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PERIODIC TRAVELLING WAVE SOLUTIONS OF A CURVATURE FLOW EQUATION IN THE PLANE

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Abstract. In the plane, we consider a curvature flow equation in heterogeneous media with periodic horizontal striations, the periodicity in space is expressed by periodic (in vertical direction) coefficients in the equation. We prove the existence and uniqueness of a curve which travels upward periodically with an average speed. At each time, the graph of the curve is a periodic undulating line at a finite distance from a straight line with a given inclination angle. We also show that the average speed depends on the inclination angle monotonously. Moreover, for homogenization problem as the spatial period tends to zero, we estimate the average speed by the inclination angle and some means of the periodic coefficients.

1. Introduction. In this paper we will be concerned with periodic travelling wave solutions of a curvature flow equation

$$V = a\left(\frac{y}{\varepsilon}\right)\kappa + b\left(\frac{y}{\varepsilon}\right) \tag{1}$$

in the xy-plane, where V is the normal velocity of a plane curve, κ is the curvature, $\varepsilon > 0$ is a parameter, a and b are 1-periodic positive functions, which are continuously differentiable. To avoid sign confusion, the normal to the curve will always be chosen upward (toward y-direction), and the sign of V and κ will be understood in accordance with this choice of the direction of the normal. Consequently, V is positive when the curve moves upward and κ is positive at those points where the curve is convex (see Figure 1).

We will only consider the case where each curve is the graph of a function $y = u(x, t)$, so (1) is equivalent to

$$u_t = a\left(\frac{u}{\varepsilon}\right)\frac{u_{xx}}{1+u_x^2} + b\left(\frac{u}{\varepsilon}\right)\sqrt{1+u_x^2}\,, \quad x \in \boldsymbol{R}\,, \quad t > 0\,. \tag{2}$$

If $b \equiv b_0 > 0$, then it is easily seen that, for any given $\alpha \in [0, \pi/2)$, the straight line $y = \tan\alpha \cdot x + (b_0/\cos\alpha)t$ is a solution of (2) with $b \equiv b_0$. This solution starts at line $l_\alpha :\ y = \tan\alpha \cdot x$ and travels in the y-direction with speed $c = b_0/\cos\alpha$; In other words, it travels in its normal direction with speed b_0. Now in (1) and (2), b is periodic, so it is natural to conjecture that for any given $\alpha \in (0, \pi/2)$, undulating the line $\tan\alpha \cdot x + (b_0/\cos\alpha)t$ may give a solution of (2). Moreover, such a solution is expected to be periodic in time according to "Periodicity in space generates a periodic-in-time regime". In the following a solution $u(x, t)$

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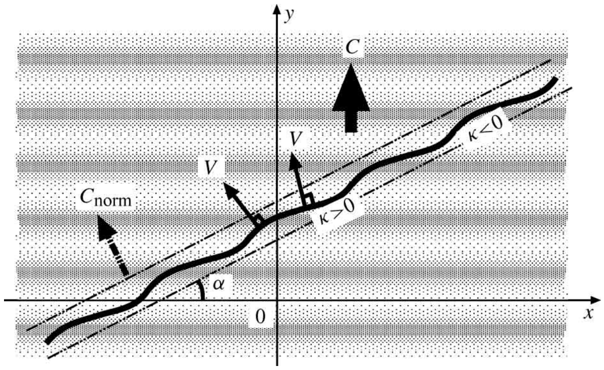

FIGURE 1. Propagation of a curve in heterogeneous media with periodic horizontal striations.

of (2) is called a *periodic travelling wave solution* if

$$u(x,t+T) = u(x,t) + \varepsilon\,, \tag{3}$$

for some $T > 0$, and $c := \varepsilon/T$ is called the *average speed of* u *in* y*-direction*.

In this paper, we try to seek for solutions of (2)–(3) of the form

$$u(x,t) \equiv v(x + ct\cot\alpha) = v(\xi) \quad \text{with} \quad v(\xi + \varepsilon\cot\alpha) = v(\xi) + \varepsilon\,, \tag{4}$$

where $\xi := x + ct\cot\alpha$ and $c = \varepsilon/T$. In this case, for each $t > 0$ the graph of u is a periodic undulating line with "inclination angle" α (see Figure 1). (It should be pointed out that the solution of (2)–(3) does not necessarily have to be such a form as v. In fact, [6] studied V-shaped travelling wave solutions of $V = \kappa + b_0$. V-shaped periodic travelling wave solutions of (1) will be the object of a forthcoming paper).

Periodic travelling wave solutions correspond to the propagation of fronts in striated media where the striations are disposed in a periodic fashion (cf. [3]). Equations in such a medium generally contain spatially periodic coefficients, like a and b in (1). Motivated by "periodicity in space generates a periodic-in-time regime", many authors studied periodic travelling wave solutions of parabolic equations with spatially periodic coefficients. Among others, [1], [2], [3] and [5] studied problems similar to ours. More precisely, [1] studied reaction diffusion equations in heterogeneous media with horizontal striations (eg., $a = a(y)$). The travelling fronts they considered are also horizontal ones. It turns out that the propagation front is flat (independent of x). [2] studied curvature flow equation (1) for $a \equiv 1$. The striation in the heterogeneous media is also horizontal ($b = b(y)$), but the propagation direction they considered is the $-x$-direction. In this case only travelling waves are possible. In [3] and [5], propagation through oblique striations is considered. The equation is a modified form of (1): replacing $\kappa = u_{xx}/(1+u_x^2)^{3/2}$ by $u_{xx}/(1+u_x^2)^{1/2}$, that is, they essentially considered semilinear parabolic equations. Though the qualitative behavior of the propagation is similar to the original problem (1), in the modified problem some technical difficulties can be avoided.

In this paper we study Problem (1), and prove the existence and uniqueness of a periodic travelling wave solution with form v as in (4). At each time, the graph of the solution is a periodic undulating line (see Figure 1) which is in a finite distance from a straight line with inclination angle α, so the propagation is just like that in oblique disposed striations as in [3] and in [5].

In the study of periodic travelling waves, another more interesting problem is to characterize the average speed. [3] gave some estimate on the average speed c as $a \equiv a_0 \to 0$, but did not give the explicit relationship among c, α and the periodic coefficient b. In this paper, we consider the homogenization problem as the period ε tends to 0, estimate c by $\cos\alpha$ and the *arithmetic means* of $1/a$ and b/a.

