymbol{Z})$. Choose a matrix $\bar{P} \in GL(n+1, \boldsymbol{R})$ diagonalizing $\bar{A}$ and define

$$\tilde{\psi}_{\boldsymbol{k}} : V = l_{\boldsymbol{k}}(\mathcal{E}^+(\boldsymbol{k})) \otimes \boldsymbol{R} \to GL(n+1, \boldsymbol{R}) \cong GL(\mathcal{O}(\boldsymbol{k}) \otimes \boldsymbol{R}) = GL(\boldsymbol{k} \otimes \boldsymbol{R})$$

by

$$\tilde{\psi}_{\boldsymbol{k}}((\mu_1, \mu_2, \dots, \mu_{n+1})) = \bar{P}\, \mathrm{diag}(e^{\mu_1}, e^{\mu_2}, \dots, e^{\mu_{n+1}})\bar{P}^{-1}.$$

The homomorphism $\tilde{\psi}_{\boldsymbol{k}} : V \to GL(\mathcal{O}(\boldsymbol{k}) \otimes \boldsymbol{R})$ is injective and does not depend on the choice of a basis of $\mathcal{O}(\boldsymbol{k})$.

Put $\Gamma_{\boldsymbol{k}} := l_{\boldsymbol{k}}(\mathcal{E}^+(\boldsymbol{k})) \ltimes_{\psi_{\boldsymbol{k}}} \mathcal{O}(\boldsymbol{k})$, $H_{\boldsymbol{k}1} := l_{\boldsymbol{k}}(\mathcal{E}^+(\boldsymbol{k})) \otimes \boldsymbol{R}$, $H_{\boldsymbol{k}0} := \mathcal{O}(\boldsymbol{k}) \otimes \boldsymbol{R}$ and $H_{\boldsymbol{k}} := H_{\boldsymbol{k}1} \ltimes_{\tilde{\psi}_{\boldsymbol{k}}} H_{\boldsymbol{k}0}$. Because the representation $\tilde{\psi}_{\boldsymbol{k}}$ is injective and is a direct sum of non-equivalent real 1-dimensional representations, the Lie group $H_{\boldsymbol{k}}$ is isomorphic to $H = \boldsymbol{R}^n \ltimes_{\psi} \boldsymbol{R}^{n+1}$. By Proposition 2.5, $\Gamma_{\boldsymbol{k}}$ is a lattice of $H_{\boldsymbol{k}} \cong H$. The following lemma follows directly from the definitions.

LEMMA 3.4. *The algebraic number field $\boldsymbol{k}(\Gamma_{\boldsymbol{k}})$ associated with $\Gamma_{\boldsymbol{k}}$ coincides with $\iota_{\boldsymbol{k}}(\boldsymbol{k})$.*

The above construction of lattices of H is described briefly by Ghys ([8, p. 298]), who attributes the idea to Haefliger. One can also find analogous descriptions in [12, p. 33, Example 7.6]. Our construction is a generalization of them. An additive subgroup $\mathcal{M} \subset \boldsymbol{k}$ of rank $n+1$ is called a *full module* of $\boldsymbol{k}$ ([3, Chapter 2.1]). Two full modules $\mathcal{M}$ and $\mathcal{M}'$ are said to be *similar* if $\mathcal{M}' = \gamma\mathcal{M}$ for some $\gamma \in \boldsymbol{k}^\times := \boldsymbol{k} \setminus \{0\}$. It is obvious that an arbitrary full module of $\boldsymbol{k}$ is similar to a full module contained in $\mathcal{O}(\boldsymbol{k})$. Let $\mathcal{E}$ be a subgroup of finite index of $\mathcal{E}^+(\boldsymbol{k})$. We say a full module $\mathcal{M}$ is *$\mathcal{E}$-invariant* if $\mathcal{E}\mathcal{M} \subset \mathcal{M}$. Using the notation in §2.1, for an $\mathcal{E}$-invariant full module $\mathcal{M}$ and a mod $\mathcal{M}$ relative cocycle $\eta \in C^1_N(l_{\boldsymbol{k}}(\mathcal{E}), \boldsymbol{k} \otimes \boldsymbol{R})$, we can define a lattice $\mathrm{Gr}(\mathcal{M}, l_{\boldsymbol{k}}(\mathcal{E}), \eta)$ of $H_{\boldsymbol{k}}$. It is easy to see that $\mathrm{Gr}(\gamma\mathcal{M}, l_{\boldsymbol{k}}(\mathcal{E}), \gamma\eta)$ is isomorphic to $\mathrm{Gr}(\mathcal{M}, l_{\boldsymbol{k}}(\mathcal{E}), \eta)$ for any $\gamma \in \boldsymbol{k}^\times$.

LEMMA 3.5. *Let Γ be a lattice of $H_{\boldsymbol{k}}$ which is commensurable with $\Gamma_{\boldsymbol{k}}$. Then there exist a subgroup $\mathcal{E}$ of finite index of $\mathcal{E}^+(\boldsymbol{k})$, an $\mathcal{E}$-invariant full module $\mathcal{M} \subset \boldsymbol{k}$ and a mod $\mathcal{M}$ relative cocycle $\eta \in C^1_N(l_{\boldsymbol{k}}(\mathcal{E}), \boldsymbol{k} \otimes \boldsymbol{R})$ such that $\Gamma = \mathrm{Gr}(\mathcal{M}, l_{\boldsymbol{k}}(\mathcal{E}), \eta)$.*

PROOF. Put $\mathcal{M} := \Gamma_0 = \Gamma \cap H_{\boldsymbol{k}0}$. Then $\mathcal{M}$ is a full module of $\boldsymbol{k}$ which is commensurable with $(\Gamma_{\boldsymbol{k}})_0 = \mathcal{O}(\boldsymbol{k})$. Let $\boldsymbol{t} \in \Gamma_1 \subset H_{\boldsymbol{k}1}$. Because $\tilde{\psi}_{\boldsymbol{k}}(\boldsymbol{t})$ preserves $\mathcal{M}$ and commutes with $\tilde{\psi}_{\boldsymbol{k}} \circ l_{\boldsymbol{k}}(\alpha)$ ($\alpha \in \mathcal{E}^+_{n+1}(\boldsymbol{k})$), by Lemma 3.1, there exists $\beta \in \mathcal{E}^+(\boldsymbol{k})$ such that $\boldsymbol{t} = l_{\boldsymbol{k}}(\beta)$. Thus we obtain $\mathcal{E} := l_{\boldsymbol{k}}^{-1}(\Gamma_1) \subset \mathcal{E}^+(\boldsymbol{k})$. Obviously $\mathcal{M}$ is $\mathcal{E}$-invariant. Hence we can write Γ as $\Gamma = \mathrm{Gr}(\mathcal{M}, l_{\boldsymbol{k}}(\mathcal{E}), \eta)$ where η is a relative cocycle. □

3.3. Proof of Theorem 1.1. We return to the case where Γ is a lattice of the Lie group H and prove Theorem 1.1. In 3.1, we defined a linear isomorphism $l_\psi : H_1 \to V \subset \boldsymbol{R}^{n+1}$.

