ABSOLUTE REGULARITY FOR CONVERGENT INTEGRALS

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The necessary and sufficient condition of absolute regularity for any sequence-to-sequence transformation was given by Knopp-Lorentz [2] and one of the present authors [4] independently. On the other hand, for any function-to-function transformation Knopp-Lorentz stated sufficient conditions for absolute regularity, but they did not prove the necessity. The object of this note is to prove this.

THEOREM 1. *In order that for any* $a(t) \in L(0, \infty)$ *the transformation*

$$\alpha(x) = \int_0^\infty b(x, t)\, a(t)\, dt$$

is defined and $b(x) \in L(0, \infty)$, *it is necessary and sufficient that*

$$\operatorname*{ess.\,sup}_{0 \leqq t < \infty} \int_0^\infty |b(x,t)|\, dx \leqq M,$$

where M *is an absolute constant.*

THE FIRST PROOF. The method of this proof is analogous to the previous paper of Sunouchi [4]. We prove only the necessity, since the sufficiency is evident.

The transformation

$$\int_0^\infty b(x, t) a(t) dt$$

is an additive and homogeneous operation from $L(0, \infty)$ into itself. Put

$$U(a) = \int_0^\infty b(x, t)\, a(t)\, dt$$

and

$$U(a) = p(a),$$

where the generic elements $a(\cdot)$ and $U(\cdot) \in L(0, \infty)$ and the norm is in the L-sense.

Then, since

$$p(a) = \int_0^\infty \left| \int_0^\infty b(x, t)\, a(t)\, dt \right| dx,$$

we get

$$\liminf_{\|a_n-a\|_L\to 0}\int_0^\infty\left|\int_0^\infty b(x,t)\,a_n(t)\,dt\right|dx$$

$$\geqq\int_0^\infty \liminf_{\|a_n-a\|_L\to 0}\left|\int_0^\infty b(x,t)\,a_n(t)\,dt\right|dx$$

$$\geqq\int_0^\infty\left|\int_0^\infty b(x,t)\,a(t)\,dt\right|dx,$$

by Fatou's lemma. That is, $p(x)$ is lower semi-continuous, so $p(x)$ is continuous from Gelfand's lemma [1]. Thus $U(a)$ is a linear bounded transformation from $L(0,\infty)$ into $L(0,\infty)$.

On the other hand the most general linear bounded transformation of $L(0,\infty)$ into itself is well known, for example see Phillips [3].

His general form is

$$U(a)=(\mathfrak{P})\int_0^\infty a(t)d\dot{x}$$

where $(\mathfrak{P})$ is the integral of Phillips' sense and

$$\dot{x}\in V^\infty(x),$$

$V^\infty(x)$ is the class of $L(0,\infty)$-valued abstract additive set functions $x(\tau)$, and

$$V^\infty(x)\equiv[x(\tau)\ |\ \|x(\tau)\|\leqq M\ |\tau|,\quad |\tau|<\infty].$$

So, in our case we can write

$$U(a)=\int_0^\infty b(x,t)a(t)\,dt=\int a(t)\,d_\tau B(x,\tau),$$

where

$$B(x,\tau)=\int_\tau b(x,t)\,dt$$

for any measurable set τ.

Consequently, if we denote by $\mathfrak{M}$ the class of all measurable sets with finite measure, then we have

$$\operatorname*{l.u.b.}_{\tau\in\mathfrak{M}}\frac{\|x(\tau)\|}{|\tau|}=\operatorname*{l.u.b.}_{\tau\in\mathfrak{M}}\frac{\int_0^\infty|B(x,\tau)|\,dx}{|\tau|}\leqq M,$$

that is,

$$\operatorname*{l.u.b.}_{\tau\in\mathfrak{M}}\frac{1}{|\tau|}\int_0^\infty\left|\int_\tau b(x,t)\,dt\right|dx\leqq M.$$

Letting $|\tau|\to 0$, we have

$$\frac{1}{|\tau|}\int_\tau b(x,t)\,dt \to b(x,t) \qquad \text{p.p. in } t,$$

and so

$$\text{l.u.b.} \int_0^\infty dx\,|\,b(x,t)\,| \leqq M, \qquad \text{p.p. in } t.$$

THE SECOND PROOF OF NECESSITY. The continuity of the functional $p(a)$ is deduced as in the preceding proof. Then we can find two positive numbers δ and N such that $p(a) < N$ for every $a(t)$, if

$$\int_0^\infty |\,a(t)\,|\,dt < \delta$$

is satisfied. For any fixed x, let E_x be the Lebesgue set in t of the function $b(x,t)$, clearly its complement CE_x is a null set. Hence, by the Fubini theorem, almost all t are the Lebesgue points of $b(x,t)$ in t for almost all x. We denote such set of t-points by T, then $|\,CT\,| = 0$. For any $\tau \in T$, put

$$\begin{aligned} a_h(t) &= \delta/h && \text{if } t \in (\tau, \tau+h), \\ &= 0 && \text{otherwise.} \end{aligned}$$

Since $\int_0^\infty |\,a_h(t)\,|\,dt \leqq \delta$, we get by the above fact

$$N > p(a_h(t)) = \int_0^\infty \left| \frac{\delta}{h}\int_\tau^{\tau+h} b(x,t)\,dt \right| dx$$

or

$$\frac{N}{\delta} > \int_0^\infty \left| \frac{1}{h}\int_\tau^{\tau+h} b(x,t)\,dt \right| dx.$$

Let $h \to 0$ and take the lower limit in each side, then we have

$$\frac{N}{\delta} > \int_0^\infty \liminf_{h\to 0} \left| \frac{1}{h}\int_\tau^{\tau+h} b(x,t)\,dt \right| dx = \int_0^\infty |\,b(x,\tau)\,|\,dx,$$

As $|\,CT\,| = 0$, this proves the necessity with $M = N/\delta$.

THEOREM 2. *In order that for any $s(t) \in BV(0,\infty)$, the transformation*

$$\alpha(x) = \int_0^\infty b(x,t)ds(t)$$

is defined and $\alpha(x) \in L(0,\infty)$, it is necessary and sufficient that

$$\operatorname*{ess.\,sup}_{0\leqq t<\infty} \int_0^\infty |\,b(x,t)\,|\,dx \leqq M.$$

PROOF. If we fix an x, then $b(x,t)$ is continuous for t. Especially we assume that $s(t)$ is absolutely continuous and its derivative is denoted by $a(t)$, then

$$\alpha(x) = \int_0^\infty b(x,t)\,ds(t) = \int_0^\infty b(x,t)\,a(t)\,dt.$$

So, by Theorem 1, we get

$$\operatorname*{ess.\,sup}_{0\leqq t<\infty} \int_0^\infty |\, b(x,t)\,|\, dx \leqq M.$$

The sufficiency is evident.

LITERATURE

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