NOTES ON FOURIER ANALYSIS (XXV): QUASI-TAUBERIAN THEOREM*)

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The first general treatment of quasi-Tauberian theorems was given by N. Wiener [16]. In this note the author proves another quasi-Tauberian theorem concerning absolute limit under Wiener's conditions. Wiener derived the Cesàro summability theorem of Fourier series from his general theorem. In this paper further applications of his theorem and analogues concerning absolute limit are given. Some of theorems proved in this paper are known and the other are new. It is interesting that these theorems are derived from two key theorems which are given in § 1. In § 2, the Cesàro and absolute Cesàro summability theorems due to Paley [11] and Bosanquet [1] [2] are derived from the key theorems. In § 3, we prove the Cesàro and absolute Cesàro summability theorems of the conjugate Fourier series. These theorems include the essential parts of the results due to Paley [11] and Bosanquet-Hyslop [5]. The Cesàro summability problems of the derived Fourier series are discussed in § 4. The essential part of the results due to Takahashi [13], Wang [14] [15], Zygmund [17], Bosanquet [3] [4] and Hyslop [7] are derived from our key theorems. It is well known that these results are interpreted as the relation between Cesàro summation and Riemann summation of the first kind and their a nalogues concerning absolute summation. In § 5, the relations between the generalized jump of a function and its Fourier constants are discussed. This problem was treated early by Zygmund [18] and Szàsz [12]. These results are known as the relation between Cesàro summation, Riemann summation of the second kind and their analogues concerning absolute summation. As applications of these theorems, we can also prove theorems analogous to the results due to Misra [9] and Moursund [10].

In the sequel we shall fully use the notations and theorems of Chapter VII in the Wiener's work [16] without references.

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1. The key theorems.

Wiener's key theorem (cf. Wiener [16], Theorem XXII′ and XXIII′) reads as follows.

Theorem 1. *Let $f(x)$ be of limited total variation over every finite interval. Let*

(1°) *$K_1(x)$ be bounded and continuous,*

(2°) $$\int_{-\infty}^{\infty} |d(K_1(x)e^{-\lambda x})| < const., \text{ and } K_1(x) \sim A_1 e^{\lambda x}$$

$$(\lambda > 0,\ A_1 \neq 0) \text{ as } x \to -\infty,$$

(3°) $$K_1(x) \in L_2(-\infty, \infty).$$

Put $k_i(u) = \int_{-\infty}^{\infty} K_i(x) e^{ux} dx\ (i = 1, 2)$ and let $k_2(u)/k_1(u)$ be analytic over $-\varepsilon \leqq Re(u) \leqq \lambda + \varepsilon$, and let it belong to L_2 over every ordinate in that strip. Then if

$$\lim_{y\to\infty} \int_0^{\infty} K_1(y-x)\, df(x) = A \int_{-\infty}^{\infty} K_1(x)\, dx,$$

it follows that

$$\lim_{y\to\infty} \int_0^{\infty} K_2(y-x)\, df(x) = A \int_{-\infty}^{\infty} K_2(x)\, dx.$$

In the hypothesis, if $K_1(x) = 0\ (x \geqq 0)$, we may replace the strip $-\varepsilon \leqq Re(u) \leqq \lambda + \varepsilon$ by the narrower strip $-\varepsilon \leqq Re(u) \leqq \varepsilon$.

The proof has been given in Wiener [16], Chapter VII.

Theorem 1′. *Under the hypothesis of Theorem 1,*

$$\int_{-\infty}^{\infty} \left| dy \int_0^{\infty} K_1(y-x)\, df(x) \right| < \infty$$

implies

$$\int_{-\infty}^{\infty} \left| dy \int_0^{\infty} K_2(y-x)\, df(x) \right| < \infty.$$

Proof. Under the hypothesis of the theorem, Wiener concludes (see [16] p. 75), that

$$(1) \quad \int_{-\infty}^{\infty} d_z R(y-z) \int_0^{\infty} K_1(z-x)\, df(x) = \int_0^{\infty} df(x) \int_{-\infty}^{\infty} K_1(y-x-z)\, dR(z).$$

Put

$$\int_0^\infty K_1(y-x)\,df(x) = F(y),$$

then

$$\int_{-\infty}^\infty |dF(y)| < \infty.$$

By the convolution theorem of Fourier transform,

$$K_2(x) = \int_{-\infty}^\infty K_1(x-z)\,dR(z),$$

where

$$R(x) = \frac{1}{2\pi i}\int_0^x d\xi \int_{-\infty}^\infty \frac{k_2(u)}{k_1(u)} e^{-u\xi} du$$

and

(2) $$\int_{-\infty}^\infty |dR(z)| < \infty. \qquad \text{(Wiener [16], pp. 72–77).}$$

It is sufficient to prove $\int_{-\infty}^\infty |dG(y)| < \infty$, where

$$G(y) = \int_0^\infty K_2(y-x)\,df(x).$$

Now

$$\begin{aligned} G(y) &= \int_0^\infty df(x) \int_{-\infty}^\infty K_1(y-x-z)\,dR(z) \\ &= \int_{-\infty}^\infty d_z R(y-z) \int_0^\infty K_1(z-x)\,df(x) \qquad \text{(by (1))} \\ &= \int_{-\infty}^\infty d_z R(y-z)\,F(z) \\ &= [R(y-z)\,F(z)]_{-\infty}^\infty - \int_{-\infty}^\infty R(y-z)\,dF(z). \end{aligned}$$

Using the hypothesis and Theorem 1, we see that $F(+\infty)$ exists (we assume $F(-\infty)$ exists, this is permissible in the following applications) and that $R(+\infty)$ and $R(-\infty)$ exist by (2). Thus the first term of the right hand side is constant, so we have

$$\int_{-\infty}^\infty |dG(y)| = \int_{-\infty}^\infty \left| d_y \int_{-\infty}^\infty R(y-z)\,dF(z) \right|.$$

But, by the unsymmetric Fubini's theorem (cf. Cameron and Martin [6]) and (2), we have

$$\int_{-\infty}^{\infty} |dG(y)| \leqq \int_{-\infty}^{\infty} |dF(z)| \int_{-\infty}^{\infty} |d_y R(y-z)| < \infty .$$

Thus we get the theorem.