Choose $\boldsymbol{t}_0 \in {H_1}^0 \cap \Gamma_1$ and put $A := \psi(\boldsymbol{t}_0)$. Choose a vector $v_0 \in (H_0 \setminus \bigcup_{i=1}^{n+1} W(i)) \cap \Gamma_0$ and define $\rho : \Gamma_0 \otimes \boldsymbol{Q} \to \boldsymbol{Q}(A)$ by $\rho(\sum_{i=0}^n a_i A^i v_0) = \sum_{i=0}^n a_i A^i$ ($a_i \in \boldsymbol{Q}$, $0 \le i \le n$). The map $\rho : \Gamma_0 \otimes \boldsymbol{Q} \to \boldsymbol{Q}(A)$ is an isomorphism of $\boldsymbol{Q}$-vector spaces, and the image $\rho(\Gamma_0)$ is commensurable with $\mathcal{O}(\boldsymbol{Q}(A))$.

By Dirichlet's unit theorem, the image $\psi(\Gamma_1)$ is a subgroup of finite index of $\mathcal{E}^+(\boldsymbol{Q}(A))$. Put $\boldsymbol{k} := \boldsymbol{k}(\Gamma) = \boldsymbol{Q}(A)$. Define $f_i \in \mathrm{Imb}(\boldsymbol{k})$ by $f_i(A) = \lambda_i(\boldsymbol{t}_0)$. Then we have $\mathrm{Imb}(\boldsymbol{k}) = \{f_1, f_2, \ldots, f_{n+1}\}$. Using these imbeddings, we defined an injective map $l_{\boldsymbol{k}} : \mathcal{E}^+(\boldsymbol{k}) \to V \subset \boldsymbol{R}^{n+1}$ in 3.2. By renumbering the indices if necessary, we may assume $l_\psi|_{\Gamma_1} = l_{\boldsymbol{k}} \circ \psi|_{\Gamma_1} : \Gamma_1 \to l_{\boldsymbol{k}}(\mathcal{E}^+(\boldsymbol{k})) \otimes \boldsymbol{R} = V$. Then the map l_ψ induces a $\boldsymbol{Q}$-vector space isomorphism $\Gamma_1 \otimes \boldsymbol{Q} \to l_{\boldsymbol{k}}(\mathcal{E}^+(\boldsymbol{k})) \otimes \boldsymbol{Q}$.

LEMMA 3.6. *There exists an isomorphism $\Psi_\Gamma : H \to H_{\boldsymbol{k}}$ such that $\Psi_\Gamma(\Gamma)$ is weakly commensurable with $\Gamma_{\boldsymbol{k}}$, where $\boldsymbol{k} := \boldsymbol{k}(\Gamma)$.*

PROOF. Define a map $\Psi : \Gamma_1 \times \Gamma_0 \to H_{\boldsymbol{k}}$ by $\Psi(\boldsymbol{t}, \boldsymbol{x}) = (l_\psi(\boldsymbol{t}), \rho(\boldsymbol{x}))$ ($\boldsymbol{t} \in \Gamma_1, \boldsymbol{x} \in \Gamma_0$). Let $\psi(\boldsymbol{t}) = f(A)$ and $\boldsymbol{x} = g(A)v_0$ ($(\boldsymbol{t}, \boldsymbol{x}) \in \Gamma_1 \times \Gamma_0$ and $f(X), g(X) \in \boldsymbol{Q}[X]$). Then we have

$$\psi_{\boldsymbol{k}}(l_\psi(\boldsymbol{t}))(\rho(\boldsymbol{x})) = (\psi_{\boldsymbol{k}} \circ l_{\boldsymbol{k}}(f(A))) \cdot g(A) = \iota_{\boldsymbol{k}}(f(A)) \cdot g(A) = f(A)g(A) = \rho(\psi(\boldsymbol{t})(\boldsymbol{x}))\,.$$

Thus the map Ψ is a homomorphism from $\mathrm{Gr}(\Gamma_0, \Gamma_1, 0) = \Gamma_1 \ltimes_\psi \Gamma_0$ to $H_{\boldsymbol{k}}$. Clearly, the homomorphism extends to an isomorphism $\Psi_\Gamma : H \to H_{\boldsymbol{k}}$. Since $\Psi_\Gamma(\mathrm{Gr}(\Gamma_0, \Gamma_1, 0))$ is commensurable with $\Gamma_{\boldsymbol{k}}$, by Proposition 2.8, $\Psi_\Gamma(\Gamma)$ is weakly commensurable with $\Gamma_{\boldsymbol{k}}$ □

PROOF OF THEOREM 1.1. By Lemma 3.3, the map in the theorem is well-defined. For each totally real algebraic number field $\boldsymbol{k}$ of degree $n+1$, choose an arbitrary isomorphism $\Phi_{\boldsymbol{k}} : H_{\boldsymbol{k}} \to H$. Then the correspondence $\boldsymbol{k} \mapsto \Phi_{\boldsymbol{k}}(\Gamma_{\boldsymbol{k}})$ induces a right (resp. left) inverse of the map in the theorem by Lemma 3.4 (resp. Lemma 3.6). □

From Lemmas 3.5 and 3.6, every lattice of H is described as follows.

PROPOSITION 3.7. *Let Γ be a lattice of H. Then there exist an isomorphism $\Psi : H \to H_{\boldsymbol{k}(\Gamma)}$, a subgroup $\mathcal{E}$ of finite index of $\mathcal{E}^+(\boldsymbol{k}(\Gamma))$, an $\mathcal{E}$-invariant full module $\mathcal{M} \subset \boldsymbol{k}(\Gamma)$ and a mod $\mathcal{M}$ relative cocycle $\eta \in C^1_N(l_{\boldsymbol{k}(\Gamma)}(\mathcal{E}), \boldsymbol{k}(\Gamma) \otimes \boldsymbol{R})$ such that $\Psi(\Gamma) = \mathrm{Gr}(\mathcal{M}, l_{\boldsymbol{k}(\Gamma)}(\mathcal{E}), \eta)$.*

3.4. The homomorphism A_Γ. Let Γ be a lattice of H. By Proposition 2.9 we identify an automorphism ϕ of Γ with its extension $\varphi \in \mathrm{Aut}(H)$.