In Section 2 we consider the case $\alpha \in (0, \pi/2)$ and α satisfies some technical conditions (see (6), (7) below). We prove the existence and uniqueness of periodic travelling wave solution with a form as v in (4), and estimate the average speed c as $\varepsilon \to 0$.

In Section 3 we consider two extreme cases: $\alpha = 0$ and $\alpha = \pi/2$. In case $\alpha = 0$, the periodic travelling wave solution is indeed a horizontal straight line, which travels in the y-direction with average speed c_0. In case when $\alpha = \pi/2$, there exists travelling wave solution (not periodic), which travels in the $-x$-direction with a speed depending on the *arithmetic means* of $1/a$ and b/a.

2. The case $\alpha \in (0, \pi/2)$.

2.1. Existence of periodic travelling wave solutions. Using $v(\xi)$ defined by (4), Problem (2)–(3) is rewritten as

$$(5) \qquad \begin{cases} c v_\xi \cot\alpha = a\left(\dfrac{v}{\varepsilon}\right) \dfrac{v_{\xi\xi}}{1 + {v_\xi}^2} + b\left(\dfrac{v}{\varepsilon}\right)\sqrt{1 + {v_\xi}^2}\,, & \xi \in \boldsymbol{R}\,, \\ v(0) + \varepsilon = v(\varepsilon\cot\alpha)\,, \quad v_\xi(0) = v_\xi(\varepsilon\cot\alpha)\,, & \xi \in \boldsymbol{R}\,. \end{cases}$$

For any given $\alpha \in (0, \pi/2)$, a pair (c, v) solving (5) is called a solution of (5). In this subsection, we prove the existence and uniqueness of the solution of (5). Without loss of generality, we assume $v(0) = 0$ throughout the paper. Our approach is inspired by [3]. Denote $b_m = \min b(s)$, $b_M = \max b(s)$. We make some technical assumptions on α:

$$(6) \qquad b_M \sin\alpha \le b_m\,,$$

$$(7) \qquad \tan^2\alpha \ge 2\left(1 - \frac{b_m}{b_M}\right)\left(1 + \frac{b_M^2}{b_m^2}\right).$$

One can see that many $\alpha \in (0, \pi/2)$ satisfy these conditions provided $b_M - b_m$ is small. For instance, (7) is true if $1 + \tan^2\alpha \ge b_M^4/b_m^4$.

THEOREM 2.1. (i) *For any* $\alpha \in (0, \pi/2)$ *satisfying* (6), *Problem* (5) *has a unique solution* $(c^*(\alpha), v^*(\xi))$. *In addition,* $v_\xi^*(\xi) > 0$ *and*

$$(8) \qquad \frac{b_m}{\cos\alpha} \le c^*(\alpha) \le \frac{b_M}{\cos\alpha}\,.$$

(ii) *If we assume further that* (7) *holds, then* $c^*(\alpha)$ *is strictly increasing in* α.

We first give a preliminary lemma.

LEMMA 2.2. *If* (c, v) *solves* (5), *then* v_ξ *is* $\varepsilon \cot\alpha$*-periodic, and* $c > 0,\ v_\xi > 0$ *in* $\boldsymbol{R}$.

PROOF. By (5), v_ξ is periodic with period $\varepsilon\cot\alpha$. So there exist $\xi_+,\ \xi_- \in [0, \varepsilon\cot\alpha)$ such that

$$v_\xi(\xi_+) = \max v_\xi(\xi) > 0\,, \quad v_\xi(\xi_-) = \min v_\xi(\xi)\,, \quad v_{\xi\xi}(\xi_+) = v_{\xi\xi}(\xi_-) = 0\,.$$

Considering the equation in (5) at $\xi_\pm$, we have

$$cv_\xi(\xi_\pm)\cot\alpha = b\left(\frac{v(\xi_\pm)}{\varepsilon}\right)\sqrt{1+v_\xi^2(\xi_\pm)} > 0\,.$$

The inequality at ξ_+ implies $c > 0$, since $v_\xi(\xi_+) > 0$, and so the inequality $cv_\xi(\xi_-) > 0$ implies $v_\xi(\xi_-) > 0$. $\square$

A direct verification shows the following

LEMMA 2.3. *There is a solution* $(c, v(\xi))$ *to* (5) *if and only if there is a solution* $(c, w(s))$ *to the following problem*

$$\begin{cases} w'(s) = \dfrac{\varepsilon(1+w^2\cot^2\alpha)}{a(s)}\,(b(s)\sqrt{\tan^2\alpha + w^2} - c) & s \in (0,1)\,, \\ w(0) = w(1)\,, \quad w > 0 \ \text{ in } [0,1]\,, \\ \displaystyle\int_0^1 w(s)ds = 1\,. \end{cases} \tag{9}$$

The transformation from $v(\xi)$ *to* $w(s)$ *is given by* $v_\xi(\xi) = (\cot\alpha\ w(s))^{-1},\ s = v(\xi)/\varepsilon$.

Hereafter $'$ denotes derivative with respect to s. To show the existence of solutions of (9) we first prove

LEMMA 2.4. *For any given* $\alpha \in (0, \pi/2)$ *and* $p \geq 0$, *the problem*

$$\begin{cases} w'(s) = \dfrac{\varepsilon(1+w^2\cot^2\alpha)}{a(s)}\,(b(s)\sqrt{\tan^2\alpha + w^2} - c)\,, & s \in (0,1)\,, \\ w(0) = w(1) = p\,, \end{cases} \tag{10}$$

has a unique solution $(c(p,\alpha), w(s,p,\alpha))$. *Moreover*,

$$b_m\sqrt{1+\max w^2(\cdot,p,\alpha)} \leq c(p,\alpha) \leq b_M\sqrt{1+\min w^2(\cdot,p,\alpha)}\,. \tag{11}$$

PROOF. Consider the initial value problem for the ordinary differential equation:

$$\begin{cases} W'(s) = \dfrac{\varepsilon(1+W^2\cot^2\alpha)}{a(s)}\,(b(s)\sqrt{\tan^2\alpha + W^2} - c) =: G(s, W, c, \alpha)\,, & s \geq 0\,, \\ W(0) = p\,. \end{cases} \tag{12}$$

Denote its solution by $W(s,p,\alpha,c)$, or, simply by $W(s)$. For each $c \geq 0$, denote the maximum existence interval of $W(s,p,\alpha,c)$ by $[0, S(c))$.

