ON THE REPRESENTATIONS OF POSITIVE DEFINITE FUNCTIONS AND STATIONARY FUNCTIONS ON A TOPOLOGICAL GROUP

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1. Introduction. Let G be a (not necessarily locally compact) topological group. A continuous (complex-valued) function $f(g)$ is called *positive definite* provided that

$$\sum_{i,j=1}^{n} f(g_i g_j^{-1})\alpha_i \overline{\alpha_j} \geqq 0$$

for any set of complex numbers $\{\alpha_i\}$ and elements $\{g_i\}$ of G. A strongly continuous function $x(g)$, being defined on G and having its values in a Hilbert space H, is called, after A. Khintchine [7], *stationary* provided that $(x(g), x(h))$ depends only on gh^{-1}.

A closed connection between positive definite functions and stationary functions is firstly pointed out by K. Fan [2], when G is the (discrete) additive group of integers. He shows that for a positive definite sequence $\{\alpha_n\}$ there exists a stationary sequence $\{x_n\}$ with $\alpha_{n-m} = (x_n, x_m)$. (Converse is naturally obvious.) Establishing this, he developed the theory of positive definite sequences parallel with that of stationary sequences, and proved several theorems without use of the Herglotz Theorem of integral representation.

On the other hand, it is known that I. Gelfand and D. Raikov [3] established the natural one-to-one correspondence between positive difinite functions and unitary representations of locally compact group, that is, for a given continuous positive definite function $f(g)$ there exists a unitary (strongly continuous) representation $U(g)$ with $f(g) = (xU(g), x)$ where x is a suitable element of the representation space. (Although Gelfand-Raikov's paper is not available to the authors, this result is reproduced in R. Godement [4] and H. Yosizawa [9]). Hence, putting $x(g) = x(1)U(g)$, $x(g)$ becomes a stationary function on G, and so Fan's theorem is generalized onto locally compact groups as follows: For any positive definite function $f(g)$ on G there is a stationary function $x(g)$ on G with $f(gh^{-1}) = (x(g), x(h))$.

It seems, therefore, Fan's method of the proof is applicable to this generalization. As it is seen in the below, this is done in § 2, with a few modification. Moreover, it generalized Gelfand-Raikov's Theorem without local compactness (Theorem 1). However, it is not so unexpected. Let G be a topological group (with or without local compactness) and $f(g)$ be a continuous positive definite function on G. Considering G as a discrete group,

$f(g)$ gives an algebraic (i. e. not necessarily continuous) unitary representation $U(g)$ of G. Therefore, if the strong continuity of $U(g)$ is deducible from the continuity of $f(g)$, then this gives the desired generalization. The proof of Fan (in a generalized form) is actually like that, and continuity problem is solved by virtue of Riesz-Kakutani's lemma, which states that a directed set of elements of a uniformly convex Banach space converges strongly if and only if it converges weakly and its norm converges. (Cf. Kakutani [6]). Thus our tactics yield essentially nothing new.

Comparing Fan's and Gelfand-Raikov's theorems, it naturally arises the following question: Is it possible to find a unitary representation $U(g)$ of the group for a given stationary function $x(g)$ such as $x(g) = x(1)U(g)$? (The converse is obvious). In the below, it is solved that the problem is possible assuming the separability of the group (Theorem 2). This is fairly done following the proof of von Neumann-Schoenberg [8], which is used in the research of screw functions in the metric geometry. (This analogy is already pointed out by K. Fan [2]). § 3 consists of the proof. From this representation, Khintchine's Theorem [7] becomes a corollary of the mean ergodic theorem of J. von Neumann.

In § 4, Fourier series of a stationary function which is representable by unitary operators, is analyzed for a (not necessarily locally compact but) commutative group. Some theorems, which are obtained by K. Fan [2] to the case of the additive group of integers, are derived by the help of the general ergodic theorem due to Alaoglu-Birkhoff [1] (Theorem 3, 4); for example, (1) the existence of the mean for a stationary function (for arbitrary G), (2) "Fourier coefficients" of such functions depends on group characters and mutually orthogonal for distinct indices, (3) "Fourier series" converges absolutely and its sum is also stationary.