Let $\mathrm{Com}(\Gamma)$ denote the set of all lattices of H commensurable with Γ and put $\mathrm{Aut}^{\mathrm{com}}(\Gamma) := \{\varphi \in \mathrm{Aut}(H)\,;\, \varphi(\Gamma) \in \mathrm{Com}(\Gamma)\}$. For each $\Gamma' \in \mathrm{Com}(\Gamma)$, we can regard $\mathrm{Aut}(\Gamma')$ as a subgroup of $\mathrm{Aut}^{\mathrm{com}}(\Gamma)$.

Define a homomorphism $\tilde{A}_\Gamma : \mathrm{Aut}^{\mathrm{com}}(\Gamma) \to \mathrm{Aut}(\boldsymbol{k}(\Gamma)/\boldsymbol{Q})$ by $\tilde{A}_\Gamma(\varphi) = \varphi_*$ (see Lemma 3.3) and define $A_\Gamma : \mathrm{Aut}(\Gamma) \to \mathrm{Aut}(\boldsymbol{k}(\Gamma)/\boldsymbol{Q})$ by $A_\Gamma = \tilde{A}_\Gamma|_{\mathrm{Aut}(\Gamma)}$.

Let $\varphi = \mathrm{hom}(\varphi_0, \varphi_1, \delta \boldsymbol{x}_0) \in \mathrm{Aut}^{\mathrm{com}}(\Gamma)$, $g(X) \in \boldsymbol{Q}[X]$ and $\boldsymbol{t} \in {H_1}^0 \cap \Gamma_1$. Then, from the compatibility condition (2.3), we have

$$\tilde{A}_\Gamma(\varphi)(g(\psi(\boldsymbol{t}))) = \varphi_0 \circ g(\psi(\boldsymbol{t})) \circ \varphi_0^{-1} = g(\psi(\varphi_1(\boldsymbol{t}))) . \tag{3.1}$$

We omit the proof of the following easy lemma. For a bijection $f : X \to Y$ and a map $g : X \to X$, let $\mathrm{Ad}(f)(g) := f \circ g \circ f^{-1} : Y \to Y$.

LEMMA 3.8. (1) *If $\Gamma' \in \mathrm{Com}(\Gamma)$, then we have $\tilde{A}_\Gamma = \tilde{A}_{\Gamma'} : \mathrm{Aut}^{\mathrm{com}}(\Gamma) \to \mathrm{Aut}(\boldsymbol{k}(\Gamma)/\boldsymbol{Q})$.*

(2) *If $\varphi : H \to H'$ is an isomorphism, then we have $A_{\varphi(\Gamma)} \circ \mathrm{Ad}(\varphi) = \mathrm{Ad}(\varphi_*) \circ A_\Gamma$.*

Let $\boldsymbol{k}$ be a totally real algebraic number field of degree $n+1$ over $\boldsymbol{Q}$. For $\sigma \in \mathrm{Aut}(\boldsymbol{k}/\boldsymbol{Q})$, let $\sigma_0 \in GL(\boldsymbol{k} \otimes \boldsymbol{R})$ (resp. $\sigma_1 \in GL(l_{\boldsymbol{k}}(\mathcal{E}^+(\boldsymbol{k})) \otimes \boldsymbol{R})$) denote the linear extension of σ (resp. $\mathrm{Ad}(l_{\boldsymbol{k}})(\sigma|_{\mathcal{E}^+(\boldsymbol{k})})$).

LEMMA 3.9. *For $i = 1, 2$, let $\Gamma_{(i)} = \mathrm{Gr}(\mathcal{M}_i, l_{\boldsymbol{k}}(\mathcal{E}_i), \eta_i) \in \mathrm{Com}(\Gamma_{\boldsymbol{k}})$ where $\mathcal{E}_i$ is a subgroup of finite index of $\mathcal{E}^+(\boldsymbol{k})$, $\mathcal{M}_i$ is an $\mathcal{E}_i$-invariant full module of $\boldsymbol{k}$ and $\eta_i \in C^1_N(l_{\boldsymbol{k}}(\mathcal{E}_i), \boldsymbol{k} \otimes \boldsymbol{R})$ is a relative cocycle. Let $\varphi = \mathrm{hom}(\varphi_0, \varphi_1, \delta \boldsymbol{x}_0)$ be an automorphism of $H_{\boldsymbol{k}}$ such that $\varphi(\Gamma_{(1)}) = \Gamma_{(2)}$. Put $\sigma := \mathrm{Ad}(\iota_{\boldsymbol{k}}^{-1})(\tilde{A}_{\Gamma_{\boldsymbol{k}}}(\varphi)) \in \mathrm{Aut}(\boldsymbol{k}/\boldsymbol{Q})$. Then we have the following.*

(1) *$\varphi_1 = \sigma_1$ and $\sigma(\mathcal{E}_1) = \mathcal{E}_2$.*

(2) *There exists $\gamma \in \boldsymbol{k}$ such that $\varphi_0 = \iota_{\boldsymbol{k}}(\gamma) \circ \sigma_0$. In particular $\mathcal{M}_2 = \gamma\sigma(\mathcal{M}_1)$.*

PROOF. By definition, $\tilde{A}_{\Gamma_{\boldsymbol{k}}}(\varphi)$ is given by $\mathrm{Ad}(\iota_{\boldsymbol{k}}^{-1})(\tilde{A}_{\Gamma_{\boldsymbol{k}}}(\varphi))|_{\mathcal{E}^+_{n+1}(\boldsymbol{k})} = \mathrm{Ad}(l_{\boldsymbol{k}}^{-1})(\varphi_1)|_{\mathcal{E}^+_{n+1}(\boldsymbol{k})}$. Therefore we get $\sigma_1 = \varphi_1$ and the assertion (1). Because the isomorphism $\varphi_0 \in GL(\boldsymbol{k} \otimes \boldsymbol{R})$ sends $\mathcal{M}_1$ to $\mathcal{M}_2$, we obtain $\varphi_0(\boldsymbol{k}) = \boldsymbol{k}$. Put $\gamma := \varphi_0(1) \in \boldsymbol{k}$. Then, for any $\alpha \in \mathcal{E}_1 \cap \mathcal{E}^+_{n+1}(\boldsymbol{k})$, we have

$$\varphi_0(\alpha) = \varphi_0(\iota_{\boldsymbol{k}}(\alpha)(1)) = \psi_{\boldsymbol{k}}(\varphi_1 \circ l_{\boldsymbol{k}}(\alpha))(\varphi_0(1)) = \iota_{\boldsymbol{k}}(\sigma(\alpha))(\gamma) = \sigma(\alpha) \cdot \gamma .$$