2. The Cesàso and absolute Cesàro summabilities of Fourier series.

Theorem 2. *Let $f(x) \in L$ with period 2π or defined over $(-\infty, \infty)$ and zero outside $(-A, A)$. Let*

$$\lim (C, m)\, \varphi(y) = \lim_{\lambda \to \infty} m\lambda \int_0^{1/\lambda} \varphi(y)(1-\lambda y)^{m-1}\, dy$$

and

$$\lim (C, m)\, \mathfrak{S}[\varphi] = \lim_{\lambda \to 8} \frac{2}{\pi} \lambda \int_0^{\infty} \varphi(y)\, dy \int_0^1 (1-z)^m \cos \lambda yz dz$$

$$= \lim_{\omega \to \infty} \omega \int_0^{\infty} \varphi(t)\, \gamma_{1+m}(\omega t)\, dt$$

where

$\varphi(y) = \{f(x+y) + f(x-y) - 2s\}/2$ *and* $\gamma_\alpha(x) = \int_0^1 (1-t)^{\alpha-1} \cos txdt$, $(\alpha > 0)$.

Then (i) if $(C, m)\, \varphi(y) \to 0$, as $y \to 0$, then $\mathfrak{S}[\varphi]$ is $(C, m+\varepsilon)$-summable to zero, where $m \geqq 1$, $\varepsilon > 0$, (ii) if $\mathfrak{S}[\varphi]$ is (C, m)-summable to zero, then $(C, m+1+\varepsilon)\, \varphi(y) \to 0$, as $y \to 0$, where $m \geqq 0$, $\varepsilon > 0$.

Proof. After Wiener we put,

$$K^{(m)}(x) = \begin{cases} (m+1)\,(1-e^x)^m e^x, & (x<0) \\ 0, & (x>0) \end{cases}$$

and

$${}^{(m)}K(x) = \frac{2}{\pi} e^x \int_0^1 (1-z)^m \cos(ze^x)\, dz \qquad (m>0),$$

then the condition (2°) in Theorem 1 is satisfied for $\lambda = 1$, and we can put $K_1(x) = K^{(m)}(x)$ or $K_1(x) = {}^{(m)}K(x)$ in Theorem 1. If we put

$$\tau_m(u) = \int_{-\infty}^{\infty} K^{(m)}(x)\, e^{ux} dx = \frac{\Gamma(m+2)\,\Gamma(u+1)}{\Gamma(u+m+2)}$$

and

$$\sigma_m(u) = \int_{-\infty}^{\infty} {}^{(m)}K(x) e^{ux}\, dx = \frac{\Gamma(m+1)}{\Gamma(m+1-u)\cos\frac{\pi u}{2}},$$

then as $|Im(u)| \to \infty$

$$\left|\frac{\sigma_m(u)}{\tau_n(u)}\right| \sim \frac{2\Gamma(m+1)}{\sqrt{2\pi}\,\Gamma(n+2)e^{R_e(u)-m}}\left|Im(u)\right|^{R_e(u)-m+n+1/2}.$$

Thus from Theorem 1 if $n > m$, then

$$(3) \qquad \lim_{y\to\infty}\int_0^\infty {}^{(m)}K(y-x)\,df(x) = A\int_{-\infty}^\infty {}^{(m)}K(x)\,dx$$

implies

$$(4) \qquad \lim_{y\to\infty}\int_0^\infty K^{(n)}(y-x)\,df(x) = A\int_{-\infty}^\infty K^{(n)}(x)\,dx,$$

while if $m > n+1$, (4) implies (3).

Theorem 2′. *Using the notations of Theorem* 2, (i) *if* $\varphi(y)$ *is* $|C, m|$ *-summable in* $(-\infty, \infty)$ $\{(-\pi, \pi)$ *for the Fourier series case*$\}$, *then* $\mathfrak{S}[\varphi]$ *is* $|C, m+\varepsilon|$*-summable, where* $m \geqq 1$ *in the Fourier integral case,* $m > 1$ *in the Fourier series case, and* (ii) *if* $\mathfrak{S}[\varphi]$ *is* $|C, m|$*-summable, then* $\varphi(y)$ *is* $|C, m+1+\varepsilon|$*-summable in* $(-\infty, \infty)$, *where* $m \geqq 0$ *in the Fourier integral case,* $m > 1$ *in the Fourier series case.*

Proof. In the proof of Theorem 2′ we use Theorem 1′ instead of Theorem 1. In the Fourier series case, since $\varphi(y)$ is $|C, m|$-summable in $(-\pi, \pi)$ but not in $(-\infty, +\infty)$ we need to prove

$$\int_1^\infty |dI(\omega)| < \infty,$$

where

$$I(\omega) = \omega\int_\pi^\infty \varphi(t)\,\gamma_{1+\beta}(\omega t)\,dt, \qquad (\beta > 0).$$

The Riesz kernel $\gamma_\alpha(x)$ satisfies the following relations (cf. Bosanquet [1]):

$$(5) \qquad \gamma_\alpha(x) + i\overline{\gamma_\alpha}(x) = \int_0^1 (1-t)^{\alpha-1}e^{ixt}\,dt \; (x \geqq 0,\ \alpha > 0)$$

$$(6) \qquad x\overline{\gamma_{1+\alpha}}(x) = 1 - \alpha\gamma_\alpha(x)$$

$$(7) \qquad x\gamma_{1+\alpha}(x) = \alpha\,\gamma_\alpha(x)$$

$$(8) \qquad \frac{d}{dx}(x\gamma_{1+\alpha}(x) = \alpha\,\overline{\gamma'_\alpha}(x)$$

$$(9) \qquad \gamma'_\beta(x) = 1/x^\rho, \text{ where } \rho = \mathrm{Min}\,(2, \beta).$$

We have

$$dI(\omega) = \frac{d}{d\omega}\left\{\omega\int_\pi^\infty \varphi(t)\,\gamma_{1+\beta}(\omega t)\,dt\right\} = \alpha\int_\pi^\infty \varphi(t)\,\overline{\gamma'_\beta}(\omega t)\,dt$$

$$= \sum_{s=1}^{\infty} \int_{(2s-1)\pi}^{(2s+1)\pi} (\omega t)^{-\rho} |\varphi(t)| dt$$

$$\leqq A\omega^{-\rho} \left\{ \sum_{s=1}^{\infty} \left[(2s-1)\pi \right]^{-\rho} \right\} \int_{-\pi}^{\pi} |\varphi(t)| dt$$

$$\leqq \omega^{-\rho} \quad (\rho > 1),$$

and then

$$\int_1^{\infty} |dI(\omega)| \leqq \int_1^{\infty} \omega^{-\rho} d\omega < \infty.$$

Thus $\mathfrak{S}[\varphi]$, where the function $\varphi(x)$ vanishes in $[-\pi, \pi]$, is $|C, \beta|$ -summable $(\beta > 1)$.

Thus the theorem is proved completely.