(a) When $c = c_1 := b_M\sqrt{\tan^2\alpha + (p+1)^2}$, we have

$$W'(s)\begin{cases} < 0\,, & \text{when} \quad W \in [-p-1, p+1]\,,\\ > 0\,, & \text{when} \quad W < -W_0 := -(c_1^2/b_m^2 - \tan^2\alpha)^{1/2}\,.\end{cases}$$

Hence $S(c_1) = \infty$, and $W(s, p, \alpha, c_1) \in (-W_0, p)$ for all $s \in (0, \infty)$. In particular, $W(1, p, \alpha, c_1) < p$.

(b) $W(s, p, \alpha, c)$ is decreasing in c. In fact, $W_c := \partial W/\partial c$ and $W_p := \partial W/\partial p$ satisfy

$$W_p' = G_W \cdot W_p\,, \quad W_p|_{s=0} = 1\,,$$

$$W_c' = G_W \cdot W_c + G_c\,, \quad W_c|_{s=0} = 0\,.$$

So $W_p = \exp(\int_0^s G_W ds) > 0$. By the variation of constant technique, we have for $\mu := W_c/W_p$,

$$\mu' = \frac{G_c}{W_p}\,, \quad \mu|_{s=0} = \left.\frac{W_c}{W_p}\right|_{s=0} = 0\,.$$

Since $G_c < 0$ we have $\mu' < 0$, and so $\mu < 0$ for $s > 0$. Hence $W_c < 0$ for any $s \in (0, S(c))$.

(c) Set $Z := \{c \in [0, c_1] \mid S(c) > 1 \text{ and } W(1, p, \alpha, c) < p\}$. Since $W_s(s, p, \alpha, 0) > 0$, we have $0 \notin Z$. The above (a) implies that $c_1 \in Z$ and (b) implies that if $c_2 \in (0, c_1) \cap Z$, then $[c_2, c_1] \subset Z$.

Fix a $c \in Z$. On the one hand, $W(s, p, \alpha, c) \geq W(s, p, \alpha, c_1) > -W_0$. On the other hand, if we put $\nu = \max_{0\leq s\leq 1} W(s, p, \alpha, c) = W(s_1, p, \alpha, c)$ for some $s_1 \in [0, 1]$ then either $\nu \leq p$ or $\nu > p$. Assume that the latter occurs: $\nu > p$. Then $s_1 \in (0, 1)$ and hence $W_s(s_1, p, \alpha, c) = 0$. Thus we have $b(s_1)\sqrt{\tan^2\alpha + \nu^2} = c$. Since $c \leq c_1$, we get $\nu^2 \leq c_1^2/b_m^2$, i.e., $W \leq c_1/b_m$. Hence $W(s, p, \alpha, c) \in [-W_0, c_1/b_m]$.

By the equation of W and by $a, b \in C^1$, there exists a positive constant $M_1 = M_1(a, b, c_1, W_0) = M_1(a, b, \alpha, p)$ such that, for any $c \in Z$, $W(s, p, \alpha, c)$ satisfies

$$\max_{s\in[0,1]} |W| \leq M_1\,, \quad \max_{s\in[0,1]} |W'| \leq M_1\,, \quad \max_{s\in[0,1]} |W''| \leq M_1\,. \tag{13}$$

(d) Denote $c_3 := \inf_Z c$. Then (13) and (b) imply that there exists a function $w(s, p, \alpha) \in C^1([0, 1])$ such that $\|W(s, p, \alpha, c) - w(s, p, \alpha)\|_{C^1} \to 0$ as $c \to c_3 + 0$. Letting $c \to c_3 + 0$ in (12) we have

$$w_s(s, p, \alpha) = G(s, w(s, p, \alpha), c_3, \alpha) \quad \text{for } s \in [0, 1]\,, \quad w(0, p, \alpha) = p\,.$$

So $w(s, p, \alpha) \equiv W(s, p, \alpha, c_3)$ on $[0, 1]$. Clearly, $W(s, p, \alpha, c_3)$ satisfies (13) and $W(1, p, \alpha, c_3) \leq p$.

(e) We show that $W(1, p, \alpha, c_3) = p$. Otherwise, $W(1, p, \alpha, c_3) < p - 2\delta_1$ for some $\delta_1 \in (0, 1)$. Denote $M_2 := \max\{|G_W| + |G_c| \mid s \in [0, 1], W \in [-M_1 - 1, M_1 + 1], c \in [0, c_1]\}$. Consider (12) for $c = c_4 := c_3 - \delta_2$, where $\delta_2 < \min\{\delta_1 \exp(-M_2),\ c_3\}$.

First we show that

$$W(s, p, \alpha, c_3) \leq W(s, p, \alpha, c_4) \leq W(s, p, \alpha, c_3) + 1 \quad \text{for } s \in [0, 1]\,. \tag{14}$$

In fact, the first inequality is true on $[0, S(c_4)) \cap [0, 1]$. The second inequality holds on $[0, s_2]$ for some small s_2. Set $s_3 := \sup\{s \mid \text{the second inequality holds on } [0, s]\}$. If $s_3 \geq 1$, then (14) is proved. If $s_3 < 1$, then $W(s_3, p, \alpha, c_4) = W(s_3, p, \alpha, c_3) + 1$. Set $\tilde{W}(s) = W(s, p, \alpha, c_4) - W(s, p, \alpha, c_3) \geq 0$. Then on $[0, s_3]$, there exists $\zeta(s)$ lying between $W(s, p, \alpha, c_4)$ and $W(s, p, \alpha, c_3)$, and $c_5 \in (c_4, c_3) \subset (0, c_1)$ such that

$$\tilde{W}'(s) = G_W(s, \zeta(s), c_4, \alpha)\tilde{W} - G_c(s, W(s, p, \alpha, c_3), \alpha, c_5)\delta_2 .$$

Since $s_3 < 1$, we have $|\tilde{W}(s_3)| \leq \delta_2 \exp(M_2) \leq \delta_1 < 1$, in contradiction to the above assumption. Therefore (14) holds.