Thus we obtain (2). □

COROLLARY 3.10. *Let $\Gamma' = \mathrm{Gr}(\mathcal{M}, l_{\boldsymbol{k}}(\mathcal{E}), 0) \in \mathrm{Com}(\Gamma_{\boldsymbol{k}})$ where $\mathcal{E}$ is a subgroup of finite index of $\mathcal{E}^+(\boldsymbol{k})$ and $\mathcal{M} \subset \boldsymbol{k}$ is an $\mathcal{E}$-invariant full module. Then we have*

$$\mathrm{Ad}(\iota_{\boldsymbol{k}}^{-1})(A_{\Gamma'}(\mathrm{Aut}(\Gamma'))) = \{\sigma \in \mathrm{Aut}(\boldsymbol{k}/\boldsymbol{Q}) \, ; \, \sigma(\mathcal{E}) = \mathcal{E} \text{ and } \sigma(\mathcal{M}) \text{ is similar to } \mathcal{M}\} .$$

Let Γ be a lattice of H. We can now determine the kernel of the homomorphism A_Γ.

COROLLARY 3.11. *Let Γ be a lattice of H and φ an automorphism of Γ. Suppose $A_\Gamma(\varphi) = \mathrm{id} \in \mathrm{Aut}(\boldsymbol{k}(\Gamma)/\boldsymbol{Q})$. Then there exists $h_0 \in H$ such that $\varphi^2 = \mathrm{Ad}(h_0)$.*

PROOF. Because $A_\Gamma(\varphi) = \mathrm{id}$, we have $\psi(\varphi_1(\boldsymbol{t})) = \psi(\boldsymbol{t})$ for any $\boldsymbol{t} \in {H_1}^0 \cap \Gamma_1$, and hence $\varphi_1 = \mathrm{id}$. Thus we can write φ as $\varphi = \mathrm{hom}(\varphi_0, \mathrm{id}, \delta \boldsymbol{x}_0) \in \mathrm{Aut}(\Gamma) \subset \mathrm{Aut}(H)$. By Proposition 3.7, there exists an isomorphism $\Psi : H \to H_{\boldsymbol{k}}$ such that $\Psi(\Gamma) =: \mathrm{Gr}(\mathcal{E}, \mathcal{M}, \eta) \in \mathrm{Com}(\Gamma_{\boldsymbol{k}})$ where $\boldsymbol{k} = \boldsymbol{k}(\Gamma)$. Applying Lemma 3.9 to $\mathrm{Ad}(\Psi)(\varphi)$, there exists $\gamma \in \boldsymbol{k}$ such that $\gamma\mathcal{M} = \mathcal{M}$ and $\mathrm{Ad}_\Psi(\varphi_0) = \iota_{\boldsymbol{k}}(\gamma)$. Because $\gamma\mathcal{M} = \mathcal{M}$, we have $\gamma \in \mathcal{E}(\boldsymbol{k})$ and hence

$\gamma^2 \in \mathcal{E}^+(\boldsymbol{k})$. Put $\boldsymbol{t}_0 := \Psi^{-1}(l_{\boldsymbol{k}}(\gamma^2)) \in H_1$ and $h_0 := (\boldsymbol{t}_0, -\boldsymbol{x}_0 - \varphi_0(\boldsymbol{x}_0)) \in H$. Then, using Lemma 2.4, it is easy to see that $\varphi^2 = \mathrm{hom}(\varphi_0^2, \mathrm{id}, \delta(\boldsymbol{x}_0 + \varphi_0(\boldsymbol{x}_0)) = \mathrm{Ad}(h_0)$. □

4. Proof of Theorem 1.2.

4.1. Proof of Theorem 1.2 for the case where $n \geq 2$. In this subsection we assume $n \geq 2$. Let $\boldsymbol{k}$ be a totally real algebraic number field of degree $n+1$. From Lemmas 3.6 and 3.8, to prove Theorem 1.2, we may assume $\Gamma = \Gamma_{\boldsymbol{k}} \subset H_{\boldsymbol{k}}$. Put $F_0 := \mathrm{Aut}(\boldsymbol{k}/\boldsymbol{Q})$, and take a subgroup F_1 of F_0. We show that there exists a subgroup Γ' of finite index of $\Gamma_{\boldsymbol{k}}$ such that $\mathrm{Ad}(\iota_{\boldsymbol{k}}^{-1})(\tilde{A}_{\Gamma_{\boldsymbol{k}}}(\mathrm{Aut}(\Gamma'))) = \mathrm{Ad}(\iota_{\boldsymbol{k}}^{-1})(A_{\Gamma'}(\mathrm{Aut}(\Gamma'))) = F_1$. By Corollary 3.10, if $F_1 = F_0$, it suffices to put $\Gamma' := \Gamma_{\boldsymbol{k}}$. In general, we seek for Γ' in the form $l_{\boldsymbol{k}}(\mathcal{E}_1) \ltimes \mathcal{O}(\boldsymbol{k})$ where $\mathcal{E}_1$ is a subgroup of finite index of $\mathcal{E}^+(\boldsymbol{k})$.

We use the following theorem due to Herbrand and Artin [1] on relative fundamental units. We say that a subset $\{\eta_1, \eta_2, \ldots, \eta_k\}$ of an abelian group (written multiplicatively) is *independent* if an arbitrary relation of the form $\prod_{i=1}^k \eta_i^{a_i} = 1$ ($a_i \in \boldsymbol{Z}, 1 \leq i \leq k$) implies $a_i = 0$ for all i.

THEOREM 4.1 (Artin). *Let $\boldsymbol{k}$ be a totally real algebraic number field and $\boldsymbol{k}_0$ a subfield of $\boldsymbol{k}$. Let $[\boldsymbol{k} : \boldsymbol{k}_0] = d$ and $[\boldsymbol{k}_0 : \boldsymbol{Q}] = r$. Suppose the extension $\boldsymbol{k}/\boldsymbol{k}_0$ is Galois with the Galois group $F_0 = \{g_j\,;\ 1 \leq j \leq d\}$.*

Then there exist $\xi_i \in \mathcal{E}(\boldsymbol{k}_0)$ $(1 \leq i \leq r-1)$ and $\varepsilon_j \in \mathcal{E}(\boldsymbol{k})$ $(1 \leq j \leq r)$ satisfying the following properties.