3. The Cesàro and absolute Cesàro summabilities of the conjugate Fourier series.

Let $f(x) \in L$ with period 2π or defined over $(-\infty, \infty)$ and zero outside $(-A, A)$. Let

$$\psi(y) = f(x+y) - f(x-y),$$

then it is known the existence of the integral

$$\psi(t) = \frac{2}{\pi} \int_t^{\to\infty} \frac{\psi(y)}{y} dy.$$

We put, after Paley [8],

$$\text{conj.} \lim_{t\to 0} (C, m)\, \psi(t) = \lim_{\lambda\to\infty} \lambda \int_0^{1/\lambda} \psi(t)(1-\lambda t)^{m-2} dt \quad (m > 1)$$

and

$$\lim\, (C, m)\, \overline{\mathfrak{S}}[f] = \lim_{\omega\to\infty} \frac{2}{\pi}\, \omega \int_0^{\infty} \psi(t)\, \overline{\gamma}_{1+\beta}(t)\, dt$$

$$= \lim_{\lambda\to\infty} \lambda \Gamma(1+m) \int_0^{\infty} (\lambda t)^{-(1+m)}\; C_{1+m}(\lambda t)\, \psi(t)\, dt$$

where $C_m(t)$ is Young's function. Then we have

Theorem 3. (i) *If* conj. $\lim\limits_{t\to 0} (C, m)\, \psi(t) = s$, *then* $\overline{\mathfrak{S}}[f]$ *is* $(C, m+\varepsilon)$ *-summable to s, where* $m \geqq 1$, $\varepsilon > 0$, *and* (ii) *if* $\overline{\mathfrak{S}}[f]$ *is* (C, m)*-summable to s, then* conj. $\lim\limits_{t\to 0} (C, m+1+\varepsilon)\, \psi(t) = s$, *where* $m \geqq 0$, $\varepsilon > 0$.

N. B. Paley proved the theorem $m \geqq 0$, but the case $1 \geqq m \geqq 0$ the definition of conj. $\lim\limits_{y\to 0} \psi(y)$ is different, so that the theorem needs to be formulated in another form, but we don't enter in this case.

If $\lambda\int_0^{1/\lambda}\psi(t)\ (1-\lambda t)^{m-2}\,dt$ is of bounded variation in $(-\infty,\infty)$ $\left\{(-\pi,\pi)\text{ for the Fourier series case}\right\}$, then we say that $\psi(t)$ is conj. $|C, m|$ -summable.

Theorem 3′. (i) *If* $\psi(t)$ *is* conj. $|C, m|$*-summable, then* $\bar{\mathfrak{S}}[f]$ *is* $|C, m+\varepsilon|$*-summable where* $m>1$ *for the Fourier integrel case,* $m>2$ *for the Fourier series case, and* (ii) *if* $\bar{\mathfrak{S}}[f]$ *is* $|C, m|$*-summable, then* $\psi(t)$ *is* conj. $|C, m+1+\varepsilon|$*-summable, where,* $m>0$ *for the Fourier integral case,* $m>1$ *for the Fourier series case.*

Proof. If $\psi(y)\sim\sum b_n \sin ny$, then we have (cf. Paley [8])

$$\sum_{n\leq\lambda} b_n\left(1-\frac{n}{\lambda}\right)^m = \frac{\lambda\Gamma(1+m)}{\pi}\int_0^{\to\infty}(\lambda t)^{-(1+m)}\ C_{2+m}(\lambda t)\ \psi(t)\,dt$$

$$= \Gamma(1+m)\,\lambda\int_0^{\to\infty}\chi_0(\lambda t)\ \Phi(t)\,dt$$

where

$$\chi_0(t) = \frac{d}{dt}\left\{t^{-m}\ C_{m+2}(t)\right\} = -\frac{d}{dt}\left\{t^{-m}\ C_m(t)\right\},$$

and

$$\chi_0(t) = O(1/t^\sigma),\quad \sigma=\min(m,2).$$

For the place of $K_1(x)$ or $K_2(x)$, we take

$$K^{(m)}(x) = \begin{cases}(m+1)(1-e^x)^m e^x, & (x<0)\\ 0, & (x>0)\end{cases}$$

and

$${}^{(m)}\bar{K}(x) = \chi_0(e^x)\,e^x.$$

Then we have

$$\int_{-\infty}^{\infty}\chi_0(e^x)\,e^x e^{iux}\,dx = \int_0^{\infty}\chi_0(t)\,t^u\,dt = \int_0^{\infty}{}^{(m-1)}K'(t)\,t^u\,dt,$$

where ${}^{(m-1)}K(t)$ is defined in Theorem 2, and the dash denotes differentiation with respect to t. By the integration by parts the above integral becomes

$$\left[{}^{(m-1)}K(t)\,t^u\right]_0^\infty - u\int_0^{\infty}{}^{(m-1)}K(t)\,t^{u-1}\,dt$$

$$= -u\int_0^{\infty}{}^{(m-1)}K(t)\,t^{u-1}\,dt,$$

since for $m>1$, ${}^{(m-1)}K(t)\,t^{1+\varepsilon}\to 0$ as $t\to\infty$.
Evaluating the integral we have

$$\frac{-u\Gamma(m)}{\Gamma(m+1-u)\cos\frac{\pi(u-1)}{2}} = \frac{u\Gamma(m)}{\Gamma(m+1-u)\sin\frac{\pi}{2}u} \equiv \bar{\sigma}_m(u), \text{ say.}$$

Similarly we have

$$\int_{-\infty}^{\infty} K^{(m)}(x)\, e^{ux}\, dx = \frac{\Gamma(m+2)\Gamma(u+1)}{\Gamma(u+m+2)} \equiv \tau_m(u), \text{ say.}$$

Since

$$\bar{\sigma}_m(u) \sim \frac{\Gamma(m)}{\sqrt{2\pi}\;|Im(u)|^{-Re(u)+m-1/2}\, e^{Re(u)-m-1}}$$

and

$$\tau_n(u) \sim \frac{\Gamma(n+2)}{|Im(u)|^{n+1}}$$

as $|Im(u)| \to \infty$, we have

$$\left|\frac{\sigma_m(u)}{\tau_n(u)}\right| \sim \frac{2\Gamma(m)}{\sqrt{2\pi}\;\Gamma(n+2)\, e^{Re(u)-m-1}} \left|Im(u)\right|^{Re(u)-m+n+3/2}$$

and $k_2(u)/k_1(u)$ belongs to L_2 over every ordinate in the strip, provided that

$$2(-m+n+3/2) < -1,$$

that is $n+2 < m$. Since other conditions are evident, conj. $\lim_{y\to 0}(C, m)\psi(y) = s$ implies

$$\lim \bar{\mathfrak{S}}[f](C, m+\varepsilon) = s \qquad (m>1).$$

The proof of Theorem 3′ is analogous.