2. A Generalization of Gelfand-Raikov's Theorem. We begin by proving, following the line of K. Fan [2] with a few modification, the next

THEOREM 1. *For any positive definite continuous function $f(g)$ on G there exists a unitary representation $U(g)$ of G such that*

$$f(g) = (xU(g), x). \tag{1}$$

Proof of the theorem requires some steps of lemmas:

LEMMA 1. *Let $f(g, h)$ $((g, h) \in G \times G)$ be a complex-valued function such that $f(g, h) = \overline{f(h, g)}$. In order that there exists a Hilbert space valued function $x(g)$ such that $f(g, h)$ is equal to the inner product $(x(g), x(h))$:*

$$f(g, h) = (x(g), \ x(h)), \tag{2}$$

it is necessary and sufficient that the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(g_i, \ h_j) \ \alpha_i \ \overline{\alpha_j} \geqq 0 \tag{3}$$

holds for any finite system of complex numbers α_i and any system of elements

g_i, h_j $(1 \leqq i,\ j \leqq n)$ of G.

PROOF. Since the necessity of the condition is evident, we shall prove the sufficiency. Let L be the set of all functions defined on G, which does not vanish only on a finite subset of G. For any function $f'(g)$ of L, the norm $\|f'\|$ will be defined by

$$(4) \qquad \|f'\|^2 = \sum_g \sum_h f(g,\ h)\, f'(g)\overline{f'(h)}$$

which is always real, non-negative in virtue of (3). That this norm has properties of usual norm is easily verified, but $\|f'\| = 0$ does not implies $f' = 0$. Moreover, it is easy to verify that in L the identity

$$(5) \qquad \|f' + f''\|^2 + \|f' - f''\|^2 = 2\|f'\|^2 + 2\|f''\|^2$$

holds. Therefore, by a theorem due to P. Jordan and J. von Neumann [3] L is considered as a subspace of the Hilbert space H, by making a suitable quotient space, with inner product

$$(6) \qquad (f', f'') = \frac{1}{4}(\|f' + f''\|^2 - \|f' - f''\|^2 + i(\|f' + if''\|^2 - \|f' - if''\|^2)).$$

On the other hand, if we define $x(g)$ such that

$$(7) \qquad x(g)\,(h) = \begin{cases} 1, & \text{for } h = g; \\ 0, & \text{for } h \neq g, \end{cases}$$

then we find, by (6), $(x(g),\ x(h)) = f(g, h)$.

LEMMA 2. *A continuous positive definite function $f(g)$ defined on G can be represented by a continuous stationary function on a suitable Hilbert space H as follows:*

$$(8) \qquad f(gh^{-1}) = (x(g),\ x(h)).$$

PROOF. It is sufficient to show that the function $x(g)$, constructed in Lemma 1, is strongly continuous and stationary. The stationarity is followed from the identity

$$(9) \qquad (x(g),\ x(h)) = f(gh^{-1}) = f(gh^{-1}\cdot 1) = (x(gh^{-1}), x(1)),$$

and the strong continuity is verified by the following two relations and Riesz-Kakutani's lemma [6]; if g_α converges to g, then

$$(10) \qquad (x(g_\alpha),\ x(1)) = f(g_\alpha) \longrightarrow f(g) = (x(g),\ x(1)),$$

$$(11) \qquad \|x(g_\alpha)\|^2 = f(1) = \|x(g)\|^2.$$

LEMMA 3. *Let $U(g)$ be an operator on H constructed in Lemma 1 which is induced by the operator $U'(g)$ on L to L:*

$$(12) \qquad U'(g):\ (f'U'(g))(h) = f'(hg),$$

then the mapping $g \to U(g)$ is a unitary representation of G.

PROOF. It is sufficient to show that the operator $U'(g)$ is an isometric operator on L.

By (6), the inner product of $f'(h)$ and $f''(h)$ of L is

$$(f', f'') = \sum_{g, h} f(gh^{-1})f'(g)\ \overline{f''(h)}.$$

The fact that $U'(g)$ is an isometric operator on L is easily seen from the following equalities,

$$\begin{aligned}(f'U'(g), f''U'(g)) &= \sum_{h, k} f(hk^{-1})f'(hg)\overline{f''(kg)} \\ &= \sum_{h', k'} f(h'g^{-1}gk'^{-1})f'(h')\overline{f''(k')} \\ &= \sum_{h', k'} f(h'k'^{-1})f'(h')\overline{f''(k')} \\ &= (f', f'').\end{aligned}$$

By using this unitary representation $U(g)$, $x(g) = x(1)U(g)$ holds, therefore

$$f(g) = (x(g), x(1)) = (x(1)U(g), x(1)).$$

This completes the proof of the theorem.