(1) *The set $\{\xi_i\,;\ 1 \leq i \leq r-1\}$ is independent and generates a subgroup of finite index of $\mathcal{E}(\boldsymbol{k}_0)$.*

(2) *For each j $(1 \leq j \leq r)$, we have $\prod_{k=1}^d g_k(\varepsilon_j) = 1$.*

(3) *For each j $(1 \leq j \leq r)$, and for each subset E_j of $\{g_k(\varepsilon_j)\,;\ 1 \leq k \leq d\}$ consisting of $d-1$ elements, the set $\{\xi_i\,;\ 1 \leq i \leq r-1\} \bigcup_{1 \leq j \leq r} E_j$ is independent and generates a subgroup of finite index of $\mathcal{E}(\boldsymbol{k})$.*

We return to our case where $\boldsymbol{k}$ is a totally real algebraic number field of degree $n+1$ and $F_0 = \mathrm{Aut}(\boldsymbol{k}/\boldsymbol{Q})$. Let $\boldsymbol{k}_0$ be the invariant field of $\boldsymbol{k}$, that is, $\boldsymbol{k}_0 = \boldsymbol{k}^{F_0} := \{\beta \in \boldsymbol{k}\,;\ g(\beta) = \beta$ for all $g \in F_0\}$, and let $r := [\boldsymbol{k}_0 : \boldsymbol{Q}]$. By the fundamental theorem of Galois theory, the extension $\boldsymbol{k}/\boldsymbol{k}_0$ is Galois with the Galois group F_0. Given a subgroup $F_1 \subset F_0$, put $\boldsymbol{k}_1 := \boldsymbol{k}^{F_1}$, $[\boldsymbol{k} : \boldsymbol{k}_1] =: e$ and $[\boldsymbol{k}_1 : \boldsymbol{k}_0] =: f$. We are assuming $n+1 = efr \geq 3$ and $\boldsymbol{k}_0 \subsetneq \boldsymbol{k}_1$, and hence $f \geq 2$.

Let $F_1 = \{\sigma_1 = \mathrm{id}, \sigma_2, \ldots, \sigma_e\}$ and let $\{\tau_1 = \mathrm{id}, \tau_2, \ldots, \tau_f\} \subset F_0$ be a complete set of representatives of the left cosets $F_1 \backslash F_0$. Then we can express F_0 as $\{\sigma_\mu \tau_\nu\,;\ 1 \leq \mu \leq e, 1 \leq \nu \leq f\}$. Applying Theorem 4.1 to our case and using the fact that $\varepsilon \in \mathcal{E}(\boldsymbol{k})$ implies $\varepsilon^2 \in \mathcal{E}^+(\boldsymbol{k})$, there exist $\{\xi_i\,;\ 1 \leq i \leq r-1\} \subset \mathcal{E}^+(\boldsymbol{k}) \cap \boldsymbol{k}_0$ and $\{\varepsilon_j\,;\ 1 \leq j \leq r\} \subset \mathcal{E}^+(\boldsymbol{k})$ which satisfy the properties (1), (2) and (3) of Theorem 4.1, where $d = ef$.

Consider the set Σ_1 given by

$$\Sigma_1 := \{\xi_1^{3(r-1)}, \xi_i, \sigma_\mu\tau_\nu(\varepsilon_1)^{\nu!}\xi_1^{r-1}, \sigma_\mu\tau_\nu(\varepsilon_j)\,; \\ 2 \leq i \leq r-1, 1 \leq \mu \leq e, 1 \leq \nu \leq f, 2 \leq j \leq r\}.$$

Let $\mathcal{E}_1$ be the subgroup of $\mathcal{E}^+(\boldsymbol{k})$ generated by Σ_1. By (3) of Theorem 4.1, we have $[\mathcal{E}^+(\boldsymbol{k}) : \mathcal{E}_1] < \infty$.

LEMMA 4.2. *The subgroup* $\mathrm{Fix}(\mathcal{E}_1) := \{g \in F_0\,;\ g(\mathcal{E}_1) = \mathcal{E}_1\}$ *coincides with* F_1.

It is obvious from the choice of Σ_1 that $F_1 \subset \mathrm{Fix}(\mathcal{E}_1)$. Take $g \in (F_1)^c \cap F_0$. Then g is described as $\sigma_{\mu_0}\tau_{\nu_0}$ where $\nu_0 \geq 2$. To prove the lemma, it is sufficient to show the following.

CLAIM 4.3. $g(\mathcal{E}_1) \not\subseteq \mathcal{E}_1$.

PROOF. Suppose $g(\sigma_1\tau_1(\varepsilon_1)\xi_1^{r-1}) = \sigma_{\mu_0}\tau_{\nu_0}(\varepsilon_1)\xi_1^{r-1} \in \mathcal{E}_1$. Then, from (3) of Theorem 4.1, there exist integers $m, n_{\mu,\nu}, a_i, b_{j,\mu,\nu}$ $(1 \leq \mu \leq e, 1 \leq \nu \leq f, 2 \leq i \leq r-1, 2 \leq j \leq r)$ such that

$$\sigma_{\mu_0}\tau_{\nu_0}(\varepsilon_1)\xi_1^{r-1} = \xi_1^{3(r-1)m} \prod_i \xi_i^{a_i} \prod_{\mu,\nu} \sigma_\mu\tau_\nu(\varepsilon_1)^{\nu!\cdot n_{\mu,\nu}} \xi_1^{(r-1)n_{\mu,\nu}} \prod_{j,\mu,\nu} \sigma_\mu\tau_\nu(\varepsilon_j)^{b_{j,\mu,\nu}}\,.$$

From (2) and (3) of Theorem 4.1, we have $a_i = 0$ $(2 \leq i \leq r-1)$, $\prod_{\mu,\nu} \sigma_\mu\tau_\nu(\varepsilon_j)^{b_{j,\mu,\nu}} = 1$ $(2 \leq j \leq r)$ and hence

$$\sigma_{\mu_0}\tau_{\nu_0}(\varepsilon_1)^{1-\nu_0!\cdot n_{\mu_0,\nu_0}} = \xi_1^{(3m-1+\sum_{\mu,\nu} n_{\mu,\nu})(r-1)} \prod_{(\mu,\nu)\neq(\mu_0,\nu_0)} \sigma_\mu\tau_\nu(\varepsilon_1)^{\nu!\cdot n_{\mu,\nu}}\,.$$