Repeating the above discussion on $[0, 1]$ we have $\tilde{W}(1) \leq \delta_1$, that is, $W(1, p, \alpha, c_4) \leq W(1, p, \alpha, c_3) + \delta_1 < p - \delta_1$. This means that $c_4 \in Z$, contradicting the definition of c_3. Therefore $W(1, p, \alpha, c_3) = p$, and so the pair $c(p, \alpha) := c_3$ and $w(s, p, \alpha) \equiv W(s, p, \alpha, c_3)$ is the unique solution of (10).

(f) Finally, we verify (11). Since $w(s, p, \alpha)$ is 1-periodic, for any $\nu \in [\min w(\cdot, p, \alpha), \max w(\cdot, p, \alpha)]$, there exist $s_4, s_5 \in [0, 1)$ such that $w(s_4, p, \alpha) = w(s_5, p, \alpha) = \nu$, $(\partial/\partial s)w(s_4, p, \alpha) \geq 0$, $(\partial/\partial s)w(s_5, p, \alpha) \leq 0$, i.e.,

$$\begin{aligned} b_m\sqrt{\tan^2\alpha + \nu^2} \leq b(s_5)\sqrt{\tan^2\alpha + \nu^2} &\leq c(p, \alpha) \\ &\leq b(s_4)\sqrt{\tan^2\alpha + \nu^2} \leq b_M\sqrt{\tan^2\alpha + \nu^2} . \end{aligned}$$

Since $\nu \in [\min w(\cdot, p, \alpha), \max w(\cdot, p, \alpha)]$ can be chosen arbitrarily, the left and the right inequalities hold indeed for any $\nu^2 \in [\min w^2(\cdot, p, \alpha), \max w^2(\cdot, p, \alpha)]$, this proves (11). □

LEMMA 2.5. *For any given $\alpha \in (0, \pi/2)$, there exists a unique $p = p(\alpha) > 0$ such that the corresponding unique solution of* (10), *that is, the pair*

$$c^*(\alpha) := c(p(\alpha), \alpha) \quad \text{and} \quad w^*(s, \alpha) := w(s, p(\alpha), \alpha) = W(s, p(\alpha), \alpha, c(p(\alpha), \alpha)) \tag{15}$$

satisfies $\int_0^1 w^*(s, \alpha)ds = 1$ and $w^*(s, \alpha) > 0$. Hence $(c^*(\alpha), w^*(s, \alpha))$ is the unique solution of (9).

PROOF. For any $p \geq 0$, recall the solution of (10) is $w(s, p, \alpha) \equiv W(s, p, \alpha, c(p, \alpha))$. Set $I(p, \alpha) = \int_0^1 w(s, p, \alpha)ds$, we need to show that there exists a unique $p = p(\alpha)$ such that $I(p(\alpha), \alpha) = 1$.

(a) By (11) we have

$$b_m\sqrt{\tan^2\alpha + p^2} \leq \begin{cases} b_M\sqrt{\tan^2\alpha + \min w^2(\cdot, p, \alpha)}\,, & \text{when} \quad \min w(\cdot, p, \alpha) > 0\,, \\ b_M \tan\alpha\,, & \text{when} \quad \min w(\cdot, p, \alpha) \leq 0\,. \end{cases}$$

The second inequality is impossible when $p > 0$ is large. Hence, when $p \to +\infty$, $\min w(\cdot, p, \alpha) > 0$ and $\min w(\cdot, p, \alpha) \to +\infty$ by the first inequality. This means that $I(p, \alpha) \to +\infty$ as $p \to +\infty$.

(b) When $p = 0$, by (11) we have $b_m\sqrt{\tan^2\alpha + \max w^2(\cdot, p, \alpha)} \leq c(p, \alpha) \leq b_M \tan\alpha$, and so by (6),

$$\max w^2(\cdot, p, \alpha) \leq \tan^2\alpha \cdot (b_M^2 - b_m^2)/b_m^2 \leq 1\,, \quad \text{i.e.}, \ |w(\cdot, p, \alpha)| \leq 1\,.$$

Clearly, $w(s, p, \alpha) \not\equiv 1$, hence $I(0, \alpha) = \int_0^1 w(s, p, \alpha)ds < 1$.

(c) Denote $\bar{W}(p,\alpha,c(p,\alpha)) := W(1,p,\alpha,c(p,\alpha))$, then by $\bar{W}(p,\alpha,c(p,\alpha)) = p$ we have $\bar{W}_p + \bar{W}_c \cdot c_p = 1$, and so $c_p = (1-\bar{W}_p)/\bar{W}_c$. Using the notation μ in the proof of Lemma 2.4, we have

$$\begin{aligned} I_p &= \int_0^1 (W_p + W_c \cdot c_p)ds = \int_0^1 \frac{W_p \bar{W}_c + W_c - W_c \bar{W}_p}{\bar{W}_c} ds \\ &= \int_0^1 \frac{\mu(1) W_p \bar{W}_p + \mu(s) W_p - \mu(s) W_p \bar{W}_p}{\mu(1)\bar{W}_p} ds > 0\,. \end{aligned}$$

The above assertions imply that there exists a unique $p(\alpha) > 0$ such that $I(p(\alpha),\alpha) = 1$. Finally, a proof similar to that in (b) also shows that $\min w(\cdot, p(\alpha),\alpha) > 0$, since $\int_0^1 w(s,p(\alpha),\alpha)ds = 1$. Thus $(c^*(\alpha), w^*(s,\alpha))$ defined by (15) is the unique solution of (9). □

PROOF OF THEOREM 2.1 (i). Lemma 2.5 shows that there exists a unique solution $(c^*(\alpha), w^*(s,\alpha))$ of (9). This solution corresponds to a solution $(c^*(\alpha), v^*(\xi))$ of (5) by Lemma 2.3, and $v^*(\xi)$ is defined implicitly by

$$\int_{v^*(0)}^{v^*(\xi)} w^*\left(\frac{v}{\varepsilon}, \alpha\right) dv = \frac{\xi}{\cot\alpha}\,.$$

The uniqueness of $v^*(\xi)$ follows from our assumption $v^*(0) = 0$ (Without this assumption, v^* will be unique up to a translation).

