REMARKS ON PREDICTION PROBLEM IN THE THEORY OF STATIONARY STOCHASTIC PROCESSES

TATSUO KAWATA[1)]

(Received February 11, 1954; in Revised form March 18, 1954)

1. Suppose that $X(t)$ is a continuous stationary process in wide sense, $E\{X(t)\} = 0, E\{|X(t)|^2\} < \infty$ and $\rho(u)$ is the correlation function $E\{X(t+u)\overline{X(t)}\}$ which is represented as

$$(1.1) \qquad \rho(u) = \int_{-\infty}^{\infty} e^{iux} dF(x),$$

$F(x)$ being a bounded, non-decreasing function.

In previous papers [1], [2], we have discussed about Wiener's prediction theory. The object of the present paper is to give some remarks on prediction problem in the case where $F(x)$ satisfies a further condition that

$$(1.2) \qquad \int_{-\infty}^{\infty} x^{2p}\, dF(x) < \infty,$$

p being a positive integer.

We shall first give some definitions, notations and some known results.

Let $K(\theta)$ be a function of bounded variation in every finite interval in $[0, \infty)$. If $\int_0^A e^{-ix\theta}\, dK(\theta)$ converges in $L_2\,(-\infty, \infty)$ with respect to $F(x)$ to a function $k(x)$ when $A \to \infty$, $K(\theta)$ is called to belong to $\mathbf{K}(0, \infty)$. That is, if

$$(1.3) \qquad \lim_{A\to\infty} \int_{-\infty}^{\infty} \left| \int_0^A e^{-ix\theta}\, dK(\theta) - k(x) \right|^2 dF(x) = 0,$$

then $K(\theta) \in \mathbf{K}(0, \infty)$ and this fact is denoted as

$$(1.4) \qquad \underset{A\to\infty}{\text{l.i.m.}}\ L_2(F) \int_0^A e^{ix\theta} dK(\theta) = k(x),$$

and $k(x)$ is called the Fourier-Stieltjes transform of $K(\theta)$ in $L_2(F)$.

It is known[3] that if $K(\theta) \in \mathbf{K}(0, \infty)$, then

$$(1.5) \qquad \underset{A\to\infty}{\text{l.i.m.}} \int_0^A X(t-\theta) dK(\theta)$$

exists. l.i.m. means the limit in variance. (1.5) is denoted as

---

[1)] This paper was written sponcered by Japanese Union of Scientists and Engineers.

(1.6) $$\mathfrak{F}_K[X(t)].$$

Next let $\{k_n(x)\}$ be a sequence of Fourier-Stieltjes transform of functions of $\mathbf{K}(0, \infty)$. If $k(x) \in L_2(F)$ is such that

$$\underset{n\to\infty}{\text{l.i.m.}}\ L_2(F)\cdot k_n(x) = k(x),$$

then $k(x)$ is called to belong to the class $\mathfrak{K}_F$[2]. And it has been shown that $\mathfrak{F}_{K_n}[X(t)]$ converges in mean (in variance) to a stationary process. This process is denoted as $\mathfrak{F}[X(t), k(\cdot)]$.

2. On ordinary Fourier transforms. Let $f(x) \in L_1(-\infty, \infty)$ and its Fourier transform be

$$F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{itx} dx. \tag{2.1}$$

It is well known that if, further, $xf(x) \in L_1(-\infty, \infty)$ then $F(t)$ is differentiable and

$$F'(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) ix e^{itx} dx.$$

Connecting this we shall prove:

LEMMA 1. *Let $ixf(x) \in L_2(-\infty, \infty)$ and its Fourier transform be $G(t)$. If $f(x) \in L_2(-\infty, \infty)$, then $\frac{1}{n} \Delta_h F(t) = \frac{F(t+h) - F(t)}{h}$ converges in L_2 to $G(t)$.*

Since $\frac{1}{h} \Delta_h F(t)$ is the Fourier transform of $f(x) \frac{e^{ixh}-1}{h}$, by Parseval relation we have

$$\begin{aligned} J &= \int_{-\infty}^{\infty} |\Delta_h F(t) - G(t)|^2 dt \\ &= \int_{-\infty}^{\infty} \left| f(x) \frac{e^{ixh}-1}{h} - ixf(x) \right|^2 dx, \end{aligned} \tag{2.2}$$

which tends to zero as $h \to 0$, for $|(e^{ixh}-1)/h|^2 \leqq x^2$.