It follows, again from (2) and (3) of Theorem 4.1, that the integers $m, n_{\mu,\nu}$ must satisfy the following equations:

$$\left(3m - 1 + \sum_{\mu,\nu} n_{\mu,\nu}\right)(r-1) = 0\,, \tag{4.1}$$

$$\nu!\cdot n_{\mu,\nu} - \nu_0!\cdot n_{\mu_0,\nu_0} + 1 = 0 \quad ((\mu,\nu) \neq (\mu_0,\nu_0))\,. \tag{4.2}$$

Case 1: $e \geq 2$. In this case we can take μ_1 such that $\mu_1 \neq \mu_0$. The equation (4.2) for $(\mu,\nu) = (\mu_1,\nu_0)$ gives

$$\nu_0!\cdot n_{\mu_1,\nu_0} - \nu_0!\cdot n_{\mu_0,\nu_0} + 1 = 0\,,$$

which is impossible because $\nu_0 \geq 2$.

Case 2: $e = 1$ and $f \geq 3$. Take ν_1 such that $\nu_1 \neq \nu_0$ and $1 < \nu_1 \leq f$. Then the equation (4.2) for $(\mu,\nu) = (1,\nu_1)$ gives

$$\nu_1!\cdot n_{1,\nu_1} - \nu_0!\cdot n_{1,\nu_0} + 1 = 0\,,$$

which is impossible because $\nu_0, \nu_1 \geq 2$.

Case 3: $e = 1$ and $f = 2$. In this case we have $r \geq 2$ from the assumption $efr \geq 3$. Then the equations (4.1) and (4.2) can be read as

$$\begin{cases} 3m - 1 + n_{1,1} + n_{1,2} = 0 \\ \qquad n_{1,1} - 2n_{1,2} + 1 = 0\,. \end{cases}$$

The subtraction of the second equation from the first shows that this is impossible. □

PROOF OF THEOREM 1.2 FOR THE CASE WHERE $n \geq 2$. Put $\boldsymbol{k} := \boldsymbol{k}(\Gamma)$. Let F be a subgroup of $F_0 := \mathrm{Aut}(\boldsymbol{k}/\boldsymbol{Q})$. As was remarked at the beginning of this subsection, we may assume that $f := [F_0 : F] \geq 2$. Take $\mathcal{E}_1 \subset \mathcal{E}^+(\boldsymbol{k})$ as in Lemma 4.2 and put $\Gamma' := l_{\boldsymbol{k}}(\mathcal{E}_1) \ltimes \mathcal{O}(\boldsymbol{k}) \subset \Gamma_{\boldsymbol{k}}$. Then we have $\mathrm{Ad}(\iota_{\boldsymbol{k}}^{-1})(\tilde{A}_{\Gamma_{\boldsymbol{k}}}(\mathrm{Aut}(\Gamma'))) = \mathrm{Ad}(\iota_{\boldsymbol{k}}^{-1})(A_{\Gamma'}(\mathrm{Aut}(\Gamma'))) = F$ by Corollary 3.10. □

4.2. The 3-dimensional case. In this subsection we treat the case where $n = 1$. That is, we consider $G = \mathrm{Aff}^+(\boldsymbol{R})$ and the 3-dimensional Lie group $H = \boldsymbol{R} \ltimes_\psi \boldsymbol{R}^2$. Let Γ be a lattice of H. Then the group $\mathrm{Aut}(\boldsymbol{k}(\Gamma)/\boldsymbol{Q})$ is isomorphic to $\boldsymbol{Z}/2\boldsymbol{Z}$. Theorem 1.2 is stated, when $n = 1$, as follows.

PROPOSITION 4.4. *Let Γ be a lattice of $H = \boldsymbol{R} \ltimes \boldsymbol{R}^2$. Then there exist lattices Γ', Γ'' commensurable with Γ such that $A_{\Gamma'}(\mathrm{Aut}(\Gamma')) \cong \boldsymbol{Z}/2\boldsymbol{Z}$ and $A_{\Gamma''}(\mathrm{Aut}(\Gamma'')) = \{\mathrm{id}\}$.*

Take $A \in SL(2, \boldsymbol{Z})$ with $\mathrm{trace}(A) > 2$. Choose $P \in GL(2, \boldsymbol{R})$ such that $P^{-1}AP = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ for some $\lambda > 0$. Define a representation $\psi_A : \boldsymbol{R} \to SL(2, \boldsymbol{R})$ by

$$\psi_A(t) = A^t := P \begin{pmatrix} \lambda^t & 0 \\ 0 & \lambda^{-t} \end{pmatrix} P^{-1} \quad (t \in \boldsymbol{R}).$$

Denote by H_A (resp. Γ_A) the Lie group $\boldsymbol{R} \ltimes_{\psi_A} \boldsymbol{R}^2$ (resp. the lattice $\boldsymbol{Z} \ltimes_{\psi_A} \boldsymbol{Z}^2$ of H_A). In order to prove Proposition 4.4, we may put $H = H_A$ and $\Gamma = \Gamma_A$ for some A (see e.g., [7, Proposition II.2.3]).

We say a matrix $B \in SL(2, \boldsymbol{Z})$ is *$\boldsymbol{Z}$-conjugate* (resp. *$\boldsymbol{Q}$-conjugate*) to $B' \in SL(2, \boldsymbol{Z})$ if there exists $P \in GL(2, \boldsymbol{Z})$ (resp. $P \in GL(2, \boldsymbol{Q})$) such that $B' = P^{-1}BP$.