4. The Cesàro and absolute Cesàro summabilities of the derived Fourier series.

Suppose that there is a polynomial

$$P(t) = \sum_{i=0}^{r-1} \frac{\theta_i}{i!} t_i$$

such that for $-\pi \leqq t \leqq \pi$,

$$\Phi_r(t) = \frac{1}{2t^r}\left[\{f(x+t) - P(t)\} + (-1)^r \{f(x-t) - P(-t)\}\right]$$

is integrable in the sense of Cauchy. In the sequel we suppose $P(t) = 0$ without any loss of generality. The (C, m)-mean of the r-th derived Fourier series $\mathfrak{S}^{(r)}[f]$ is denoted by

$$C_m^{(r)}(\omega) = (-1)^r \frac{2}{\pi}\,\omega^{r+1} \int_0^{\infty} \gamma_{1+m}^{(r)}(\omega t)\, \varphi(t)\, dt \qquad (m>r),$$

where

$$\gamma_m(t) = \Gamma(m)\, t^{-m}\, C_m(t) = \int_0^1 (1-z)^{m-1} \cos(tz)\, dz$$

and

$$(10) \qquad \gamma_{1+m}^{(r)}(t) = \int_0^1 (1-z)^m z^r \cos(tz + \tfrac{1}{2}\pi r)\, dz$$

$$= \frac{(-1)^r}{t^r} \sum_{i=0}^{r} \binom{r}{i} \frac{\Gamma(m+r-i+1)}{\Gamma(m-i+1)} \gamma_{1+m-i}(t).$$

(cf. Jacob [8]). For the the sake of brevity, we investigate the case $r = 1$. Then

$$C'_m(\omega) = (-1)\frac{2}{\pi}\omega^2 \int_0^\infty \gamma'_{1+m}(\omega t)\,\varphi(t)\,dt$$

$$= \frac{2}{\pi}\omega^2 \int_0^\infty \varphi(t)\,dt \int_0^1 (1-z)^m z \sin(z\omega t)\,dz$$

$$= \frac{2}{\pi}\int_0^\infty \Phi(t)\,dt \int_0^1 \omega^2 (1-z)^m zt \sin(z\omega t)\,dz$$

$$= \frac{2}{\pi}\int_{-\infty}^\infty \Phi(e^{-T})\,dT \int_0^1 e^{2(W-T)}(1-z)^m z \sin(ze^{W-T})dz, \ (w=e^W).$$

Let

$${}^{(m)}K(x) = \int_0^1 e^{2x}(1-z)^m z \sin(ze^x)\,dz,$$

then

$$(11)\quad \sigma_m(u) \equiv \int_{-\infty}^\infty {}^{(m)}K(x)\,e^{ux}\,dx = \int_0^\infty t^{u+1}\,dt \int_0^1 (1-z)^m z \sin zt\,dz$$

$$= \int_0^\infty t^{u+1}\gamma'_{1+m}(t) \qquad \text{(by (10))}$$

$$= \int_0^\infty t^u\,dt\left[\frac{\Gamma(m+2)}{\Gamma(m+1)}\gamma_{1+m}(t) + \frac{\Gamma(m+1)}{\Gamma(m)}\gamma_m(t)\right]$$

$$= \frac{\Gamma(m+2)}{\Gamma(m+1-u)\cos\frac{\pi u}{2}} + \frac{\Gamma(m+1)}{\Gamma(m-u)\cos\frac{\pi u}{2}}$$

$$= \frac{\Gamma(m+1)(2m+1-u)}{\cos\frac{\pi u}{2}\,\Gamma(m-u+1)}.$$

In order to deduce this formula, it is necessary to make some restriction on m and u, but the resulting formula is valid in the analytic domain of u by the principle of analytic continuation.

Put

$$(C, m)\,\Phi(t) = \omega\int_0^{1/\omega} (1-\omega t)^{n-1}\,\Phi(t)\,dt,$$

then the kernel becomes

$$K^{(m)}(x) = (m+1)(1-e^x)^m e^x$$

and

$$\tau^{(m)}(u) = \int_{-\infty}^{\infty} K^{(m)}(x)\, e^{ux}\, dx = \frac{\Gamma(m+2)\,\Gamma(u+1)}{\Gamma(u+m+2)} \qquad (m > -1).$$

When $\sigma_m(u)$ is numerator the range of analyticity of $k_2(u)/k_1(u)$, is sufficient to be $-\varepsilon < Re(u) < \varepsilon$, since $K^{(m)}(x) = K_1(x) \equiv 0\ (x>0)$ in Theorem 1, but when $\sigma_m(u)$ is denominator the range is $-\varepsilon < Re(u) < 2+\varepsilon$, by ${}^{(m)}K(x) \equiv K_1(x) \sim e^{2x}$ as $x \to \infty$ in Theorem 1. In the latter case m should be limited to be > 1, for the poles of $\Gamma(m-u+1)$. Since the order of asymptotic formula of $\sigma_m(u)/\tau_n(u)$ as $|Im(u)| \to \infty$, is $|Im(u)|^{Re(u)-m+n+1/2+1}$. by Theorem 1, in order to conclude $\tau_n(u)/\sigma_m(u) \in L_2$, it is sufficient

$$2\left(Re(u) - m + n + \frac{3}{2}\right) < -1 \qquad (m > -1)$$

that is, $n + 2 < m$.

Similarly $\sigma_m(u)/\tau_n(u) \in L_2$, when

$$-2\left(Re\,(u) - m + n + \frac{1}{2} + 1\right) < -1 \qquad (m < 1),$$

that is $n > m - 1$.

Thus we get the following theorem.

Theorem 4. (i) *If* $\Phi(t) = \frac{1}{2t}\{f(x+f) - f(x-t)\}$ *is* (C, m)*-summable to zero as* $t \to 0$, *then* $\mathfrak{S}'[f]$ *is* $(C, m+1+\varepsilon)$*-summable to zero, where* $m > 0$, $\varepsilon > 0$, *and* (ii) *if* $\mathfrak{S}'[f]$ *is* (C, m)*-summable to zero, then* $(C, m+\varepsilon)$ $\Phi(t) \to 0$ *as* $t \to 0$ *where* $m > 1$, $\varepsilon > 0$.

Theorem 4'. (i) *If* $\Phi(t)$ *is* $|C, m|$*-summable in* $(-\pi, \pi)$, *then* $\mathfrak{S}'[f]$ *is* $|C, m+1+\varepsilon|$*-summable, where* $m > 1$, *and* (ii) *if* $\mathfrak{S}'[f]$ *is* $|C, m|$*-summable, then* $\Phi(t)$ *is* $|C, m+\varepsilon|$*-summable, where* $m > 1$.