As mentioned in the proof of the previous lemma, $\int_0^1 w^*(s,\alpha)ds = 1$ implies $w^*(s,\alpha) > 0$. So we indeed have $0 < \min w^* < 1 < \max w^*$. By (11) we have

$$\frac{b_m}{\cos\alpha} \le b_m\sqrt{\tan^2\alpha + (\max w^*)^2} \le c^*(\alpha) \le b_M\sqrt{\tan^2\alpha + (\min w^*)^2} \le \frac{b_M}{\cos\alpha}\,.$$

REMARK 2.6. By the last inequality in the above proof, we know that $|w^*| \le M_3$ for some $M_3 = M_3(\alpha, b)$ and $c^*(\alpha) \le b_M/\cos\alpha$. Hence $|G(s, w^*, c^*, \alpha)| \le \varepsilon M_4$ for some $M_4 = M_4(\alpha, a, b)$, that is, $|w^{*\prime}| \le \varepsilon M_4$. This means that $\max w^*(\cdot,\alpha) - \min w^*(\cdot,\alpha) \le \varepsilon M_4$. Combining with $\int_0^1 w^*(s,\alpha)ds = 1$ we have $|w^* - 1| \le \varepsilon M_4$ on $[0,1]$. Therefore,

$$(1 - M_4\varepsilon)\tan\alpha \le v^*_\xi = \frac{\tan\alpha}{w^*} \le (1+2M_4\varepsilon)\tan\alpha \quad \text{as long as} \quad \varepsilon \le \frac{1}{2M_4}\,. \tag{16}$$

PROOF OF THEOREM 2.1 (ii). First we show

$$G_\alpha = \varepsilon\frac{\cos\alpha}{a(s)\sin^3\alpha}[b(s)(\tan^2\alpha - 2w^{*2})\sqrt{\tan^2\alpha + w^{*2}} + 2c^* w^{*2}] > 0\,.$$

Set

$$G_1 = \left(2c^* - \frac{2b_m}{b_M} b(s)\sqrt{\tan^2\alpha + w^{*2}}\right) w^{*2}\,,$$

$$G_2 = b(s)\left(\tan^2\alpha - 2w^{*2} + \frac{2b_m}{b_M} w^{*2}\right)\sqrt{\tan^2\alpha + w^{*2}}\,.$$

By (11), $c^*/b_m \ge \sqrt{\tan^2\alpha + w^{*2}}$, so $G_1 \ge 2c^* w^{*2}\,(1 - b(s)/b_M) \ge 0$.

Since $0 < \min w^* < 1 < \max w^*$, (11) implies that $\max w^{*2} < [(b_M^2 - b_m^2)\tan^2\alpha + b_M^2]/b_m^2$. So by assumptions (6) and (7) we have

$$\begin{aligned}\frac{G_2}{b(s)\sqrt{\tan^2\alpha + w^{*2}}} &= \tan^2\alpha - 2\left(1 - \frac{b_m}{b_M}\right)w^{*2} \\ &> \tan^2\alpha - 2\left(1 - \frac{b_m}{b_M}\right)\left(\frac{b_M^2 - b_m^2}{b_m^2}\tan^2\alpha + \frac{b_M^2}{b_m^2}\right) \\ &\geq \tan^2\alpha - 2\left(1 - \frac{b_m}{b_M}\right)\left(1 + \frac{b_M^2}{b_m^2}\right) \geq 0\,.\end{aligned}$$

Hence $G_\alpha > 0$.

Next we prove $dc^*(\alpha)/d\alpha > 0$. Since

$$\int_0^1 W(s, p(\alpha), \alpha, c^*(\alpha))ds = 1\,, \quad \bar{W}(p(\alpha), \alpha, c^*(\alpha)) = p(\alpha)\,.$$

Differentiating them with respect to α we have

$$\frac{dp(\alpha)}{d\alpha} = \frac{\displaystyle\int_0^1 W_\alpha ds + \frac{dc^*(\alpha)}{d\alpha}\int_0^1 W_c ds}{\displaystyle -\int_0^1 W_p ds}\,, \quad \frac{dc^*(\alpha)}{d\alpha}\cdot \bar{W}_c = (1 - \bar{W}_p)\frac{dp(\alpha)}{d\alpha} - \bar{W}_\alpha\,.$$

Therefore,

$$\frac{dc^*(\alpha)}{d\alpha} = \frac{\displaystyle\int_0^1 (\bar{W}_p W_\alpha - \bar{W}_\alpha W_p - W_\alpha)ds}{\displaystyle\int_0^1 (\bar{W}_c W_p - W_c \bar{W}_p + W_c)ds}\,.$$

Note that

$$\frac{\partial}{\partial s}W_\alpha = G_W \cdot W_\alpha + G_\alpha\,, \quad W_\alpha|_{s=0} = 0\,.$$

Hence, if we write $W_\alpha = \rho(s)W_p$, then

$$\rho'(s) = G_\alpha / W_p\,, \quad \rho(0) = 0\,.$$

Since $G_\alpha > 0$ and $W_p > 0$ we have $\rho' > 0$, and so $\rho(s) > 0$ on $(0, 1]$. Thus

$$\frac{dc^*(\alpha)}{d\alpha} = \frac{\displaystyle\int_0^1 [(\rho(s) - \rho(1))\bar{W}_p W_p - \rho(s)W_p]ds}{\displaystyle\int_0^1 [(\mu(1) - \mu(s))\bar{W}_p W_p + \mu(s)W_p]ds} > 0\,.$$

In other words, $c^*(\alpha)$ is strictly increasing in α.

2.2. Estimate of average speed. In this section we consider homogenization problem as $\varepsilon \to 0$, and estimate $c^*(\alpha)$ by using an average method.

LEMMA 2.7. *Let $u_1, u_2, \varepsilon_1, \varepsilon_2$ be constants satisfying $u_2 > u_1, \varepsilon_2 > \varepsilon_1 \geq 0$. For any $\varepsilon \in [\varepsilon_1, \varepsilon_2]\backslash\{0\}$, let $u(x) \in \boldsymbol{R}$ be the solution of*

$$\frac{du}{dx} = \varepsilon f(\varepsilon, x, u)\,, \quad u(0) = u_0 \in (u_1, u_2)\,, \tag{17}$$

where $f \in C^1([\varepsilon_1, \varepsilon_2] \times [0, \infty) \times [u_1, u_2])$, and is X-periodic in x. Define f_0 as $f_0(\varepsilon, v) = (1/X)\int_0^X f(\varepsilon, x, v)dx$. Let $v(x) \in \boldsymbol{R}$ be the solution of

$$\frac{dv}{dx} = \varepsilon f_0(\varepsilon, v)\,, \quad v(0) = u_0\,. \tag{18}$$

Then there exist $C_1 = C_1(\varepsilon_1, \varepsilon_2, u_1, u_2, u_0) > 0$ and $C_2 = C_2(\varepsilon_1, \varepsilon_2, u_1, u_2, u_0) > 0$ such that

$$|u(x) - v(x)| \leq C_2\varepsilon \quad \text{for } 0 \leq x \leq C_1/\varepsilon\,.$$

PROOF. In the special case where f and f_0 are independent of ε, this lemma is the well known average method (cf. [4] or [7]). Our proof is similar to that for the special case.

First, if we set $F = F(\varepsilon_1, \varepsilon_2, u_1, u_2) := \max_\Omega |f(\varepsilon, x, u)|$, where $\Omega := [\varepsilon_1, \varepsilon_2] \times [0, \infty) \times [u_1, u_2]$, then the general theory of ordinary differential equations implies that (17) (resp. (18)) has a unique solution $u(x) \in [u_1, u_2]$ (resp. $v(x) \in [u_1, u_2]$) for $x \in [0, C_1/\varepsilon]$, where $C_1 = C_1(\varepsilon_1, \varepsilon_2, u_1, u_2, u_0) := \min\{u_2 - u_0, u_0 - u_1\}/F$.

Set $\delta(x, v) = \int_0^x [f(\varepsilon, z, v) - f_0(\varepsilon, v)]dz$, and $w(x) = v(x) + \varepsilon\delta(x, v(x))$ for $x \in [0, C_1/\varepsilon]$. Then $\delta(x, v)$ and $\delta_v(x, v)$ are X-periodic in x and $|\delta(x, v(x))| \leq 2FX$,

$$|\delta_v(x, v(x))| = \left|\int_0^x [f_v(\varepsilon, z, v(x)) - (f_0)_v(\varepsilon, v(x))]dz\right| \leq 2F_1X\,,$$

where $F_1 = F_1(\varepsilon_1, \varepsilon_2, u_1, u_2) := \max_\Omega |f_v(\varepsilon, x, v)|$. Since

$$\frac{dw}{dx} = \varepsilon f_0(x, v) + \varepsilon[f(\varepsilon, x, v) - f_0(\varepsilon, v)] + \varepsilon^2\delta_v\, f_0(x, v)\,,$$

we have

$$\begin{aligned}\frac{du}{dx} - \frac{dw}{dx} &= \varepsilon f(\varepsilon, x, u) - \varepsilon f(\varepsilon, x, v) - \varepsilon^2\delta_v f_0(x, v)\\ &= \varepsilon[f(\varepsilon, x, u) - f(\varepsilon, x, w)] + \varepsilon[f(\varepsilon, x, w) - f(\varepsilon, x, v)] - \varepsilon^2\delta_v f_0(x, v)\,.\end{aligned}$$

Thus

$$\left|\frac{du}{dx} - \frac{dw}{dx}\right| \leq \varepsilon F_1|u - w| + 4FF_1X\varepsilon^2\,.$$

Hence

$$|u - w| = \left|\int_0^x \left(\frac{du}{dx} - \frac{dw}{dx}\right)dx\right| \leq \varepsilon F_1 \int_0^x |u(x) - w(x)|dx + 4FF_1X\varepsilon^2 x\,,$$

that is,

$$|u - w| + 4FX\varepsilon \leq \varepsilon F_1 \int_0^x [|u(x) - w(x)| + 4FX\varepsilon]dx + 4FX\varepsilon\,.$$

By Gronwall's lemma we have $|u - w| + 4FX\varepsilon \leq 4FX\varepsilon\exp(\varepsilon F_1 x)$. Then

$$|u(x) - v(x)| \leq 2FX\varepsilon + 4FX\varepsilon\big(e^{F_1C_1} - 1\big) =: C_2\varepsilon \quad \text{for } x \in [0, C_1/\varepsilon]\,. \qquad \square$$

THEOREM 2.8. *Assume* (6) *holds. Then*

$$c^*(\alpha) = \frac{\bar{d}}{\bar{a}\ \cos\alpha} + O(\varepsilon) \quad \text{as } \varepsilon \to 0\,,$$

where

$$\bar{a} = \int_0^1 \frac{ds}{a(s)}\,, \quad \bar{d} = \int_0^1 \frac{b(s)}{a(s)} ds\,.$$

PROOF. Recall $(c^*(\alpha), w^*(s,\alpha))$ is the unique solution of (9), which corresponds to the unique solution $(c^*(\alpha), v^*(\xi))$ of (5). For simplicity, we write $c^*(\alpha) = c^*$ in the following. Set

$$\theta = \frac{\xi}{\varepsilon}\,, \quad \varphi(\theta) = \frac{v^*(\xi)}{\varepsilon} = \frac{v^*(\varepsilon\theta)}{\varepsilon}\,.$$

At the beginning of Subsection 2.1 we assume $v^*(0) = \varphi(0) = 0$, then φ satisfies

$$\text{(19)} \quad \begin{cases} a(\varphi)\dfrac{\varphi_{\theta\theta}}{1+\varphi_\theta{}^2} = \varepsilon\big(c^* \cot\alpha\ \varphi_\theta - b(\varphi)\sqrt{1+\varphi_\theta{}^2}\big)\,, \quad \theta \in \boldsymbol{R}\,, \\ \varphi(0) = 0\,,\ \ \varphi(\cot\alpha) = 1\,,\ \ \varphi_\theta(0) = \varphi_\theta(\cot\alpha) = 1/(p(\alpha)\cot\alpha) =: \psi_0\,,\ \ \theta \in \boldsymbol{R}\,, \end{cases}$$

where $p(\alpha) = w^*(0,\alpha) > 0$ (see Lemmas 2.4 and 2.5 above). Denote $\varphi_\theta = \psi > 0$. Then

$$\psi_\theta = \varepsilon\big(c^* \cot\alpha\ \psi - b(\varphi)\sqrt{1+\psi^2}\big)\frac{1+\psi^2}{a(\varphi)}$$

and hence (19) implies that $\psi(\varphi)$ is a 1-periodic function and it solves

$$\text{(20)} \qquad \begin{cases} \dfrac{d\psi}{d\varphi} = \varepsilon\left(\dfrac{c^*\cot\alpha}{a(\varphi)} - \dfrac{b(\varphi)}{a(\varphi)}\dfrac{\sqrt{1+\psi^2}}{\psi}\right)(1+\psi^2)\,, \quad \varphi \in [0,\infty)\,, \\ \psi|_{\varphi=n} = \psi_0\,, \quad n \in \boldsymbol{N}\,. \end{cases}$$

By Remark 2.6 we have $\psi = \varphi_\theta = v^*_\xi \in [(1 - M_4\varepsilon)\tan\alpha, (1 + 2M_4\varepsilon)\tan\alpha]$. Then for $h := (1 - \sin\alpha)/(3\sin\alpha)$ and for some $M_5 = M_5(\alpha, a, b) > 0$ we have

$$\text{(21)} \qquad -3h - M_5\varepsilon \le 1 - \sqrt{1+\psi^2}/\psi \le -3h + M_5\varepsilon \quad \text{for } \varphi \in [0,\infty)\,,$$

as long as ε is sufficiently small.