REMARK. If $ixf(x) \in L_2(-\infty, \infty)$, then $f(x) \in L_1$ in the vicinity of infinity. Hence further if $f(x) \in L_2(-\infty, \infty)$, $f_1(x) \in L_1(-\infty, \infty)$ and the Fourier transform $F(t)$ is continuous.

The following lemma is immediate from Lemma 1.

LEMMA 2. *If $F(t) = 0$, for $t < 0$, then $G(t) = 0$ almost everywhere for $t < 0$.*

For $\Delta_h F(t) = 0$ for $t < -h$, if $h > 0$, and if $h_1 > h$, then

$$\lim_{h\to 0} \int_{-\infty}^{-h_1} \left| \frac{1}{h} \Delta_h F(t) - G(t) \right|^2 dt = 0,$$

whence

---

2) It is evident that if we have only to define $\mathfrak{K}_F$, it suffices to take more special class instead of K.

$$\int_{-\infty}^{-h_1} |G(t)|^2 dt = \lim_{h\to 0} \int_{-\infty}^{-h_1} \left| \frac{1}{h} \Delta_h F(t) \right|^2 dt = 0.$$

Hence $G(t) = 0$ almost everywhere in $t < -h_1$. Since h_1 is arbitrary positive number, $G(t) = 0$ almost everywhere in $t < 0$.

LEMMA 3. *Under the assumptions of Lemma* 1,

$$F(t) - F(0) = \int_0^t G(u)du$$

By Lemma 1, $\frac{1}{h}\Delta_h F(t)$ converges to $G(t)$ in L_2. Hence by weak convergence

$$\lim_{h\to 0} \int_0^t \frac{1}{h} \Delta_h F(u)du = \int_0^t G(u)du$$

But

$$\frac{1}{h}\int_0^t \Delta_h F(u)du = \frac{1}{h}\left(\int_0^t F(u+h)du - \int_0^t F(u)\, du\right)$$

(2.3)

$$= \frac{1}{h}\int_t^{t+h} F(u)du - \frac{1}{h}\int_0^h F(u)du.$$

Since $F(t)$ is continuous for $f(x) \in L_1$, the right of (2.3) converges to $F(t) - F(0)$.

3. Derivatives of a stationary process. Let $F(x)$ be the spectral function of a continuous stationary process $X(t)$. If

$$\int_{-\infty}^{\infty} x^2 dF(x) < \infty, \tag{3.1}$$

then $X'(t)$ exists in the sense that

$$\operatorname*{l.i.m.}_{h\to 0} \frac{X(t+h) - X(t)}{h} = X'(t).$$

$X'(t)$ is a stationary process and its spectral function is $\int_{-\infty}^{x} x^2 dF(x)$. This is well known[2]. Repeated applications of this fact show immediately that

If

$$\int_{-\infty}^{\infty} x^{2p} dF(x) < \infty, \tag{3.2}$$

p being a positive integer, then

$$X^{(k)}(t) = \operatorname*{l.i.m.}_{h\to 0} \frac{X^{(k-1)}(t+h) - X^{(k-1)}(t)}{h}, \quad (k = 1, 2, \cdots p)$$

exists, the spectral function of this stationary process is $\int_{-\infty}^{x} x^{2k} dF(x)$, *and the correlation function of* $X^{(p)}(t)$ *is* $(-1)^p \rho^{(2p)}(u)$, $\rho(u)$ *being the correlation function of* $X(t)$.