More generally in the case of the r-th derived series, instead of (11), we have from (10) for $\sigma_m^{(r)}(u)$ which denotes the Mellin transform of the $C_m^{(r)}$-kernel,

$$\sigma_m^{(r)}(u) = \int_0^{\infty} t^u \left\{\sum_{i=0}^{r} \binom{r}{i} \frac{\Gamma(m+r-i+1)}{\Gamma(m-i+1)}\, \gamma_{1+m-i}(t)\right\} dt$$

$$= \frac{\Gamma(m+1)\, P_r(u)}{\cos\frac{\pi u}{2}\, \Gamma(m+1-u)},$$

where $P_r(u)$ is a polynomial of u of order r. When $\sigma_m^{(r)}(u)$ is denominator, $k_1(u)/k_1(u)$ is sufficient to be analytic in $-\varepsilon < Re\,(u) < r+1+\varepsilon$ by $K_1(x) \sim e^{(r+1)x}$ as $x \sim \infty$, and it is sufficient to suppose $m > r$. Let $\tau_n^{(r)}$ be the Mellin transform of $\Phi_r(t)$, then $\tau_n^{(r)}(u)/\sigma_m^{(r)}(u) \in L_2$, when $2\{Re(u) - m +$

$n+\frac{1}{2}+r\} < -1$, that is $n+r<m$, but $\sigma_m^{(r)}(u)/\tau_n^{(r)}(u) \in L_2$, when$-2\left\{Re(u)-m+\frac{1}{2}+r\right\} < -1\ (m>r)$, that is $n > m-r$.

Theorem 5. (i) *If* $\lim\limits_{t\to 0}\ (C, m)\ \Phi_r(t) = 0$, *then* $\mathfrak{S}^{(r)}(f)$ *is* $(C, m+r+\varepsilon)$*-summable to zero, where* $m>0$, *and* (ii) *if* $\mathfrak{S}^{(r)}[f]$ *is* (C, m)*-summable to zero, then* $\lim\ (C, m-r+1+\varepsilon)\ \Phi_r(t) = 0$ *where* $m > r \geqq 1$.

Theorem 5′. (i) *If* $\Phi_r(t)$ *is* $|C, m|$*-summable in* $(-\pi, \pi)$, *then* $\mathfrak{S}^{(r)}[f]$ *is* $|C, m+r+\varepsilon|$*-summable, where* $m>1$ *and* (ii) *if* $\mathfrak{S}^{(r)}[f]$ *is* $|C, m|$*-summable, then* $\Phi_r(t)$ *is* $|C, m-r+1+\varepsilon|$*-summable in* $(-\pi,\pi)$, *where* $m > r > 0$.

5. Relations between generalized jump of a function and its Fourier coefficients and certain of its applications.

Let $\psi(t) = \psi_r(t) = f(x+t) - (-1)^r f(x+t)$ and let $\Psi_r\ (t) = \psi_r(t)/r!\,t^r$. If $\lim\limits_{t\to 0} \Psi_r(t) = B_r$, Zygmund [15] called it the r-th jump of $f(x)$ at x and proved that

$$\frac{1}{\pi} B_k = \lim_{n\to\infty}\ (C,\ \alpha)\ (a_n \cos nx + b_n \sin nx)^{(k+1)}, \qquad (\alpha > k+1)$$

which denotes the $(k+1)$-th derivative for x. We shall generalize this result as follows. According to Zygmund [15],

$$\lim_{n\to\infty}\ (C, \alpha)\ (a_n \cos nx + b_n \sin nx)^{(k+1)}$$

$$= \frac{(-1)^{(k+1)}}{\pi} \lim_{n\to\infty} \frac{A_n^{\alpha-1}}{A_n^{\alpha}} \int_0^{\pi} \psi_k(t) \frac{d^{k+1}}{dt^{k+1}} K_n^{(\alpha-1)}(t)\, dt \ (\text{where } K_n^{(\alpha)}(t)$$

is (C, α)-kernel)

$$= \lim_{\varepsilon\to 0} \varepsilon \int_0^{\infty} \frac{\psi_k(t)}{t^k}\, t^k \frac{1}{\varepsilon} \frac{d^{k+1}}{dt^{k+1}} \left[\int_0^1 (1-\lambda)^{\alpha-1} \cos \frac{\lambda t}{\varepsilon} \right] d\lambda$$

$$= \lim_{\varepsilon\to 0} \int_0^{\infty} \frac{\psi_k(t)}{t^k} t^k \int_0^1 (1-\lambda)^{\alpha-1} \frac{\lambda^{k+1}}{\varepsilon^{k+1}} \cos\left(\frac{\lambda t}{\varepsilon} + \frac{k+1}{2}\pi\right) d\lambda$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^{\infty} \frac{\psi_k(t)}{t^k} \left(\frac{t}{\varepsilon}\right)^k \int_0^1 (1-\lambda)^{\alpha-1} \lambda^{k+1} \cos\left(\frac{\lambda t}{\varepsilon} + \frac{k+1}{2}\pi\right) d\lambda.$$

Comparing with Theorem 1, the kernel

$${}^{(m)}K(x) = e^{(k+1)x} \int_0^1 (1-\lambda)^{\alpha-1} \lambda^{k+1} \cos\left(\lambda e^x + \frac{k+1}{2}\pi\right) d\lambda$$

is of the same order as the kernel of derived series. For instance, let $k=0$, then

$${}^{(m)}K(x) = e^x \int_0^1 (1-\lambda)^{\alpha-1} \lambda \sin(\lambda e^x)\, d\lambda,$$

$$\sigma_m(u) = \int_{-\infty}^{\infty} K(x) e^{ux}\, dx = \int_{-\infty}^{\infty} e^{ux} e^{x} \int_0^1 (1-\lambda)^{\alpha-1} \sin(\lambda e^x)\, d\lambda$$

$$= \int_0^{\infty} t^u\, dt\, \gamma'_m(t) = \frac{\Gamma(m)\,(2m-u)}{\cos\frac{\pi u}{2}\Gamma(m-u+1)}.$$

Theorem 6. (i) *If* $\lim_{t\to 0} (C,\alpha)\, \psi_x(t) = f(x+t) - f(x-t) = j,$ *then* $\lim_{n\to\infty} (C, 1+\alpha+\varepsilon)\ (nb_n \cos nx - na_n \sin nx) \to j/\pi$ *where* $\alpha \geqq 0,\ \varepsilon > 0,$ *and conversely* (ii) *If* $\lim_{n\to\infty} (C,\alpha)\ (nb_n \cos nx - na_n \sin nx) \to j/\pi,$ *then* $\lim_{t\to 0} (C, \alpha+\varepsilon)\ \psi_x(t) \to j,$ *where* $\alpha > 1$.