LEMMA 4.5. *Let $B \in SL(2, \boldsymbol{Z})$* ($\mathrm{trace}(B) > 2$). *Then $A_{\Gamma_B}(\mathrm{Aut}(\Gamma_B)) \neq \{\mathrm{id}\}$ if and only if B is $\boldsymbol{Z}$-conjugate to B^{-1}.*

PROOF. An automorphism φ of Γ_B induces $\varphi_0 \in \mathrm{Aut}((\Gamma_B)_0) \cong GL(2, \boldsymbol{Z})$ and $\varphi_1 \in \mathrm{Aut}((\Gamma_B)_1) \cong GL(1, \boldsymbol{Z})$. Suppose there exists $\varphi \in \mathrm{Aut}(\Gamma_B)$ such that $A_{\Gamma_B}(\varphi) \neq \{\mathrm{id}\}$. Then, from the compatibility condition (2.3), we have $\varphi_1 = -\mathrm{id}$, and hence $B^{-1} = \varphi_0 B \varphi_0^{-1}$. The converse statement is proved similarly. □

The $\boldsymbol{Z}$-conjugacy classes of matrices in $SL(2, \boldsymbol{Z})$ are studied explicitly by Fukuhara and Sakamoto [6]. Each matrix $B \in SL(2, \boldsymbol{Z})$ with $\mathrm{trace}(B) > 2$ is $\boldsymbol{Z}$-conjugate to a form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $a \geq c \geq d \geq 0$ and $a \geq b \geq d$. For a matrix $B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of the above form, let

$$\frac{a}{c} = l_1 + \cfrac{1}{l_2 + \cfrac{1}{\cdots + \cfrac{1}{l_{2r-1} + \cfrac{1}{l_{2r}}}}}, \quad (r, l_1, l_2, \ldots, l_{2r} \in \boldsymbol{Z}_+)$$

be the continued fraction expansion of a/c into even number of terms. Then B is $\boldsymbol{Z}$-conjugate to B^{-1} if and only if the sequence $(l_1, l_2, \ldots, l_{2r})$ coincides with the inverse sequence $(l_{2r}, \ldots, l_2, l_1)$ up to a repetition of the cyclic permutation of this order ([6, p. 316]).

Let $B \in SL(2, \boldsymbol{Z})$. Suppose there exist $p \in \boldsymbol{Z}$ and $P \in GL(2, \boldsymbol{Q})$ such that $B = P^{-1}A^p P$. Then we can define an isomorphism $\Phi_{A^p B} : H_B \to H_A$ by $(t, \boldsymbol{x}) \mapsto (pt, P\boldsymbol{x})$ $((t, \boldsymbol{x}) \in H_B)$. It is easy to see that $\Phi_{A^p B}(\Gamma_B)$ is commensurable with Γ_A.

PROOF OF PROPOSITION 4.4. The existence of a lattice Γ' follows from the argument in the first paragraph of 4.1.

Take a matrix $A \in SL(2, \boldsymbol{Z})$ with $\tau := \mathrm{trace}(A) > 2$ and put $\Gamma = \Gamma_A$. Put

$$A'' := \begin{pmatrix} \tau^6 - 6\tau^4 + 9\tau^2 - \tau - 1 & (\tau - 2)(\tau^4 + \tau^3 - 4\tau^2 - 2\tau + 5) \\ \tau^2 & \tau - 1 \end{pmatrix}.$$

Then we have

$$\frac{\tau^6 - 6\tau^4 + 9\tau^2 - \tau - 1}{\tau^2} = \tau^4 - 6\tau^2 + 8 + \cfrac{1}{1 + \cfrac{1}{\tau - 2 + \cfrac{1}{\tau + 1}}}.$$

The quadruple $(\tau^4 - 6\tau^2 + 8, 1, \tau - 2, \tau + 1)$ is not equal to $(\tau + 1, \tau - 2, 1, \tau^4 - 6\tau^2 + 8)$ up to cyclic permutations because $1 \le \tau - 2 < \tau + 1 < \tau^4 - 6\tau^2 + 8$. By the criterion of Fukuhara and Sakamoto, the matrix A'' is not $\boldsymbol{Z}$-conjugate to its inverse.

If λ, λ^{-1} are eigenvalues of A, then $\lambda + \lambda^{-1} = \tau$. So,

$$\mathrm{trace}(A^6) = \lambda^6 + \lambda^{-6} = (\lambda^3 + \lambda^{-3})^2 - 2 = (\tau^3 - 3\tau)^2 - 2\,.$$

Thus we have $\mathrm{trace}(A'') = (\tau^3 - 3\tau)^2 - 2 = \mathrm{trace}(A^6)$. It is easy to see that both A'' and A^6 are $\boldsymbol{Q}$-conjugate to their common companion matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & \mathrm{trace}(A'') \end{smallmatrix}\right)$ (see e.g., [13, p. 249]).

Put $\Gamma'' := \Phi_{A^6 A''}(\Gamma_{A''})$. Then Γ'' is commensurable with $\Gamma = \Gamma_A$ and $A_{\Gamma''}(\mathrm{Aut}(\Gamma'')) = \{\mathrm{id}\}$ by Lemma 4.5. This completes the proof of Proposition 4.4. □

5. Codimension one homogeneous actions of $\mathrm{Aff}^+(\boldsymbol{R})^n$**.** Let $\Phi_i : G_i \times M_i \to M_i$ $(i = 1, 2)$ be an action of a Lie group G_i on a manifold M_i. They are said to be C^r*-conjugate* $(0 \le r \le \omega)$ if there exist an isomorphism $\rho : G_1 \to G_2$ and a C^r-diffeomorphism $f : M_1 \to M_2$ such that $\Phi_2(\rho(g), f(x)) = f(\Phi_1(g, x))$ for all $(g, x) \in G_1 \times M_1$.

Let $\Gamma \subset H$ be a lattice and let $G' \subset H$ be a subgroup isomorphic to $G = \mathrm{Aff}^+(\boldsymbol{R})^n$. Denote by $\Phi_{(G', \Gamma)}$ the homogeneous action of G' on H/Γ. The following proposition is much stronger than what is needed for the proof of Theorem 1.3. But we give a proof here because the assertion seems to be of some independent interest.

PROPOSITION 5.1. *Let G_i $(i = 1, 2)$ be subgroups of H which are isomorphic to $G = \mathrm{Aff}^+(\boldsymbol{R})^n$ and let Γ_i $(i = 1, 2)$ be lattices of H. Suppose the homogeneous action $\Phi_{(G_1, \Gamma_1)}$ is C^0-conjugate to $\Phi_{(G_2, \Gamma_2)}$ by a homeomorphism $f : H/\Gamma_1 \to H/\Gamma_2$ and an isomorphism $\rho : G_1 \to G_2$. Let $\tilde{f} : H \to H$ be a lift of f. Then there exists $h_0 \in H$ such that $\varphi := L_{h_0} \circ \tilde{f}$ is an automorphism of H and $\rho = \mathrm{Ad}(h_0^{-1}) \circ \varphi_{|G_1}$ where L_{h_0} denotes the left translation of H by h_0. In particular the homeomrphism f is analytic.*

PROOF. Because every isomorphism between lattices of H extends uniquely to an automorphism of H (Proposition 2.9), by composing an automorphism of H, we may assume that $\Gamma_1 = \Gamma_2$ and $f : H/\Gamma_1 \to H/\Gamma_1$ is isotopic to the identity. Then we have $\sup\{d(h, \tilde{f}(h))\,;\, h \in H\} < \infty$ where d is a right invariant metric of H. Put $h_0 := \tilde{f}(e)^{-1}$ and $\varphi := L_{h_0} \circ \tilde{f}$ where e is the unit element of H. It suffices to show that $\varphi = \mathrm{id}$.