We shall prove that

$$\text{(3.3)} \qquad \underset{h\to 0}{\text{l.i.m.}}\, h^{-p}\left[\sum_{k=0}^{p}\binom{p}{k}(-1)^{p-k}X(t+kh)\right] = X^{(p)}(t)$$

under the condition (3.2).

Let this statement holds for any stationary process with the condition (3.2) for $p = r$. And if it should be proved that (3.3) holds for $p = r+1$ under (3.2) with $p = r+1$, then our statement holds generally by induction. Hence it is sufficient to show that

$$\text{(3.4)} \qquad \underset{h\to 0}{\text{l.i.m.}}\, \left\{h^{-(r+1)}\Delta_h{}^{(r+1)}X(t) - h^{-r}\Delta_h^{(r)} X'(t)\right\} = 0,$$

where

$$\Delta_h X(t) = \Delta_h^{(1)}X(t) = X(t+h) - X(t),$$
$$\Delta_h{}^{(r+1)}X(t) = \Delta_h\Delta_h{}^{(r)}X(t).$$

For $X'(t)$ is a stationary process whose spectral function is $\int_{-\infty}^{x} x^2 dF(x) = F_1(x)$, and

$$\int_{-\infty}^{\infty} x^{2r}\, dF_1(x) = \int_{-\infty}^{\infty} x^{2(r+1)} dF(x) < \infty.$$

Now we can easily prove that, if $Z(t)$ is a stationary process, with $\int_{-\infty}^{\infty} x^2 dF_Z(x) < \infty$, $F_Z(x)$ being the spectral function of Z, then

$$\text{(3.5)} \qquad E\left\{\left|\frac{1}{h}\Delta_h Z(t)\right|^2\right\} = -\frac{1}{h^2}\{\varphi(h) - 2\varphi(0) + \varphi(-h)\}$$
$$= -\frac{1}{h^2}\Delta_h^{(2)}\varphi(-h),$$

where φ is the correlation function of $Z(t)$,

$$\text{(3.6)} \qquad E\{|Z(t)|^2\} = \lim_{h\to 0} E\left\{\left|\frac{1}{h}\Delta_h Z(t)\right|^2\right\} = -\varphi''(0),$$

$$\text{(3.7)} \qquad E\left\{\frac{1}{h}\Delta_h Z(t)\cdot\overline{Z(t)}\right\} = \lim_{\varepsilon\to 0} E\left\{\frac{1}{h}\Delta_h Z(\cdot t)\,\frac{1}{\varepsilon}\Delta_\varepsilon\overline{Z(t)}\right\} \cdot \frac{1}{h}\{\varphi'(h) - \varphi'(0)\}$$

and

$$\text{(3.8)} \qquad E\left\{\Delta_h Z(t+u)\Delta\overline{Z_h(t)}\right\} = \Delta_h^{(2)}\varphi(u-h).$$

Under these preliminaries, we shall prove (3.4). Since the finite linear combination of $X(t+d_i)$ is also a stationary process, we can take $\Delta_h^{(k)}X(t)$ for $Z(t)$ above. And we have, by (3.5) and (3.7)

$$E\left\{|h^{-1}\Delta_h^{(1)}X(t)|^2\right\} = -\frac{1}{h^2}\Delta_h^{(2)}\rho(-h)$$

$$E\left\{|h^{-2}\Delta_h^{(2)}X(t)|^2\right\} = \frac{1}{h^4}\Delta_h^{(2)}\Delta_h^{(2)}\rho(-2h) = \frac{1}{h^4}\Delta_h^{(4)}\rho(-2h)$$

and at last

$$F\{|h^{-r}\Delta_h^{(r)}X(t)|^2\} = (-1)^r h^{-2r}\Delta_h^{(2r)}\rho(-rh). \tag{3.9}$$