Theorem 6′. (i) *If* $\psi_x(t)$ *is* $|C,\alpha|$*-summable in* $(-\pi,\pi)$, *then* $(nb_n \cos nx - na_n \sin nx)$ *is* $|C, 1+\alpha+\varepsilon|$*-summable, where* $\alpha > 1$ *and* $\varepsilon > 0$ *and* (ii) *if* $(nb_n \cos nx - na_n \sin nx)$ *is* $|C,\alpha|$*-summable, then* $\psi_x(t)$ *is* $|C,\alpha+\varepsilon|$ *summable, where* $\alpha > 1$.

N. B. If $f(t) \sim \sum_{n=1}^{\infty} b_n \sin nt$, then

$$\frac{1}{t}\int_0^t f(u)\, du = \frac{F(t) - F(0)}{t} = \sum_{n=1}^{\infty} \frac{b_n (\cos nt - 1)}{nt}$$

$$= 2\sum_{n=1}^{\infty} \frac{nb_n \sin^2 \frac{1}{2} nt}{\frac{1}{2} n^2 t}.$$

This is the $(R', 2)$-summation of nb_n. Theorem 6 and 6′ denote the relation between $(R', 2)$ and $(C, 2+\varepsilon)$, and $|R', 2|$ and $|C, 2+\varepsilon|$, respectively. More generally

Theorem 7. *If* $\Phi_k(t)$ *is* (C,α)*-summable to* j *as* $t \to 0$, *then* $(a_n \cos nx + b_n \sin nx)^{(k+1)}$ *is* $(C, k+1+\alpha+\varepsilon)$*-summable to* j/π, *where* $\alpha > 0,\ \varepsilon > 0,$ *and* (ii) *if* $(a_n \cos nx + b_n \sin nx)^{(k+1)}$ *is* (C,α)*-summable to* j/π, *then* $\Phi_k(t)$ *is* $(C, \alpha - k + 1 + \varepsilon)$*-summable to* j, *where* $\alpha > k,\ \varepsilon > 0$.

Theorem 7′. (i) *If* $\Phi_k(t)$ *is* $|C,\alpha|$*-summable in* $(-\pi,\pi)$, *then* $(a_n \cos nx + b_n \sin nx)^{(k+1)}$ *is* $|C, k+1+\alpha+\varepsilon|$*-summable, where* $\alpha > 1,\ \varepsilon > 0,$ *and* (ii) *if* $(a_n \cos nx + b_n \sin nx)^{(k+1)}$ *is* $|C,\alpha|$*-summable, then* $\Phi_k(t)$ *is* $|C, \alpha - k + 1 + \varepsilon|$*-summable, where* $\alpha > k,\ \varepsilon > 0$.

From these theorems, we can see the relation between the generalized jump and generalized Gibbs' phenomenon by Szàsz's Theorem [12].

Theorem 8. *If the conjugate Fourier series* $\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx)$ *is*

Abel-summable and $\lim\limits_{t\to 0}(C,\alpha)\psi(t)=0$, *then the series is* $(C,\alpha+\varepsilon)$*-summable and exists* $\lim\limits_{\epsilon\to 0}(C,\alpha)\dfrac{1}{\pi}\displaystyle\int_{\epsilon}^{\infty}\frac{\psi(t)}{t}$.

Theorem 8′. *If the conjugate Fourier series* $\sum\limits_{n=1}^{\infty}(b_n\cos nx-a_n\sin nx)$ *is* $|A|$*-summable and* $\psi(t)$ *is* $|C,\alpha|$*-summable in* $(-\pi,\pi)$, *then the series is* $|C,\alpha+\varepsilon|$*-summable* $(\alpha>1)$ *and the integral* $\dfrac{1}{\pi}\displaystyle\int_{\epsilon}^{\infty}\frac{\psi(t)}{t}$ *is* $|C,\alpha|$*-summable.*

Proof. Since $(C,\alpha)\ \psi(t)\to 0$, $(nb_n\cos nx-na_n\sin nx)$ is $(C,1+\alpha+\varepsilon)$-summable to zero by Theorem 6. Using the well known Tauberian theorem, the Abel summability of the series implies $(C,\alpha+\varepsilon)$-summability. The existence of $\lim\limits_{\epsilon\to 0}(C,\alpha)\dfrac{1}{\pi}\displaystyle\int_{\epsilon}^{\infty}\frac{\psi(t)}{t}$ is due to Paley [11].

Theorem 9. *When* $(C,\alpha)\ \psi(t)\to 0$ *as* $t\to 0$, *in order that the conjugate Fourier series* $\sum\limits_{n=1}^{\infty}(b_n\cos nx-a_n\sin nx)$ *is summable to* $+\infty$ *by Abel's method, the necessary and sufficient condition is* $\lim\limits_{\epsilon\to 0}(C,\alpha)\displaystyle\int_{\epsilon}^{\infty}\frac{\psi(t)}{t}\,dt=+\infty$.

Theorem 9′. *When* $\psi(t)$ *is* $|C,\alpha|$*-summable in* $(-\pi,\pi)$, *in order that the conjugate Fourier series is not summable by* $|A|$*-method, the necessary and sufficient condition is*

$$\frac{1}{\pi}\int_{\epsilon}^{\infty}\frac{\psi(t)}{t}\,dt \text{ is not } |A|\text{-summable } (-\pi,\pi) \text{ where } \alpha>1.$$

Proof. The necessity is analogous to the proof of the Theorem 9 The sufficiency is due to Theorem 1, where $A=+\infty$.