COROLLARY. *If f and f' are continuous p.d. functions on G, then*

(13) $$|f(g)| \leqq f(1),$$

and the pointwise product of f and f' is also p.d.

PROOF. Inequality (13) is easily verified by using the Schwarz inequality for $f(g) = (x(1)U(g), x(1))$. To prove the last part of the corollary, it is sufficient to show that

(14) $$\sum_i \sum_j ff'(g_i g_j^{-1})\, \alpha_i \overline{\alpha_j} \geqq 0$$

for any finite system of $g_i \in G$ and complex numbers α_i.

Let $x(g)$ be a stationary function, by which $f(g)$ is represented: $f(gh^{-1}) = (x(g), x(h))$, and $H'(\subseteqq H)$ be a closed linear manifold spanned by the set $x(g_1), \cdots, x(g_n)$. If the set $\{e_k; k = 1, \cdots, p\}$, $(p \leqq n)$ is a base of H', then

$$x(g_i) = \sum_{k=1}^{p} \lambda_{i,k} e_k \qquad (i = 1, \cdots, n)$$

and

$$f(g_i g_j^{-1}) = (x(g_i), x(g_j)) = \left(\sum_{k=1}^{p} \lambda_{i,k} e_k, \sum_{k=1}^{p} \lambda_{j,k} e_k\right) = \sum_{k=1}^{p} \lambda_{i,k} \overline{\lambda_{j,k}}.$$

Therefore,

$$\sum_i \sum_j f(g_i g_j^{-1})\, f'(g_i g_j^{-1})\, \alpha_i \overline{\alpha_j} = \sum_{k=1}^{p} \left(\sum_{i,j} f'(g_i g_j^{-1})\, (\lambda_{i,k} \alpha_i)\, (\overline{\lambda_{j,k} \alpha_j}) \right) \geqq 0$$

by the positive definiteness of f'.

3. The Representation of a Stationary Function. Firstly, we shall prove the following

THEOREM 2. *In a separable group, any continuous stationary function*

$x(g)$ is representable by a unitary representation $U(g)$ of G:

$$x(g) = x(1)U(g). \tag{15}$$

PROOF. Let H_1 be the closed linear subset of H, which is spanned by all $\{x(g);\ g \in G\}$.

Write a dense subset of G as a sequence, $g_1 = 1,\ g_2,\ g_3, \cdots,$ and then orthogonalize the sequence

$$x(g_1),\ x(g_2), \cdots,\ x(g_n), \cdots$$

by the Gram-E.Schmidt procedure, thus we obtain the normalized orthogonal set

$$u_1,\ u_2, \cdots,\ u_n, \cdots .$$

Thus

$$u_i = \sum_{j=1}^{i} \alpha_{ij}\, x(g_j), \tag{16}$$

where all α_{ij} are complex scalars. The $x(g_i)$; $i = 1,\ 2, \cdots,$ are linear aggregates of the u_i's, hence (owing to $x(g)$'s continuity in g) all $x(g)$, $g \in G$, are limit points of such linear aggregates. Therefore all $x(g)$ belong to the closed linear set which is spanned by $u_1, u_2, \cdots,$ and therefore coincides with H_1. Hence

$$x(g) = \sum_i a_i(g)\, u_i \tag{17}$$

where $a_i(g)$ are complex continuous functions of g.