Moreover

$$E\{h^{-r}\Delta_h^{(r)}X(t+u)\overline{h^{-r}\Delta_h^{(r)}X(t)}\} = (-1)^r h^{-2r}\Delta_h^{(2r)}\rho(u-rh). \tag{3.10}$$

And

$$E\{|^{-(r+1)}\Delta_h{}^{(r+1)}X(t) - h^{-r}\Delta_h^{(r)}X'(t)|^2\}$$
$$= F\{h^{-2r}[h^{-1}\Delta_h\cdot\Delta_h^{(r)}X(t) - \{\Delta_h^{(r)}X(t)\}']^2\}$$

which by taking $\Delta_h^{(r)}X(t)$ for $Z(t)$ again, applying (3.5) (3.6) and (3.7), and using (3.9) we can write as

$$(-1)^{r+1}h^{-2(r+1)}\Delta_i^{(2(r+1))}\rho(-(r+1)h) \quad -(-1)^r h^{-2r}\Delta_h^{(2r)}\rho''(-rh)$$
$$+\frac{1}{h}(-1)^r h^{-2r}\Delta_h^{(2r)}\rho'(-(r-1)h) + (-1)^r h^{-2r}\Delta_h^{(2r)}\rho'(-(r+1)h)$$
$$-2(-1)^r h^{-2r}\Delta_h^{(2r)}\rho'(-rh)\}.$$

By letting $h\to 0$, it is easily verified that the limit is

$$(-1)^{r+1}\rho^{2(r+1)}(0) + (-1)^{r+1}\rho^{(2r+1)}(0) + (-1)^r\rho^{(2r+1)}(0) + \rho^{(2r+1)}(0) = 0.$$

Thus we have proved (3.3).

4. A differential operator. In this section we also assume that the spectral function $F(x)$ of a continuous stationary process $X(t)$ satisfies

$$\int_{-\infty}^{\infty} x^{2p}dF(x) < \infty, \tag{4.1}$$

p a positive integer. We shall prove that $X^{(p)}(t)$ *can be expressed as* $\mathfrak{F}[X(t), k(\cdot)]$ *for some* $k(x)\in\mathfrak{K}_F$.

Let the function $K_n(\theta)$ of bounded variation be defined as

$$\begin{aligned} K_n(\theta) &= 0 \text{ at } \theta = 0 \\ &= (-n)^p\sum_{k=0}^{j}\binom{p}{k}(-1)^{p-k}, \text{ for } \frac{j}{n}<\theta\leqq\frac{j+1}{n}, \\ &\qquad\qquad j=0,1,\ldots,p-1, \\ &= (-1)^p\sum_{k=0}^{p}\binom{p}{k}(-1)^{p-k}, \text{ for } \frac{p}{n}<\theta<\infty. \end{aligned} \tag{4.2}$$

Then

$$\mathfrak{F}_{K_n}[X(t)] = \int_0^{\infty} X(t-\theta)dK_n(\theta)$$

$$= (-n)^p \sum_{k=0}^{p} \binom{p}{k} (-1)^{p-k} X\left(t - \frac{k}{n}\right).$$

By (3.3),

(4.3) $$\underset{n\to\infty}{\mathrm{l.i.m.}}\ \mathfrak{F}_{K_n}[X(t)] = X^{(p)}(t).$$

The Fourier-Stieltjes transform $L_2(F)$ of $K_n(\theta)$, is $k_n(x)$ which converges to $(ix)^p$. Further we have

$$|k_n(x) - (ix)^p|^2 \leqq 2n^{2p}\left\{\left(1 - \cos\frac{x}{n}\right)^2 + \sin\frac{x}{n}\right\}^p + |x|^{2p} \leqq c|x|^{2p}.$$