6. Cesàro and absolute Cesàro summability of the derived series of conjugate Fourier series.

Let

$$f(x)\sim\sum_{n=0}^{\infty}(a_n\cos nx+b_n\sin nx)\equiv\sum_{n=0}^{\infty}A_n(x),$$

and its conjugtea series be

$$\sum_{n=1}^{\infty}(b_n\cos nx-a_n\sin nx)\equiv\sum_{n=1}^{\infty}B_n(x),$$

then its derived series is

$$\sum_{n=1}^{\infty} - (n\, a_n \cos nx + n\, b_n \sin nx) \equiv \sum_{n=1}^{\infty} - n A_n(x).$$

The (C, k)-mean of this series is (cf. Sayers [20])

$$\tau_k(\omega) = \frac{\omega}{(k+1)\pi} \int_0^{\infty} \varphi(t) \frac{d}{dt} \left[\omega t\, \gamma_{k+2}(\omega t)\right] dt,$$

where

$$\varphi(t) = f(x+t) + f(x-t) - 2f(x).$$

Then we have

$$\tau_k(\omega) = \lim_{N\to\infty} \frac{\omega}{(k+1)\pi} \int_0^{N} \varphi(t) \frac{d}{dt} \left[\omega t\, \gamma_{k+2}(\omega t)\right] dt \tag{12}$$

$$= \lim_{N\to\infty} A\omega \int_0^{N} \varphi(t) \frac{d}{dt} \left[(\omega t)^{-1-k}\; C_{k+2}(\omega t)\right] dt$$

$$= \lim_{N\to\infty} A\left[\omega^2 \int_0^{N} \varphi(t)\, \overline{\gamma_k'}(\omega t)\, dt\right]$$

$$= \lim_{N\to\infty} A \int_0^{N} (\omega t)^2 \frac{\varphi(t)}{t^2} \gamma_k'(\omega t)\, dt,$$

where

$$\overline{\gamma_k'}(t) = \frac{d}{dt} \left(t^{-1-k}\; C_{k+2}(t)\right).$$

We have

$$\int_t^{\infty} \frac{|\varphi(t)|}{t^2} dt \leqq \frac{1}{[t]^2} \int_{[t]}^{[t]+2\pi} |\varphi(t)|\, dt + \frac{1}{([t]+2\pi)^2} \int_{[t]+2\pi}^{[t]+4\pi} |\varphi(t)|\, dt + \cdots \leqq C.$$

Hence, if we put

$$\int_t^{\infty} \frac{\varphi(t)}{t^2} dt \equiv \Psi_1(t),$$

then $\varepsilon^2 \Psi_1(\varepsilon) \to 0$, as $\varepsilon \to +0$, applying the second mean value theorem. Then, by the integration by parts,

$$\tau_k(\omega) = \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} A\left[-\Psi_1(u)(\omega u)^2 \overline{\gamma_k'}(\omega u)\right]_{\varepsilon}^{N}$$

$$+ A\omega \int_0^{N} \Psi_1(t)\, \chi(\omega t)\, dt,$$

where

$$\chi(t) = \frac{d}{dt} \left(t^2\, \overline{\gamma_k'}(t)\right).$$

Since, for $k > 0$,

$$t^2\, t^{-1-k} C_{k+2}(t) = t^3\, t^{-(k+2)} C_{k+2}(t) = O\left(t^3\, t^{-k-2}\, t^k\right) = O(t),$$

as $t \to \infty$, and

$$\int_t^\infty \frac{\varphi(t)}{t^2}\, dt = o(t),$$

as $t \to \infty$, we get

$$\tau_k(\omega) = \lim_{N\to\infty} A\omega \int_0^N \Psi_1(t)\, \chi(\omega t)\, dt = A\int_0^\infty \Psi_1(u/\omega)\, \chi(u)\, du.$$

But

$$\begin{aligned}\chi(u) &= \frac{d}{dt}\left\{t^2 \frac{d}{dt}\,[t\, \gamma_{k+2}(t)]\right\}\\ &= \frac{d}{dt}\left\{t^2\, \gamma_{k+2}(t) + t^3 \frac{d}{dt}\, \gamma_{k+2}(t)\right\}\\ &= \frac{d}{dt}\left\{2t^2\, \gamma_{k+2}(t) + t^3 \frac{d}{dt}\, \gamma_{k+2}(t) - t^2\, \gamma_{k+2}(t)\right\}\\ &= \frac{d}{dt}\left\{t \frac{d}{dt}\, (t^2\, \gamma_{k+2}(t)) - t^2\, \gamma_{k+2}(t)\right\}.\end{aligned}$$

If we put

$$\frac{d}{dt}\, (t^2\, \gamma_{k+2}(t)) = \chi_0(t),$$

then

$$\chi(t) = \frac{d}{dt}\, \{t\, \chi_0(t)\} - \chi_0(t) = t\chi_0'(t).$$

Consequently,

$$\begin{aligned}\tau_k(\omega) &= A\int_0^\infty \Psi_1\!\left(\frac{t}{\omega}\right) t\, \chi_0'(t)\, dt \\ &= A\omega\int_0^\infty \Psi_1(u)\, \omega u\, \chi_0'(\omega u)\, du,\end{aligned} \tag{13}$$

where

$$\chi_0'(t) = \frac{d^2}{dt^2}\, \{t^{-k}\, C_{k+2}(t)\} = -\frac{d^2}{dt^2}\, \{t^{-k}\, C_k(t)\} = -\frac{d^2}{dt^2}\, (\gamma_k(t))$$

and

$$\gamma_{k+1}''(t) = \frac{1}{t^2}\left\{\frac{\Gamma(k+3)}{\Gamma(k+1)}\, \gamma_{1+k}(t) + 2\frac{\Gamma(k+2)}{\Gamma(k)}\, \gamma_k(t) + \frac{\Gamma(k+1)}{\Gamma(k-1)}\, \gamma_{k-1}(t)\right\}.$$

If we put as the kernel of the key theorems in § 1,

$${}^{(k)}K(x) = e^{2x}\, \chi_0'(e^x),$$

then its Mellin transform is

$$\begin{aligned}\int_{-\infty}^\infty {}^{(k)}K(x)\, e^{ux}\, dx &= \int_{-\infty}^\infty e^{2x}\, e^{ux}\, \chi_0'(e^x)\, dx\\ &= \int_0^\infty t^{u+1}\, \chi_0'(t)\, dt\\ &= \int_0^\infty t^{u-1}\left\{\frac{\Gamma(k+2)}{\Gamma(k+1)}\, \gamma_k(t) + 2\,\frac{\Gamma(k+1)}{\Gamma(k-1)}\, \gamma_{k-1}(t) + \frac{\Gamma(k)}{\Gamma(k-2)}\, \gamma_{k-2}(t)\right\}dt\end{aligned}$$

$$= \frac{\Gamma(k)\{u^2 - 4(k-1)(u-1) - (u-1) + 4(k-1)^2 + 4(k-1) + 1\}}{\Gamma(k+1-u)\cos \pi u/2} \equiv \bar{\sigma}_k(u),$$

say.

As $|Im(u)| \to \infty$, its asymptotic behaviour is

$$\bar{\sigma}_m(u) \sim A|Im(u)|^{Re(u)-m+1+1/2}.$$

The ordinary Cesàro kernel has the Mellin transform

$$\tau^{(n)}(u) \sim B|Im(u)|^{-n-1},$$

as $|Im(u)| \to \infty$.