Denote by $\omega(\varphi)$ the solution of

$$\text{(22)} \qquad \begin{cases} \dfrac{d\omega}{d\varphi} = \varepsilon\left(\bar{a}c^*\cot\alpha\ - \bar{d}\dfrac{\sqrt{1+\omega^2}}{\omega}\right)(1+\omega^2) =: F_0(\omega)\,, \quad \varphi \ge 0\,, \\ \omega|_{\varphi=0} = \psi_0\,. \end{cases}$$

Notice that c^* in (20) and (22) depends on ε; but since $c^* \in [b_m/\cos\alpha, b_M/\cos\alpha]$, we have $\sup\{|F_0(\omega)/\varepsilon| \mid \varepsilon > 0, \omega \in [\psi_0/2, 2\psi_0]\} < +\infty$. Recall also the remark on ψ immediately after (20), so that $[\psi_0/2, 2\psi_0] \subset [(\tan\alpha)/4, 4\tan\alpha]$ since $M_4\varepsilon \le 1/2$.

Applying Lemma 2.7 to (20) and (22), we have $C_1, C_2 > 0$ depending on α, a, b but independent of ε such that ω exists on $[0, \mathcal{C}]$ (where $\mathcal{C}$ denotes the largest integer less than or equal to C_1/ε), $\omega(\varphi) \ge \psi_0/2$, and

$$\text{(23)} \qquad |\psi(\varphi) - \omega(\varphi)| \le C_2\varepsilon \quad \text{for } \varphi \in [0, \mathcal{C}]\,.$$

Combining with (21) we have, for sufficiently small $\varepsilon > 0$,

$$1 - \sqrt{1+\omega^2}/\omega \le -2h \quad \text{for } \varphi \in [0, \mathcal{C}]\,. \tag{24}$$

Case 1. If $\bar{a}c^* \cot\alpha \le \bar{d}(1+h)$, then by (24) we have

$$F_0(\omega) \le \varepsilon\bar{d}\left(1 + h - \frac{\sqrt{1+\omega^2}}{\omega}\right)(1+\omega^2) \le -\varepsilon\bar{d}h(1+\omega^2) < -\varepsilon\bar{d}h \quad \text{for } \varphi \in [0, \mathcal{C}]\,.$$

Using this inequality and integrating the equation in (22) on $[0, \mathcal{C}]$ we have $\omega(\mathcal{C}) - \omega(0) \le -\varepsilon\bar{d}h\mathcal{C} < -\bar{d}hC_1/2$ as long as ε is small. This is in contradiction to (23), since $\omega(0) = \psi_0 = \psi(\mathcal{C})$. Therefore, Case 1 is impossible.

Case 2. We assume $\bar{a}c^* \cot\alpha > \bar{d}(1+h)$ and denote by l the root of $\bar{a}c^* \cot\alpha = \bar{d}\sqrt{1+l^2}/l$, then $\sqrt{1+l^2}/l > 1+h$ and so $0 < l < h^{-1}$. Note that

$$\zeta_0 := \left.\frac{1}{\zeta^2\sqrt{1+\zeta^2}}\right|_{\zeta=\tan\alpha+h^{-1}} \le \frac{1}{\zeta^2\sqrt{1+\zeta^2}} \quad \text{for all } \zeta \in (0, \tan\alpha + h^{-1}]\,. \tag{25}$$

Subcase 2.1. Assume $\psi_0 \ge l + 2C_2\varepsilon/(C_1\zeta_0\bar{d})$. Then

$$\frac{F_0(\psi_0)}{1+\psi_0^2} = \varepsilon\left(\bar{a}c^*\cot\alpha - \bar{d}\frac{\sqrt{1+\psi_0^2}}{\psi_0}\right) > \varepsilon\left(\bar{a}c^*\cot\alpha - \bar{d}\frac{\sqrt{1+l^2}}{l}\right) = 0\,.$$

So $\omega \ge \psi_0$ and hence, for some ζ lies between l and ψ_0,

$$\begin{aligned} F_0(\omega) \ge F_0(\psi_0) &\ge \varepsilon\left(\bar{a}c^*\cot\alpha - \bar{d}\frac{\sqrt{1+\psi_0^2}}{\psi_0}\right) \\ &= \varepsilon\bar{d}\left(\frac{\sqrt{1+l^2}}{l} - \frac{\sqrt{1+\psi_0^2}}{\psi_0}\right) = \varepsilon\bar{d}\frac{\psi_0 - l}{\zeta^2\sqrt{1+\zeta^2}} \ge \varepsilon\bar{d}\frac{2C_2\varepsilon}{C_1\bar{d}} = \frac{2C_2}{C_1}\varepsilon^2\,. \end{aligned}$$

Therefore,

$$\omega(\mathcal{C}) \ge \omega(0) + \frac{2C_2}{C_1}\varepsilon^2 \cdot \mathcal{C} \ge \psi(\mathcal{C}) + \frac{2C_2}{C_1}\varepsilon^2 \cdot \left(\frac{C_1}{\varepsilon} - 1\right) = \psi(\mathcal{C}) + 2C_2\varepsilon - \frac{2C_2}{C_1}\varepsilon^2\,,$$

this contradicts (23) when ε is sufficient small.

Subcase 2.2. Assume $\psi_0 \le l - 2C_2\varepsilon/(C_1\zeta_0\bar{d})$. Then a similar discussion as in Subcase 2.1 also induces a contradiction.

Subcase 2.3. If $l - 2C_2\varepsilon/(C_1\zeta_0\bar{d}) < \psi_0 < l + 2C_2\varepsilon/(C_1\zeta_0\bar{d})$, then

$$\frac{d\omega}{d\varphi} = \varepsilon\bar{d}\left(\frac{\sqrt{1+l^2}}{l} - \frac{\sqrt{1+\omega^2}}{\omega}\right)(1+\omega^2) = \varepsilon\bar{d}\frac{1+\omega^2}{\zeta^2\sqrt{1+\zeta^2}}(\omega - l) = \varepsilon D(\varphi)(\omega - l)\,,$$

where $\zeta(\varphi)$ lies between l and $\omega(\varphi)$, and $D(\varphi) = \bar{d}(1+\omega^2)\zeta^{-2}(1+\zeta^2)^{-1/2}$ satisfies $|D(\varphi)| \le M_6(\alpha, \bar{d})$. Therefore, $\omega(\varphi) - l = (\omega(0) - l)\exp\left(\int_0^\varphi \varepsilon D(\varphi)d\varphi\right)$ on $[0, \mathcal{C}]$, and so

$$|\omega(\varphi) - l| \le |\psi_0 - l|\exp(\varepsilon M_6\varphi) \le \frac{2C_2\varepsilon}{C_1\zeta_0\bar{d}}\exp(M_6C_1) = M_7\varepsilon \quad \text{for } \varphi \in [0, \mathcal{C}]\,,$$

where $M_7 = 2C_2(C_1\zeta_0\bar{d})^{-1}\exp(M_6C_1)$. This means that $\omega = l + O(\varepsilon)$, and by (23) we have $\psi = l + O(\varepsilon)$. Since we have known $\psi = \tan\alpha + O(\varepsilon)$ above, we have $l = \tan\alpha + O(\varepsilon)$, that is,

$$\bar{a}c^*\cot\alpha = \bar{d}\frac{\sqrt{1+\tan^2\alpha}}{\tan\alpha} + O(\varepsilon) \quad \text{or}, \quad c^* = \frac{\bar{d}}{\bar{a}\,\cos\alpha} + O(\varepsilon) \quad \text{as } \varepsilon \to 0. \qquad \square$$

REMARK 2.9. In §2.1 we give the unique solution $(c^*(\alpha), v^*(\xi))$ of (5). Then u^* defined by v^* as in (4) is a solution of (2)–(3). It is easily seen that the graph of v^* is a periodic undulating line at finite distance from the line $\xi\tan\alpha$. So, for each $t > 0$, the graph of $u^*(x, t)$ is a periodic undulating line near the graph of $x\tan\alpha + h(t)$ for some $h(t)$. We call $(-\sin\alpha, \cos\alpha)$ the *global normal direction* of the graph of u^*, since $(-\sin\alpha, \cos\alpha)$ is the normal vector of line $x\tan\alpha + h(t)$. Along this direction, the average speed of u^* is

$$c^*_{\text{norm}} = c^*(\alpha)\cos\alpha = \bar{d}/\bar{a} + O(\varepsilon). \tag{26}$$

In other words, the homogenized average speed in *global normal direction* does not depend on α, just like the trivial case $b \equiv b_0$.

3. Two extreme cases.

3.1. The case $\alpha = 0$. Define $u_0(t) := \varepsilon F^{-1}(t/\varepsilon)$, where $F(s) = \int_0^s b(\tau)^{-1}d\tau$ is an increasing function and F^{-1} is its inverse function. Then $u_0(t)$ is a solution of (2)–(3), for each $t > 0$, its graph is a horizontal line. The average speed c_0 of u_0 is the harmonic mean of b: $c_0 = \left(\int_0^1 b(\tau)^{-1}d\tau\right)^{-1}$.

On the other hand, since (6) holds even for $\alpha \to 0^+$, the results in §2.1 (except for Theorem 2.1 (ii)) remain to hold for $\alpha \to 0^+$. One can see that, formally, the graph of $u^*(x, t) := v^*(x + c^*(\alpha)t\cot\alpha)$ tends to the graph of $u_0(t)$ as $\alpha \to 0^+$. Moreover,

$$c^*(\alpha) = \frac{\bar{d}}{\bar{a}\cos\alpha} + O(\varepsilon) \to \frac{\bar{d}}{\bar{a}} + O(\varepsilon) \quad \text{as } \alpha \to 0^+.$$

But this limit is different from c_0 even in the special case $a \equiv 1$. We believe that, this difference comes from our estimate for $c^*(\alpha)$. Because in the proof of Theorem 2.8, when we use average method we indeed require that $\cot\alpha = O(1)$. However, in the limit process $\alpha \to 0^+$, $\cot\alpha$ may becomes larger and larger and even $\cot\alpha \cdot \varepsilon \gg 1$. In such a case, average method is not valid, and the estimate for $c^*(\alpha)$ given as above is not accurate.

3.2. The case $\alpha = \pi/2$. As $\alpha \to \pi/2$, (6) is not satisfied, Theorem 2.1 does not include the existence of periodic travelling wave solutions in this case. On the other hand, when $\alpha = \pi/2$, [2] proved the existence of travelling wave solutions of (1) with the form $x = -\tilde{u}(y) - \tilde{c}t$ (*not periodic*), which is a solution of

$$\begin{cases} \tilde{c} = a\left(\dfrac{y}{\varepsilon}\right)\dfrac{\tilde{u}_{yy}}{1+\tilde{u}_y^2} + b\left(\dfrac{y}{\varepsilon}\right)\sqrt{1+\tilde{u}_y^2}, \quad y \in \boldsymbol{R},\ t > 0, \\ \tilde{u}(0) = \tilde{u}(\varepsilon), \quad \tilde{u}_y(0) = \tilde{u}_y(\varepsilon). \end{cases}$$

Though the equation considered in [2] is the case $a \equiv 1$, the result is also true for periodic a. Moreover, when $a \equiv 1$, [2] gave a rough estimate for $\tilde{c}$: $b_m < \tilde{c} < b_M$.

In fact, as in the proof of Theorem 2.8, using average method one can show that $\tilde{c} = \bar{d}/\bar{a} + O(\varepsilon)$. On the other hand, (26) implies that the periodic travelling wave solution of the form $v^*(\xi)$ travels in its *global normal direction* with average speed $\bar{d}/\bar{a} + O(\varepsilon)$, approximating $\tilde{c}$.

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