And hence

(4.4) $$\underset{n\to\infty}{\mathrm{l.i.m.}}\ L_2(F)\ k_n(x) = (ix)^p.$$

By the fact stated in the last part of §1, we have

(4.5) $$X^{(p)}(t) = \mathfrak{F}[X(t), k(\cdot)]\ ,$$

where

$$k(x) = (ix)^p.$$

5. Optimum prediction operator. Assume through this section that the spectral function $F(x)$ is absolutely continuous,

$$F(x) = \Phi(x)$$

and

(5.1) $$\int_{-\infty}^{\infty} \frac{\log \Phi(x)}{1 + x^2} dx < \infty.$$

Then

(5.2) $$\Phi(x) = |\Psi(x)|^2,$$

where the Fourier transform in ordinary L_2 sense of $\Psi(x)$

$$\psi(t) = \underset{A\to\infty}{\mathrm{l.i.m.}}\ \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \Psi(x) e^{ixt} dx$$

satisfies

(5.3) $$\psi(t) = 0, \qquad t < 0,$$

almost everywhere. We have in a previous paper proved that if

(5.4) $$\frac{1}{\Psi(x)} \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \psi(\alpha + t) e^{-ixt} dt = h(x),\ \ \alpha > 0,$$

(the integral is taken in L_2 sense) is a function of $\mathfrak{K}_F$[3], then $\mathfrak{F}[X(t), h(\cdot)]$ becomes the optimum predictor of $X(t+\alpha)$ when $X(t+\alpha)$ is to be estimated by $\mathfrak{F}[X(t), k(\cdot)]$, $k(x) \in \mathfrak{K}_F$.

It is the object of the present section is to express (5.4) in another form, under the condition that,

(5.5) $$\int_0^{\infty} |x|^{2p} \Phi(x) dx < \infty.$$

[3] $h(x)$ in (5.4) is, in fact, a function of $\mathfrak{K}_F$. This circumstance was investigated by K. Takano, Note on Wiener's prediction theory, Annales of the Institute of Stat. Math., 5 (1954).

Following theorems are given, essentially by N. Wiener [4], but we shall prove in a more rigorous manner.

THEOREM 1. *Let* (5.5) *hold* $p \geqq 1$. *If*

$$r(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{\Psi(x)} \int_0^\infty e^{-itx} dt \int_{-\infty}^\infty \Psi(u) e^{iut} \Big[e^{i\alpha u} - 1 - i\alpha u - \dots - \frac{(i\alpha)^{p-1} u^{p-1}}{(p-1)!} \Big] du \tag{5.6}$$

is of $\mathfrak{R}_F$, *then* $h(x)$ *in* (5.4) *is the optimum predictor and is represented as*

$$h(x) = 1 + ix\alpha + \frac{\alpha^2}{2!}(ix)^2 + \dots + \frac{\alpha^{p-1}}{(p-1)!}(ix)^{p-1} + r(x). \tag{5.7}$$

The outer integral in the right hand side of (5.6) is taken as L_2-sense, and the inner integral is absolutely convergent for $p \geqq 1$.

We consider $\Psi(x)$ in (5.2). Then

$$|x^{2p}\Phi(x)| = |x^p \Psi(x)|^2 \tag{5.8}$$

and by Lemma 2, the Fourier transform of $x^k\Psi(x)$ vanishes for $x < 0$, for $k = 1, 2, \dots, p$, and we have

$$\psi^{(k)}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty (ix)^k \Psi(x) e^{ixt} dx, \quad k = 1, 2, \dots, p-1. \tag{5.9}$$

$(ix)^k\Psi(x) \in L_1$ $(ix)^k\Psi(x)$ also belongs to L_2 and hence we can consider it is the Fourier transform (inverse transform) of $\psi^{(k)}(t)$.