When $\bar{\sigma}_m(u)$ is the denominator, the range of analyticity of $k_2(u)/k_1(u)$ is sufficient to be $-\varepsilon < Re(u) < 2 + \varepsilon$ since ${}^{(m)}K(x) \equiv K_1(x) \sim e^{x}$ as $x \to \infty$ in Therem 1. Hence in this case m should be limited to be > 1.

If

$$2(Re(u) - m + 3/2 + n + 1) < -1,$$

that is

$$n + 3 < m,$$

then

$$\tau^{(n)}(u)/\bar{\sigma}_m(u) \in L^2$$

and if

$$-2(Re(u) - m + 3/2 + n + 1) < -1, \qquad (m > 1),$$

that is,

$$n > m - 2,$$

then

$$\bar{\sigma}_m(u)/\tau^{(n)}(u) \in L^2.$$

Thus we have the following theorem.

Theorem 10. *If we put*

$$\Psi_1(t) = \int_t^\infty \frac{\varphi(t)}{t^2} dt$$

then, (i) *if* $\lim_{t\to 0} (C, m)\ \Phi_1(t) = s$ *(that is* conj. $\lim_{t\to 0} (C, m+1)\varphi(t)/t = s$), *then* $\mathfrak{S}'[f]$ *is* $(C, m+2+\varepsilon)$*-summable to* s, *where* $m \geqq 0$, $\varepsilon > 0$, *and* (ii) *if* $\mathfrak{S}'[f]$ *is* (C, m)*-summable to* s, *then* $\lim_{t\to 0} (C, m-1+\varepsilon)\ \Phi_1(t) = s$ {*that is,* conj. $\lim_{t\to 0} (C, m+\varepsilon)\varphi(t)/t = s$, *where* $m > 1$ *and* $\varepsilon > 0$.

Theorem 10′. *The same is true for the absolute Cesàro summability, where in the case* (i) $m \geqq 0$ *must be replaced by* $m \geqq 1$.

N. B. When

$$F(x) \sim \sum_{n=1}^\infty \frac{a_n \cos nx + b_n \sin nx}{n} \qquad (F(x) \in L(0, 2\pi)),$$

A. Zygmund [19] proved that if

$$\frac{-1}{\pi} \lim_{\alpha\to+0} \int_\alpha^\pi \frac{F(x+t) + F(x-t) - 2F(x)}{\left(2\sin\frac{1}{2}t\right)^2} dt$$

exists then $\sum a_n$ is summable $(K, 2)$. Hence we get the following Corollary of Theorem 10.

Corollary. *If* $\sum_{n=1}^{\infty} \frac{a_n}{n} \cos nx$ *is a Fourier series, then the* $(K, 2)$*-summability of* $\sum_{n=1}^{\infty} a_n$ *implies its* $(C, 2+\varepsilon)$*-summability.*

Similarly estimating as in § 5, we have

$$\lim_{n\to\infty} (C,\alpha)(-b_n \cos nx + a_n \sin nx)$$

$$= \lim_{n\to\infty} \frac{2}{\pi} \frac{A_n^{\alpha-1}}{A_n^{\alpha}} \int_0^{2\pi} \varphi_x(t) \frac{d}{dt} \overline{K}_n^{(\alpha-1)}(t)\, dt \qquad \text{(where } \overline{K}_n^{(\alpha-1)}(t) \text{ is}$$

conjugate $(C, \alpha-1)$-kernel)

$$= \lim_{\varepsilon\to 0} A\varepsilon \int_0^{\infty} \varphi_x(t) \frac{1}{\varepsilon} \frac{d}{dt} [\overline{\gamma}_{\alpha-1}(t/\varepsilon)]\, dt \text{ (where A is a constant)}$$

$$= \lim_{\omega\to\infty} A\omega \int_0^{\infty} \varphi_x(t)\, \overline{\gamma}'_{\alpha-1}(\omega t)\, dt$$

$$= \lim_{\omega\to\infty} \frac{A}{\omega} \int_0^{\infty} (\omega t)^2 \frac{\varphi_x(t)}{t^2} \overline{\gamma}'_{\alpha-1}(\omega t)\, dt.$$

Comparing with (12), put

$$\int_t^{\infty} \frac{\varphi_x(t)}{t^2} dt = \Psi_1(t),$$

then we have, similarly as (13),

$$\lim_{n\to\infty} (C,\alpha)(-b_n \cos nx + a_n \sin nx)'$$

$$= \lim_{\omega\to\infty} A\omega \int_0^{\infty} \Psi_1(u)\, u\, X_1'(\omega u)\, du,$$

where

$$X_1'(t) = \frac{d}{dt}(t^2 \gamma_{\alpha+1}(t)).$$

Hence the kernel is

$${}^{(\alpha)}K(x) = e^x X_1'(e^x),$$

which is the same order as the kernel of the derived series of the conjugate series. Thus we get

Theorem 11. (i) *If*

$$\lim_{t\to 0} (C,\alpha) \frac{2}{\pi} t \int_t^{\infty} \frac{\varphi_x(t)}{t^2} dt = s,$$

then $\lim_{n\to\infty} (C, \alpha + 2 + \varepsilon)\ (na_n \cos nx + nb_n \sin nx) = s$, *where* $\alpha \geqq 0,\ \varepsilon > 0$ *and conversely* (ii) *if* $\lim_{n\to\infty} (C, \alpha)\ (n a_n \cos nx + nb_n \sin nx) = s$, *then* $\lim_{t\to 0} (C, \alpha - 1 + \varepsilon)\ \frac{2}{\pi} t \int_t^{\infty} \frac{\varphi_x(t)}{t^2} dt = s$, *where* $\alpha > 1,\ \varepsilon > 0$,

Theorem 11′. *The same is true for the absolute Cesàro summability.*

N. B. If

$$\lim_{\alpha\to+0} \frac{2\sin(t/2)}{\pi} \int_{\alpha}^{\infty} \frac{\varphi_x(t)}{t^2} dt = \lim_{\alpha\to+0} \sum_{n=1}^{\infty} nA_n(x)\ \frac{2\sin(\alpha/2)}{\pi n} \int_{\alpha}^{\pi} \frac{4\sin^2 \frac{1}{2} nt}{4\sin^2 \frac{t}{2}} dt$$

exists at $x = 0$, we say that the sequence (nA_n) is $(K', 2)$-summable. This is the conjugate analogue of $(R', 2)$-summability. Then Theorem 11 implies that $(K', 2)$-summability implies $(C, 2 + \varepsilon)$-summability, provided that $\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ is Fourier series. Concerning this sort of summability we shall return in another occasion.

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