Combining (16) and (17) we get

$$x(g) = \sum_i a_i(g) \left\{ \sum_{j=1}^{i} \alpha_{ij} x(g_j) \right\}. \tag{18}$$

Consider the equation

$$x(gh) = \sum_i a_i(g) \left\{ \sum_{j=1}^{i} \alpha_{ij} x(g_j h) \right\}. \tag{19}$$

(19) can be written as a relation of the

$$(x(g'h),\ x(g''h)) \qquad \text{for } g',\ g'' = g,\ g_1,\ g_2, \cdots,$$

and the

$$a_i(g), \qquad \alpha_{ij}.$$

By the stationarity of $x(g)$, this means a relation of the

$$(x(g'h),\ x(g''h)) = f(g'hh^{-1}g''^{-1}) = f(g'g''^{-1})$$

and the

$$a_i(g), \qquad \alpha_{ij}.$$

Hence (19) is independent of h. But for $h = 1$ (19) coincides with (18), and hence is true. Therefore they hold for all h.

Let

$$u_i(h) = \sum_{j=1}^{i} \alpha_{ij} x(g_j h). \tag{20}$$

Then (19) gives

(21) $$x(gh) = \sum_i a_i(g)\, u_i(h).$$

By the same method as the proof of (19), we can prove

(22) $u_1(h),\ u_2(h),\ \cdots$ is a normalized orthogonal set for all h.

Consider a fixed h. The $u_1(h),\ u_2(h),\ \cdots$ span the same closed, linear set as the $x(g_1h),\ x(g_2h),\ \cdots$. Owing to the continuity of $x(g)$ in g, this is the same set as spanned by the $x(gh)$, $g \in G$, or, if we write g for gh, by the $x(g)$, $g \in G$. In other words:

(23) $u_1(h),\ u_2(h),\ \cdots$ span the closed linear set H_1 for all h.

By (22), (23) the equations

(24) $$u_iU(h) = u_i(h), \qquad \text{for } i = 1, 2, \cdots,$$

define a unitary transformation $U(g)$ in H_1. Then that the representation $g \to U(g)$ is a desired one follows from (18) and (21).

COROLLARY (Khintchine). *If $x(n)$ is a stationary sequence on the additive group of integers, then its arithmetic mean*

$$y(n) = (2n+1)^{-1} \sum_{p=-n}^{n} x(p)$$

converges strongly, that is,

$$\lim_{n, m \to \infty} \|y(n) - y(m)\|^2 = 0.$$

PROOF. Since by Theorem 2 the stationary sequence $x(n)$ is of form $x(0)U^n$, this is an immediate consequence of the celebrated mean ergodic theorem of J. von Neumann.

4. Fourier Series of a Stationary Function. In the following we shall consider a continuous stationary function $x(g)$ on G with range in H, which is representable by a unitary representation $U(g)$ of G: $x(g) = x(1)U(g)$. By Theorem 2, this class includes all stationary functions on separable groups.

For the following discussion, we recall the following well-known Lemma due to L. Alaoglu and G. Birkhoff [1]:

LEMMA 4. *Let H be a Hilbert space and $\mathfrak{U} = \{U\}$ be a group of unitary operators on H; let F be a closed linear manifold of H spanned by the set $\{x;\ xU = x \text{ for all } U \in \mathfrak{U}\}$. Then for any $x \in H$, the smallest closed convex set K_x which contains $\{xU;\ U \in \mathfrak{U}\}$ meets with F by a unique point x_0 which is same time (a) the projection of x in F, (b) the point which has the smallest norm in K_x.*

By the use of the preceding Lemma 4 for a given representable stationary function $x(g)$, there exists a projection P on H such that $PU(g) = U(g)P = P$, $x(g) = x(1)U(g)$ for all $g \in G$. Define the *mean* by

$$\int x(g)\,dg = x(g)P = x(1)U(g)P = x(1)P, \tag{25}$$

then we have:

LEMMA 5. *For a representable stationary function* $x(g)$,

$$\left(\int x(g)\,dg,\ x(h)\right) = \left(x(h),\ \int x(g)\,dg\right) = \left\|\int x(g)\,dg\right\|^2$$

for all h *of* G.

PROOF. $\left(\int x(g)\,dg,\ x(h)\right) = (x(1)P,\ x(1)U(h)) = (x(1),\ x(1)U(h)P)$

$$= (x(1),\ x(1)P) = \|x(1)P\|^2 = \left\|\int x(g)\,dg\right\|^2.$$

THEOREM 3. *Let* G *be abelian,* $x(g)$ *be a representable stationary function:* $x(g) = x(1)U(g)$, *and* $\chi(g)$ *be a continuous character of* G. *Then*

$$x(\chi) = \int \overline{\chi(g)}\,x(1)U(g)\,dg = x(1)P_\chi$$

defines a projection P_χ *in* H *and* $P_\chi \cdot P_{\chi'} = 0$ *for any different characters* χ, χ' *of* G.