Now put

$$\frac{1}{\Psi(x)} \frac{1}{\sqrt{2\pi}} \int_0^\infty \psi(\alpha + t) e^{-ixt} dt - 1 - i\alpha x - \dots - \frac{\alpha^{p-1}(ix)^{p-1}}{(p-1)!} r(x). \tag{5.10}$$

We have

$$\begin{aligned} r(x) &= \frac{1}{\Psi(x)} \Big[\frac{1}{\sqrt{2\pi}} \int_0^\infty \psi(\alpha + t) e^{-ixt} dt \\ &\qquad - \Psi(x) - \alpha \cdot ix\Psi(x) - \dots - \frac{\alpha^{p-1}}{(p-1)!}(ix)^{p-1}\Psi(x) \Big] \qquad (5.11) \\ &= \frac{1}{\Psi(x)} \frac{1}{\sqrt{2\pi}} \int_0^\infty \Big\{ \psi(t+\alpha) - \psi(t) - \alpha\psi'(t) - \dots \\ &\qquad - \frac{\alpha^{p-1}}{(p-1)!} \psi^{(p-1)}(t) \Big\} e^{-ixt} dt \end{aligned}$$

(the integral being taken as L_2 sense)

$$\begin{aligned} &= \frac{1}{\Psi(x)} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ixt} dt \Big[\int_{-\infty}^\infty \Big\{ \Psi(u) e^{iu\alpha} - \Psi(u) - \alpha iu \Psi(u) \\ &\qquad - \dots - \frac{\alpha^{p-1}}{(p-1)!}(iu)^{p-1} \Big\} e^{iut} du \Big] \end{aligned}$$

which proves (5.6).

If $r(x) \in \mathfrak{R}_F$ then $h(x) \in \mathfrak{R}_F$, because in (5.10) $(ix)^k$ is a function of $\mathfrak{R}_F$ as was shown in §4. Thus our theorem is proved.

THEOREM 2. *$r(x)$ in Theorem 1 can be represented as*

$$(5.12) \qquad r(x) = \frac{1}{\Psi(x)} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ixt} dt \int_0^\alpha du_1 \int_0^{u_1} \cdots \int_0^{u_{p-1}} \psi^{(p)}(t+u_p) du_p,$$

$\psi^{(p)}(t)$ being defined as the limit in mean $h \to 0$ of $\frac{1}{h} \Delta_h \psi^{(p-1)}(t)$.

Clearly we have

$$\psi(t+\alpha) = \psi(t) + \alpha\psi'(t) + \frac{\alpha^2}{2!}\psi''(t) + \cdots + \frac{\alpha^{p-2}}{(p-2)!}\psi^{(p-2)}(t)$$
$$+ \int_0^\alpha du_1 \int_0^{u_1} \cdots \int_0^{u_{p-2}} \psi^{(p-1)}(t+u_{p-1}) du_{p-1},$$

and by Lemma 3

$$\psi^{(p-1)}(t) = \int_0^t \psi^{(p)}(u_p) du_p.$$

These in connection with (5.11), proves the theorem. Theorem 1 can be also stated as

THEOREM 3. *If $h(x) \in \mathfrak{K}_F$, or $r(x) \in \mathfrak{K}_F$, then $X(t+\alpha)$ is best predicted by*

$$(5.31) \qquad X(t) + \alpha X'(t) + \cdots + \frac{\alpha^{p-1}}{(p-1)!} X^{(p-1)}(t) + \mathfrak{F}[X(t), r(\cdot)].$$

In conclusion, I should like to express my hearty thanks to Prof. G. Sunouchi for his kind criticism and valuable suggestions. Lemma 1 was improved and I add some footnotes by his suggestion.

REFERENCES

[1] T. KAWATA, Stationary process and harmonic analysis, Kôdai Math. Sem. Rep. no. 2, 1953.
[2] T. KAWATA, On Wiener's prediction theory, Rep. Stat. Appl. Res. J. U. S. E. 2(1953).
[3] T. KAWATA, loc. cit. [1].
[4] N. WIENER, Extrapolation, interpolation and smoothing of stationary process, New York, 1949,

DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY.