FIBRED RIEMANNIAN SPACES WITH ISOMETRIC PARALLEL FIBRES

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(Received September 20, 1954)

Recently Y. Mutô [5] has shown interesting results about the fibred Riemannian spaces, and A. G. Walker [7] has dealt with the fibring of the manifold with a Riemannian metric which reduces locally to a product of two Riemannian metrices. Among the various types of fibred Riemannian spaces introduced by Y. Mutô, the one with isometric parallel fibres is especially interesting. In this paper we shall attempt to detail some of its properties. For this purpose, it is important to know the bundle structure of the fibred Riemannian space with isometric parallel fibres, but this problem is completely solved by Walker. Starting from a lemma which is covered by the Walker's, we shall study properties of fibred Riemannian spaces with isometric parallel fibres.

For notations and concepts concerning the fibre bundles we follow N. Steenrod [6]. Throughout the whole discussion let the indices run as follows:

$$a, b, c, \ldots. = 1, 2, \ldots., n;\ i, j, k, \ldots. = n+1,\ n+2,\ \ldots.,\ n+m;$$
$$M, N, \ldots. = 1, 2, \ldots., n+m.$$

1. Differentiable fibre bundles. First of all, we shall recall some properties of the differentiable fibre bundles, especially their systems of local coordinates of a special type. In a differentiable fibre bundle $\mathfrak{B}$ of class C^r, its bundle space B, base space X and fibre space Y are all differentiable manifolds of class C^r, and the structure group G of $\mathfrak{B}$ is a group of differentiable transformations of Y onto itself.

In the bundle space B there exists a system of coordinate neighbourhoods $\{W\}$ such that a system of coordinates (x^a, y^i) is defined in each neighbourhood W of $\{W\}$, and moreover the system of equations $x^a =$ const. and $y^i =$ const. give a portion of a fibre and a local slice respectively.

Let p: $B \to X$ be the projection of the bundle structure of $\mathfrak{B}$. So the collection $\{p(W)\}$ of open sets $p(W)$ is a system of neighbourhoods on the base space X and (x^a) is a system of coordinates in $p(W)$. Now we shall denote the open covering $\{p(W)\}$ of X by $\{U_\lambda\}$, say $p(W) = U_\lambda$, where $\{U_\lambda\}$ may be assumed to be a system of coordinate neighbourhoods of the fibre bundle $\mathfrak{B}$. The coordinate function of $\mathfrak{B}$ is given by

$$\phi_\lambda:\ U_\lambda \times Y \to p^{-1}(U_\lambda)$$

for any neighbourhood U_λ of $\{U_\lambda\}$. Further, a mapping

$$\phi_{\lambda,x}:\ Y \to p^{-1}(x)\cdot$$

for any point $x \in U_\lambda$ is defined as follows:

$$\phi_{\lambda,x}(y) = \phi_\lambda(x, y), \ \ y \in Y.$$

Then there exists a mapping

$$p_\lambda : p^{-1}(U_\lambda) \to Y,$$

which is defined by

$$p_\lambda(b) = \phi_{\lambda,x}^{-1}(b), \ \ b \in p^{-1}(U_\lambda), \quad x = p(b) \in U_\lambda.$$

Here it is easily seen that the set $p_\lambda(W) \subset Y$ is open in Y. Thus we are able to take the collection $\{p_\lambda(W)\}$ of such open sets $p_\lambda(W)$ as a system of coordinate neighbourhoods on the fibre space Y, and (y^i) is a system of coordinates in $p_\lambda(W)$. A system of local coordinates on B just considered is called a system of *favourable coordinates* by Y. Mutô. Similarly, such a coordinate neighbourhood W is called *a favourable neighbourhood.*

Let (x^a, y^i) and $(\bar{x}^a, \bar{y}^i)$ be two systems of favourable coordinates at a point of B. Then there exists a transformation, between these systems, expressed by the equations

$$\begin{aligned} \bar{x}^a &= \bar{x}^a(x^1, x^2, \ldots, x^n), \\ \bar{y}^i &= \bar{y}^i(x^1, x^2, \ldots, x^n; y^{n+1}, y^{n+2}, \ldots, y^{n+m}), \end{aligned} \tag{1}$$

whose classes are C^r obviously. The first system of equations (1) gives a transformation of coordinates in the base space X, and the second one for fixed (x^a) is nothing but a local expression of a coordinate transformation of the fibre bundle $\mathfrak{B}$, that is,

$$\bar{y} = \gamma_{\lambda\mu}(x)(y),$$

where $y, \bar{y} \in Y$, $x \in U_\lambda \cap U_\mu \subset X$ and U_λ, U_μ are two intersecting coordinate neighbourhoods of the fibre bundle $\mathfrak{B}$. It is well known that the mapping $\gamma_{\lambda\mu}$: $U_\lambda \cap U_\mu \to G$, which associates an element $\gamma_{\lambda\mu}(x) \in G$ to each point $x \in U_\lambda \cap U_\mu$, is differentiable one of class C^r.

To avoid complexity, the words "differentiable fibre bundle" will be simply replaced by "fibre bundle" in the following sections. We assume hereafter that the classes of differentiability of fibre bundles, manifolds, mappings and so on are sufficiently high.

2. Fibred Riemannian spaces. We shall give the definition of the fibred Riemannian space and its fundamental properties in this section.

An $(n+m)$-dimensional Riemannian space B is called a *fibred Riemannian space,* if its underlying manifold has a bundle structure $\mathfrak{B} = \{B, X, Y, G\}$, where the base space X and the fibre space Y are supposed to be manifolds of n and m dimensions respectively.

There exists a field F of n-dimensional plane-elements which are orthogonal to the tangent space of the fibre at each point of B. Each fibre Y in the given fibred Riemannian space B has a Riemannian metric induced from the metric of B by the inclusion mapping. Consequently any fibre Y will be considered as a Riemannian space having such an induced metric.

By definition, the fibred Riemannian space B is called to have *holonomic* *fibres* according to Y. Mutô, if the field F is completely integrable.

We shall introduce some concepts and notations. Let us take a piece-wise differentiable curve C of the base space X. Let x_0 and x_1 be its initial and terminal points respectively. Then there exists an integral curve $\widetilde{C}$ of the field F which covers the given curve C, if the initial point b_0 of $\widetilde{C}$ is given on the fibre Y_0 over the point x_0. Suppose that b_1 is the terminal point of the integral curve $\widetilde{C}$. Then the point b_1 is on the fibre Y_1 covering the point x_1. Thus there exists a correspondence which associates a point b_1 of Y_1 to a given point b_0 of Y_0, when a curve C is given on the base space X.

This correspondence defines a mapping $\varphi(C)$: $Y_0 \to Y_1$, and the mapping $\varphi(C)$ is obviously differentiable. Especially, if we take a closed piece-wise differentiable curve C passing through a fixed point x_0, then the mapping $\varphi(C)$ maps the fibre Y_0 onto itself. The totality of such mappings $\varphi(C)$ has a group structure and the group thus obtained is a group H_0 of transformations of the manifold Y_0. The group H_0 just introduced is called the *holonomy group* of the given fibred Riemannian space at the point x_0.

It is easily seen that the holonomy group H_x at any point x of X is isomorphic to H_0. The following fact is easily proved:

The fibred Riemannian space B has a bundle structure $\mathfrak{B}$ which has H_0 as its structure group, if the group H_0 is the holonomy group of B at a point $x_0 \in X$.

3. I.P.F. Riemannian spaces. A fibred Riemannian space B is called to have *isometric fibres*, if the mapping $\varphi(C)$: $Y_0 \to Y_1$ defined in § **2** is an isometric correspondence between two fibres Y_0 and Y_1 for any piece-wise differentiable curve C on X.

By definition, a fibred Riemannian space is called to have *isometric parallel fibres* by Y. Mutô, if it has holonomic and isometric fibres, and it is denoted by *I. P. F. Riemannian space* for the sake of convenience. The following theorem has been given by Y. Mutô.

THEOREM 1. *In an I. P. F. Riemannian space B, there exists a system of favourable coordinates (x^a, y^i) having the following properties at each point of B.*

i) *With respect to such a system of coordinates the Riemannian space B has a decomposed metric:*

$$ds^2 = g_{ab}(x)dx^a dx^b + g_{ij}(y)dy^i dy^j. \tag{2}$$

ii) *The system of equations $x^a = const.$ gives a portion of a fibre and the system of equations $y^i = const.$ gives a local slice which is a local integral variety of the field F of plane-elements.*

The Walker's Theorem 2 [7] covers the following lemma.

LEMMA 1. *An I. P. F. Riemannian space B has a bundle structure $\mathfrak{B} = \{B, X, Y, G\}$ having the following properties:*

1°. *The base space X and the fibre space Y are Riemannian spaces.*

2°. *The structure group G of $\mathfrak{B}$ is a Lie group of isometric homeomorphisms* *acting on* Y.

3°. *The coordinate transformations* $\gamma_{\lambda\mu}$ *of the fibre bundle* $\mathfrak{B}$ *are constant functions on* $U_\lambda \cap U_\mu$, *where* U_λ, U_μ *are two coordinate neighbourhoods of the fibre bundle* $\mathfrak{B}$ *such that* $U_\lambda \cap U_\mu \neq \phi$.

Here, we shall give a sketch of the proof of Lemma 1. Let us consider a closed piece-wise differentiable curve C passing through a point $x_0 \in X$ on the base space X, then there exists an isometric mapping $\varphi(C)$ of Y_0 onto itself, where Y_0 is the fibre over the point x_0. It follows obviously from the complete integrability of the field F that two mappings $\varphi(C)$ and $\varphi(C')$ corresponding to two closed curves C and C' respectively are identical, if C and C' are homotopic. Thus the isometric mapping $\varphi(C): Y_0 \to Y_0$ depends only on homotopy class of the closed curve C.

Consequently, we denote this transformation of Y_0 by $s(\alpha)$, where α is the homotopy class of the closed curve C. The totality of these transformations $s(\alpha)$, when α runs over the homotopy group $\pi_1(X)$ of X, forms the holonomy group H_0 of the given I. P. F. Riemannian space B at the point x_0. Then the holonomy group of B is a homomorphic image of the homotopy group $\pi_1(X)$ of X. By some elementary consideration, we can conclude that the structure group G of the fibre bundle B is reducible to the group H_0. Hence, Lemma 1 holds good.

It is remarkable that the structure group of an I. P. F. Riemannian space is isomorphic to a factor group of the homotopy group $\pi_1(X)$ of the base space X. From this remark we have the following result:

COROLLARY. *If an I. P. F. Riemannian space has a simply connected base space, then it is reducible to a product of two Riemannian spaces which are isometric and homeomorphic to the base space and the fibre space respectively.*

At the last of this section we shall seek for a proposition equivalent to the condition 3° of Lemma 1. Let us consider a fibre bundle $\mathfrak{B}$ and its base space X. Let $\widetilde{X}$ be a covering space of X, and $\widetilde{\mathfrak{B}}$ be an induced fibre bundle of the fibre bundle $\mathfrak{B}$ by the projection of the covering structure of $\widetilde{X}$ over X. Then we can prove the following lemma:

LEMMA 2. *A fibre bundle* $\mathfrak{B}$ *has the property* 3° *of Lemma* 1, *if and only if there is a suitable covering space* $\widetilde{X}$ *of its base space* X *and the induced fibre bundle* $\widetilde{\mathfrak{B}}$ *is equivalent to a product bundle.*

PROOF. If a fibre bundle $\mathfrak{B}$ has the property 3° of Lemma 1, then the induced bundle $\widetilde{\mathfrak{B}}_0$ over the universal covering space of $\widetilde{X}_0$ of X is obviously equivalent to a product bundle. Conversely, if an induced bundle $\widetilde{\mathfrak{B}}$ is equivalent to a product bundle $\widetilde{X} \times Y$, it has a cross-section $\mathfrak{M}$, defined by $\widetilde{X} \times y$, passing through any point (x, y) of $\widetilde{B} = \widetilde{X} \times Y$. Let $\rho: \widetilde{X} \to Y$ and $\widetilde{\rho}$: $\widetilde{\mathrm{B}} \to B$ be the natural mappings of coverigns, where B is the bundle space of $\mathfrak{B}$. Suppose that $p: B \to X$ and $\widetilde{p}: \widetilde{B} \to \widetilde{X}$ are the projection of the bundles

$\mathfrak{B}$ and $\widetilde{\mathfrak{B}}$ respectively. Then there exists a relation among them, that is,

$$p\widetilde{\rho} = \rho\widetilde{p}.$$

Putting $\mathfrak{M} = \widetilde{\rho}(\widetilde{\mathfrak{M}})$, then $\mathfrak{M} \subset B$. Moreover,

$$p(\mathfrak{M}) = p\widetilde{\rho}(\widetilde{\mathfrak{M}}) = \rho\widetilde{p}(\widetilde{\mathfrak{M}}) = \rho(\widetilde{X}) = X$$

by virtue of the above relation. Hence $\mathfrak{M}$ is a covering of the space X and its covering projection is given by $p: \mathfrak{M} \rightarrow X$ which is the restriction of $p: B \rightarrow X$ on $\mathfrak{M}$. Moreover, it is easily seen that there exists one and only one subvariety $\mathfrak{M}$ passing through any given point of B. Consequently, it follows that the bundle $\mathfrak{B}$ has a discrete group as its structure group. Therefore, $\mathfrak{B}$ has the property 3° of Lemma 1. Hence Lemma 2 is proved.

If an I. P. F. Riemannian space B satisfies the condition of Lemma 2, then the induced fibre bundle $\widetilde{\mathfrak{B}} = \{\widetilde{B}, \widetilde{X}, Y, G\}$ over $\widetilde{X}$ is a product bundle. Thus, the bundle space $\widetilde{B} = \widetilde{X} \times Y$ of $\widetilde{\mathfrak{B}}$ is a covering space of B, and B has a Riemannian metric of the type (2) which is the Pythagorean sum of the metrices of Y and X. Here, the projection of the covering structure of $\widetilde{B}$ over B is locally an isometric correspondence, that is, the covering $\widetilde{B}$ over B is an *isometric covering.*

4. Inverse problem. We shall now consider the inverse proposition of Lemma 1.

THEOREM 2. *Let us suppose that a differentiable manifold B has a bundle structure satisfying the condition* 1°, 2° *and* 3° *of Lemma* 1. *Then there exists such a Riemannian metric on B that the fibred Riemannian space B with this metric is an I. P. F. Riemannian space.*

PROOF. Let $\mathfrak{B} = \{B, X, Y, G\}$ be the given fibre bundle. Then the fibre space Y and the base space X are both Riemannian spaces and the structure group G is a Lie group of isometric homeomorphisms of Y onto itself.

Let us suppose that $\{U_\lambda\}$ is a system of coordinate neighbourhoods of the given bundle $\mathfrak{B}$ and the mappings $\gamma_{\lambda\mu}: U_\lambda \cap U_\mu \rightarrow G$ are the coordinate transformations of $\mathfrak{B}$. According to the condition 3°, the elements $\gamma_{\lambda\mu}(x) \in G$ is a fixed element for any point $x \in U_\lambda \cap U_\mu$. Thus, in the transformation (1) of two systems of favourable coordinates, the second equations

$$\bar{y}^i = \bar{y}^i(x^1, x^2, \ldots, x^n ; y^{n+1}, y^{n+2}, \ldots, y^{n+m})$$

contain no variable x^i and, in consequence, the transformation (1) has the following expression:

$$\begin{aligned} \bar{x}^a &= x^a(x^1, x^2, \ldots, x^n), \\ \bar{y}^i &= \bar{y}^i(y^{n+1}, y^{n+2}, \ldots, y^{n+m}). \end{aligned} \tag{3}$$

At the present step, we shall introduce a reduced Riemannian metric on the bundle space B. We take a favourable neighbourhood W defined in § **1**. Let $p(W) = U_\lambda$ be a coordinate neighbourhood of the fibre bundle $\mathfrak{B}$ and $p_\lambda(W) = V$ be a neighbourhood of the fibre space Y. Moreover, if a system of favourable coordinates is given by (x^a, y^i) in W, then (x^a) and (y^i) are a system of coordinates in U_λ and V respectively. Take a point b having the coordinates (x^a, y^i) in W; then the point $x = p(b)$ in U_λ and the point $y = p_\lambda(b)$ in V have respectively (x^a) and (y^i) as their coordinates.

Let $\| g_{ab}(x) \|$ be the matrix-representation of the metric tensor of the Riemannian space X at the point x with respect to the system of coordinates (x^a) and $\| g_{ij}(y) \|$ be the similar one related to the Riemannian space Y.

Using the above two matrices, let us consider the following matrix:

$$\| g_{MN}(b) \| = \begin{Vmatrix} g_{ab}(x) & 0 \\ 0 & g_{ij}(y) \end{Vmatrix}$$

at the point $b \in W$. Naturally, the matrix $\| g_{MN}(b) \|$ defines a Riemannian metric on W with respect to (x^a, y^i). If we can show that the matrix $\| g_{MN}(b) \|$, which is given in each W, define a tensor on B, then the proof of the Theorem 2 is completed. This assertion is a consequence of the two facts that the transformation (1) of the systems of favourable coordinates has the special type (3), and that $\| g_{MN}(b) \|$ has a completely reduced type. Thus we have the Theorem 2.

As an additional result we have the following corollary:

COROLLARY. *Let us suppose that a fibre bundle $\mathfrak{B} = \{B, X, Y, G\}$ has a compact Lie group G as its structure group and it satisfies the condition* 3° *of Lemma* 1. *Then the bundle space B is an I. P. F. Riemannian space by a suitable metrization.*

In fact, a Riemannian metric is defined on the base space X by a method of N. Steenrod [6] and the similar process can be applied to the fibre space Y. On the other hand, since the compact Lie group G is a group of transformations operating on Y, then we are able to construct a Riemannian metric on Y which is invariant under the action of G by the average process. Consequently, we have the corollary from Theorem 2.

As a consequence of Lemma 1 and Theorem 2 we have easily the following result:

The bundle space of a fibre bundle $\mathfrak{B}$ is an I. P. F. Riemannian space by a suitable metrization, if and only if $\mathfrak{B}$ has the properties 1°, 2° *and* 3° *of Lemma* 1.

5. The Betti numbers of the I. P. F. Riemannian spaces. When an I. P. F. Riemannian space B is compact, the fibre space Y and the base space X are both compact. It is easily seen that X is orientable for orientable B and Y. Similarly, when B and X are orientable, Y is so. Here we shall prove the following theorem.

THEOREM 3. *Consider a compact orientable I. P. F. Riemannian space B. If either the fibre space or the bace space is orientable, and moreover if its structure group G is a subgroup of a connected group of isometries on Y, then the p-th Betti numbers $R_p(B)$ of B is equal to the p-th Betti numbers $R_p(X \times Y)$ of the product space of X and Y for all integers p such that*

$0 \leqq p \leqq \dim B$.

To prove Theorem 3, we have to remark the following lemma due to K. Yano [8].

LEMMA 3. *The harmonic forms on a compact orientable Riemannian space are invariant under a connected group of isometries of this space.*

PROOF OF THEOREM 3. The following inequalities are well known for any fibre bundle $\mathfrak{B} = \{B, X, Y\}$. (See [3] and [4]). That is,

$$R_p(B) \leqq R_p(X \times Y) \qquad (0 \leqq p \leqq \dim B). \tag{4}$$

Consequently, if we get the inequalities

$$R_p(B) \geqq R_p(X \times Y) \qquad (0 \leqq p \leqq \dim B), \tag{5}$$

then we have the required Theorem 3.

Now, we are going to show the inequalities (5). First of all, it is usefull to associate a harmonic form on B to a given harmonic form on Y. If we take a harmonic form ω on Y, then we can define a differential form ω_λ on $U_\lambda \times Y$ as follows:

$$\omega_\lambda = \delta\rho_\lambda(\omega).$$

Here $\delta\rho_\lambda$ is the dual of $d\rho_\lambda$ which is the differential of the projection $\rho_\lambda : U_\lambda \times Y \to Y$ and U_λ is a coordinate neighbourhood of the fibre bundle $\mathfrak{B} = \{B, X, Y\}$. Let a homeomorphism

$$\phi_\lambda^{-1} : p^{-1}(U_\lambda) \to U_\lambda \times Y$$

be the inverse mapping of the local isometric homeomorphism ϕ_λ of the fibre bundle $\mathfrak{B}$. Denote the differential of the homeomorphism ϕ_λ^{-1} by $d(\phi_\lambda^{-1})$, and the dual mapping of $d(\phi_\lambda^{-1})$ by $\delta(\phi_\lambda^{-1})$. Now, we shall define a differential form $\widetilde{\omega}_\lambda$ on $p^{-1}(U_\lambda)$ by the relation

$$\widetilde{\omega}_\lambda = \delta(\phi_\lambda^{-1})(\omega_\lambda).$$

For a point $b \in p^{-1}(U_\lambda) \cap p^{-1}(U_\mu)$ the point $x = p(b)$ is in $U_\lambda \cap U_\mu$, where U_λ, U_μ are two intersecting coordinate neighbourhoods of the bundle $\mathfrak{B}$. Then, at the point b two differential forms $\widetilde{\omega}_\lambda(b)$ and $\widetilde{\omega}_\mu(b)$ are defined and they are related as follows. That is, by a simple consideration, we have

$$\widetilde{\omega}_\mu = \delta(\phi_\lambda^{-1})\cdot\delta(\gamma_{\lambda\mu}(x))(\omega),$$

where $\delta(\gamma_{\lambda\mu}(x))$ is the dual of the differential of the transformation $\gamma_{\lambda\mu}(x)$: $Y \to Y$. But, since ω is a harmonic form, it is invariant under the structure group G by means of Lemma 3 just mentioned. Then we have

$$\widetilde{\omega}_\mu = \delta(\phi_\lambda^{-1})(\omega).$$

Hence, it follows that

$$\widetilde{\omega}_\mu(b) = \widetilde{\omega}_\lambda(b).$$

From the above discussion, it is easily seen that a differential form $\widetilde{\omega}$ on B is defined by the differential form $\widetilde{\omega}_\lambda$ on $p^{-1}(U_\lambda)$. On the other hand, Christoffel's symbols of B satisfy the relations

$$\left\{ \begin{matrix} M \\ i\,a \end{matrix} \right\} = 0,$$

since the metric is decomposed as shown in (2) of Theorem 1. Using these relations, we have by a slight calculation that the differential form $\widetilde{\omega}$ is harmonic as a consequence of the fact that the form ω is harmonic.

Furthermore, it is not difficult to show that the two harmonic forms $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ respectively corresponding to harmonic forms ω_1 and ω_2 on Y are linearly independent, if ω_1 and ω_2 are so.

Next, there corresponds a harmonic form $\widetilde{\Omega}(b)$ on B with a given one $\Omega(x)$ on X in such a manner that $\widetilde{\Omega} = \delta p(\Omega)$, where δp is the dual mapping of the differential of the projection $p: B \to X$ of the fibre bundle $\mathfrak{B}$. By the same reason as above, the linear independency between the several harmonic forms on X is preserved by this correspondence. Finally, it is easily seen that the two harmonic forms $\widetilde{\omega}$ and $\widetilde{\Omega}$ thus introduced are linearly independent to each other.

Summing up the above results, we arrive at the required inequalities (5) by virtue of Hodge's theorem. Thus Theorem 3 is proved.

Next, we have the following theorem:

THEOREM 4. *Let B be an I. P. F. Riemannian space satisfying the conditions of Theorem* 3. *Then every fibre of B is not homologous to zero in B. And all of them are contained in a fixed homology class of B, when B is connected.*

PROOF. Let us suppose that a fibre Y_b passing through a point b of B is homologous to zero in B. Let ω be the differential form on Y expressing the volume element of the fibre space Y. Then ω is harmonic. Corresponding to ω, a harmonic form $\widetilde{\omega}$ of degree m ($m = \dim Y$) on B is defined by the same method as we used in the proof of Theorem 3. Since Y_b is homologous to zero and $\widetilde{\omega}$ is harmonic, then we have

$$V(Y_b) = \int_{Y_b} \widetilde{\omega} = 0.$$

On the other hand, $V(Y_b)$ is not equal to zero, since the integral $V(Y_b)$ expresses obviously the total volume of Y_b. This result constradicts the preceding conclusion. Hence, the first part of Theorem 4 holds true.