PROOF. The existence of $x(\chi)$ is certified by the fact that $y(g) = \overline{\chi(g)}x(1)U(g)$ is also a representable stationary function. Since we can assume that H is spanned by $\{x(g);\ g \in G\}$ without loss of generality, it is sufficient to show that $x(1)P_\chi P_{\chi'} = 0$.

Now,

$$\begin{aligned} x(1)P_\chi P_{\chi'} &= \int \overline{\chi(g)}x(1)\,P_{\chi'}U(g)\,dg \\ &= \int (\overline{\chi(g)}/\overline{\chi'(g)})\,\overline{\chi'(g)}\,x(1)\,P_{\chi'}U(g)dg \\ &= \int \overline{\chi''(g)}\,y(g)dg, \end{aligned}$$

where $\chi''(g) = \chi(g)/\overline{\chi'(g)}$ is a character, and $y(g) = \overline{\chi'(g)}\,x(1)P_{\chi'}U(g)$. Since $P_{\chi'}\overline{\chi'(g)}\,U(g) = P_{\chi'}$ by the mean ergodic theorem of Alaoglu-Birkhoff, $y(g) = x(1)P_{\chi'}$, that is, it is constant on G. Clearly, $x \to \chi(g)x$ is a unitary transformation, whence Lemma 4 is also applicable. On the other hand, if $\chi(g)$ is non-trivial, K_x contains the origin as a fix-point, that is, the last term of the above equality vanishes. This proves the theorem.

By the preceding theorem, $\{P_\chi;\ \chi,\ \text{character}\}$ is a system of mutually orthogonal projections on H, therefore

$$\sum_\chi \|x(1)P_\chi\|^2 \leqq \|x(1)\|^2,$$

and

$$\sum_{\chi} \chi(g)\, x(1)\, P_\chi$$

converges absolutely (with respect to g) to $y(g)$. Now, we may define the *Fourier series* of a stationary function $x(g)$ such that

$$x(g) \sim \sum_{\chi} \chi(g)\, x(1)\, P_\chi .$$

THEOREM 4. *In an abelian group G, consider a representable stationary function $x(g)$ and its Fourier series*

$$y(g) = \sum_{\chi} \chi(g)\, x(1)\, P_\chi .$$

Then, $y(g)$ and $z(g) = x(g) - y(g)$ are also stationary, and the Fourier series of $z(g)$ does vanish.

PROOF. Except the stationarity of $z(g)$, all other statements are easily verified. While about $z(g)$;

$$(z(g),\ z(h)) = (x(g) - y(g),\ x(h) - y(h))$$

$$= \Big(x(g) - \sum_{\chi} \chi(g)\, x(1) P_\chi,\ x(h) - \sum_{\chi} \chi(h)\, x(1)\, P_\chi\Big)$$

$$= (x(g),\ x(h)) - \sum_{\chi} (\chi(g) x(1) P_\chi, x(h)) - \sum_{\chi} (x(g),\ \chi(h) x(1) P_\chi) + (y(g),\ (h)).$$

In the right hand side, the first and the fourth term have the stationary property, and about the second term,

$$\sum_{\chi} (\chi(g)\, x(1)\, P_\chi,\ x(h)) = \sum_{\chi} \chi(g)\, (x(1) P_\chi,\ x(1)\, U(h))$$

$$= \sum_{\chi} \chi(g)\, \overline{\chi(h)}\, (x(1)\, P_\chi,\ \overline{\chi(h)}\, x(1)\, U(h))$$

$$= \sum_{\chi} \chi(gh^{-1}) \Big(\int \overline{\chi(h)}\, x(1)\, U(h)\, dh,\ \overline{\chi(h)}\, x(1)\ U(1)\Big)$$

$$= \sum_{\chi} \chi(gh^{-1}) \Big\| \int \chi(h)\, x(1)\, U(h)\, dh \Big\|^2. \qquad \text{(By Lemma 5).}$$

Since the third term can be discussed similarly, this completes the proof.

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