It is easy to see that $\rho = \mathrm{Ad}(h_0^{-1}) \circ \varphi|_{G_1}$ and $\varphi(gh\gamma) = \varphi(g)\varphi(h)\gamma$ where $g \in G_1$, $h \in H$ and $\gamma \in \Gamma_1$. Suppose there exists $g \in G_1$ such that $\varphi(g) \neq g$. Then the set $\{d(g^n, \varphi(g^n))\,;\, n \in \boldsymbol{Z}\} = \{d(g^n, \varphi(g)^n)\,;\, n \in \boldsymbol{Z}\} \subset \boldsymbol{R}$ is easily seen to be unbounded. This contradiction shows that $\varphi|_{G_1} = \mathrm{id}_{G_1}$. Because $G_1\Gamma_1$ is dense in H ([7]) and $\varphi|_{G_1\Gamma_1} = \mathrm{id}_{G_1\Gamma_1}$, we obtain $\varphi = \mathrm{id}$. Hence we have proved the proposition. □

Denote by $\mathcal{C}(G)$ the set of all inner conjugacy classes of subgroups of H which are isomorphic to G. By Proposition 2.3, we have $\mathcal{C}(G) = \{[G(i)]\,;\, 1 \leq i \leq n+1\}$, where $G(i) = H_1 \ltimes_\psi W(i)$. Choose $\boldsymbol{t}_0 \in {H_1}^0 \cap \Gamma_1$ and let $A := \psi(\boldsymbol{t}_0) \in \boldsymbol{k}(\Gamma)$. Recall that $f_i \in \mathrm{Imb}(\boldsymbol{k}(\Gamma))$ is the imbedding defined by $f_i(A) = \lambda_i(\boldsymbol{t}_0)$, which is the eigenvalue of $\psi(\boldsymbol{t}_0)|_{W_i}$. To prove Theorem 1.3 we consider the following maps.

(1) $S : \mathrm{Aut}(H) \to \mathrm{Aut}(\mathcal{C}(G))$ defined by $S(\varphi)([G']) = [\varphi(G')]$ $([G'] \in \mathcal{C}(G))$.

(2) The bijection $\iota : \mathcal{C}(G) \to \mathrm{Imb}(\boldsymbol{k}(\Gamma))$ given by $\iota([G(i)]) = f_i$ $(1 \leq i \leq n+1)$.

(3) $R_\Gamma : \mathrm{Aut}(\boldsymbol{k}(\Gamma)/\boldsymbol{Q}) \to \mathrm{Aut}(\mathrm{Imb}(\boldsymbol{k}(\Gamma)))$ defined by $R_\Gamma(\sigma)(f) := f \circ \sigma^{-1}$ $(\sigma \in \mathrm{Aut}(\boldsymbol{k}(\Gamma)/\boldsymbol{Q}),\, f \in \mathrm{Imb}(\boldsymbol{k}(\Gamma))$.

LEMMA 5.2. *We have* $\mathrm{Ad}(\iota) \circ S|_{\mathrm{Aut}(\Gamma)} = R_\Gamma \circ A_\Gamma$.

PROOF. Let $\varphi \in \mathrm{Aut}(\Gamma)$. Suppose $\varphi([G(i)]) = [G(j)]$. Then we have $\varphi_0(W_i) = W_j$. From the compatibility condition (2.3), we obtain $f_i \circ A_\Gamma(\varphi)^{-1}(\psi(\boldsymbol{t}_0)) = \lambda_i(\psi(\varphi_1^{-1}(\boldsymbol{t}_0))) = \lambda_j(\psi(\boldsymbol{t}_0))$. It follows that $\iota \circ S(\varphi)([G(i)])) = R_\Gamma(A_\Gamma(\varphi))(f_i)$. □

PROOF OF THEOREM 1.3. By Proposition 2.3, there are at most $n+1$ homogeneous actions $\Phi_{(G(i),\Gamma)}$ $(1 \leq i \leq n+1)$ of $G = \mathrm{Aff}^+(\boldsymbol{R})^n$ on H/Γ up to inner conjugacy. By Propositions 5.1 and 2.9, the set $\mathrm{Conj}(H/\Gamma)$ is in bijective correspondence with the set $\left\{\Phi_{(G(i),\Gamma)}\,;\, 1 \leq i \leq n+1\right\}/\sim$ where the equivalence relation $\sim$ is defined by $\Phi_{(G(i),\Gamma)} \sim \Phi_{(G(j),\Gamma)}$ if and only if there exists $\varphi \in \mathrm{Aut}(H)$ such that (i) $\varphi(\Gamma) = \Gamma$ and (ii) $S(\varphi)[G(i)] = [G(j)]$. By Lemma 5.2, the condition (ii) is equivalent to the condition (ii′) $R_\Gamma(A_\Gamma(\varphi))(f_i) = f_j$. Thus $\mathrm{Conj}(H/\Gamma)$ is in bijective correspondence with the quotient set $\mathrm{Imb}(\boldsymbol{k}(\Gamma))/R_\Gamma \circ A_\Gamma(\mathrm{Aut}\,\Gamma)$. Because R_Γ gives a free action of $\mathrm{Aut}(\boldsymbol{k}(\Gamma)/\boldsymbol{Q})$ on $\mathrm{Imb}(\boldsymbol{k}(\Gamma))$, we obtain the first assertion of the theorem. The second assertion follows from Theorem 1.2 and the first assertion. □

REMARK 5.3. The number $|\,\mathrm{Aut}(\boldsymbol{k}(\Gamma)/\boldsymbol{Q})|$ divides $n+1$ because $\boldsymbol{k}(\Gamma)$ is an extension field of degree $n+1$ over $\boldsymbol{Q}$.

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DEPARTMENT OF MATHEMATICS
TOIN UNIVERSITY OF YOKOHAMA
KUROGANECHO, AOBA, YOKOHAMA 225–8502
JAPAN

E-mail address: tsuchiya@cc.toin.ac.jp

DEPARTMENT OF MATHEMATICS
INTERNATIONAL CHRISTIAN UNIVERSITY
OSAWA, MITAKA, TOKYO 181–8585
JAPAN

E-mail address: yamakawa@icu.ac.jp