The second part follows easily from the fact that any two consecutive fibres are mutually homologous in B, since B is locally homeomorphic to the product of Y and an open subset of X by virtue of the local product representation of B. Thus Theorem 4 is proved completely.

6. Examples and applications. We shall consider some examples of the I. P. F. Riemannian spaces and their applications.

6.1. The torus and the Klein bottles are simple examples of the I. P. F.

Riemannian space. In these examples the base spaces are circles, and then the universal covering spaces of these base spaces are contractible to a point. Generally, if a fibre bundle $\mathfrak{B}$ has a base space whose universal covering space is constractible to a point, then the bundle $\mathfrak{B}$ has the property 3° of Lemma 1.

6.2. Recently C. Ehresmann [2], S. S. Chern [1] and other authors have dealt with the infinitesimal connection in a fibre bundle. In this place we attempt to characterize the fibre bundle which admits a locally flat infinitesimal connection.

The infinitesimal connection in a fibre bundle $\mathfrak{B}$ is introduced in two manners. In the first method, it is defined by a differential form with some special properties on the bundle space. In the other method, the connection is defined using a field of n-planes of a special type in the bundle space, where n is the number of dimension of the base space.

In the latter definition the *local flatness* is defined by the complete integrability of the n-field defining the connection.

Here, we shall recall the second definition of the infinitesimal connection. In a principal fibre bundle $\mathfrak{B}=\{B, X, G, G\}$ an infinitesimal connection is defined by a field F of n-planes which satisfies the following two conditions:

i) The tangent space of B at any point b of B is spanned by the plane-element $F(b)$ of the field F and the tangent plane of the fibre at the point b.

ii) The field F is invariant under the right translations of the principal bundle $\mathfrak{B}$.

It is well known that *a principal bundle $\mathfrak{B}$ admitting a locally flat connection has the property* 3° *of Lemma* 1. (See [1]).

Conversely, it is interesting to ask whether a principal bundle having the property 3° of Lemma 1 admits a locally flat connection or not. To attack this problem, we shall assume that the structure group of $\mathfrak{B}$ is a compact group G. Then it follows from the corollary of Theorem 2 that the bundle space B is an I. P. F. Riemannian space by a suitable metrization. Since the group G is compact, the group G has a Riemannian metric g_{ij} on its group space which is invariant under the left- and the right-translation by G itself. Now, using this invariant metric g_{ij} on the fibre space G, we can introduce a metric g_{MN} of the I. P. F. Riemannian space B. In this case it is easily seen that all of the right-translation of the principal bundle $\mathfrak{B}$ are isometries of the I. P. F. Riemannian space B.

Let b a point of B and $F(b)$ be an n-plane orthogonal to the tangent plane of the fibre through the point b. Then a field F of n-planes is defined on B by the correspondence $b \rightarrow F(b)$. We shall show that a locally flat infinitesimal connection is defined by the field F of n-planes thus introduced.

In fact, the field F satisfies obviously the condition i). Since the Riemannian metric g_{MN} is invariant under the right-translations of $\mathfrak{B}$, the field F is also invariant. Consequently, the condition ii) is fulfilled by the field F. Hence the field F defines an infinitesimal connection on $\mathfrak{B}$. Finally, it is obvious that the field F is completely integrable, and then the connection defined by F is locally flat. Thus we have a locally flat connection on $\mathfrak{B}$. The following theorem is obtained from the discussions above.

THEOREM 5. *A principal fibre bundle with a compact structure group admits a locally flat infinitesimal connection when and only when it has the property* 3° *of the Lemma* 1.

The author wishes to express his thanks to Prof. H. Hombu, Prof. S. Hokari and his colleague M. Obata for their advices.

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